EAIFR, Kigali

Lecture notes
2026

Electromagnetism and Relativity

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April 2026

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1.1 Introduction and basic tools

The nineteenth century was a crucial period for physics, since it was during this time that electric and magnetic phenomena were finally unified in Maxwell's equations. Unfortunately, this framework turned out to be incompatible with Newtonian dynamics, whose success had provided a precise description of mechanics at all scales, at least within its domain of validity.

Reconciling these two approaches ultimately required abandoning Newtonian mechanics in favor of Einstein's theory of relativity. In these lecture notes, we review a variety of electromagnetic phenomena.

Electric and magnetic phenomena were studied extensively in the nineteenth century, leading to a continuous description in terms of classical fields: the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$. Charged particles act as sources of these fields, and their space-time evolution is governed by

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0}, \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned} \tag{1.1}$$

Here, ρ and $\mathbf{J}$ denote the electric charge density and the electric current density, respectively. The operator $\boldsymbol{\nabla} \equiv \frac{\partial}{\partial \mathbf{r}}$ is the nabla operator, while μ_0 and ϵ_0 are the vacuum permeability and vacuum permittivity, with $\mu_0 \epsilon_0 c^2 = 1$.

Some basic remarks:

- When all quantities are time independent, the electric and magnetic fields become uncoupled. This leads to electrostatics and magnetostatics.
- Each field satisfies one equation with a source term and one additional equation that plays the role of a constraint.
- Can we prove that, for a given distribution of charges and/or currents, the electric and magnetic fields are uniquely determined?
- Are these equations compatible with Newtonian mechanics?

All of these questions will be addressed in the remainder of these notes.

1.2 Recap on vector calculus

Starting from a vector field $\mathbf{A}(\mathbf{r}, t)$ and a scalar field $\phi(\mathbf{r}, t)$, one has

- $$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0} \tag{1.2}$$
  This identity means that if the curl of a vector field vanishes, then the field can be expressed as the gradient of a scalar field.

- $$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0 \tag{1.3}$$
  This identity means that if a vector field has zero divergence, then it can be expressed as the curl of another vector field.

- $$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} \tag{1.4}$$

- $$\boldsymbol{\nabla} \cdot (\phi \mathbf{A}) = \mathbf{A} \cdot \boldsymbol{\nabla}\phi + \phi\, \boldsymbol{\nabla} \cdot \mathbf{A} \tag{1.5}$$

- $$\boldsymbol{\nabla} \times (\phi \mathbf{A}) = \boldsymbol{\nabla}\phi \times \mathbf{A} + \phi\, \boldsymbol{\nabla} \times \mathbf{A} \tag{1.6}$$

- $$\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) \tag{1.7}$$

- $$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = -\mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) \tag{1.8}$$

- $$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} \tag{1.9}$$

A second series of identities concerns integrals of vector and scalar fields. Let $\mathbf{A}(\mathbf{r}, t)$ be a vector field and $\phi(\mathbf{r}, t)$, $\psi(\mathbf{r}, t)$ two scalar fields. Let V be a three-dimensional volume with volume element $d\mathbf{r}^3$, and let S be a closed two-dimensional surface bounding V, with area element da and outward unit normal $\mathbf{n}$. Then

- The divergence theorem:
  $$\int_V \boldsymbol{\nabla} \cdot \mathbf{A}\, d\mathbf{r}^3 = \int_S \mathbf{A} \cdot \mathbf{n}\, da \tag{1.10}$$
  Basically, this theorem expresses that the volume integral of the divergence is equal to the flux of the vector field through the bounding surface S. In electrostatics, this theorem is often referred to as Gauss's theorem.

- $$\int_V \boldsymbol{\nabla}\phi\, d\mathbf{r}^3 = \int_S \phi\, \mathbf{n}\, da \tag{1.11}$$
  A similar theorem holds for a scalar field.

- $$\int_V \boldsymbol{\nabla} \times \mathbf{A}\, d\mathbf{r}^3 = \int_S \mathbf{n} \times \mathbf{A}\, da \tag{1.12}$$
  Be careful with this identity because the cross product is not commutative.

- Green's first identity:
  $$\int_V \left(\phi \boldsymbol{\nabla}^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi\right) d\mathbf{r}^3 = \int_S \phi\, \mathbf{n} \cdot \boldsymbol{\nabla}\psi\, da \tag{1.13}$$

- Green's theorem:

$$\int_V \left(\phi \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \phi\right) d\mathbf{r}^3 = \int_S \left(\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi\right) \cdot \mathbf{n}\, da \tag{1.14}$$

These theorems can be supplemented by identities involving an open surface S, where C is the contour bounding it, with line element $d\mathbf{l}$. The normal vector $\mathbf{n}$ is defined according to the right-hand rule with respect to the orientation of the line integral around C.

- Stokes's theorem:

$$\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{n}\, da = \oint_C \mathbf{A} \cdot d\mathbf{l} \tag{1.15}$$

-

$$\int_S \mathbf{n} \times \boldsymbol{\nabla} \phi\, da = \oint_C \phi\, d\mathbf{l} \tag{1.16}$$

1.3 Electrostatics

We first consider the situation in which all quantities are time independent. Maxwell's equations then simplify to

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.17}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{1.18}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0 \tag{1.19}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1.20}$$

The electric and magnetic fields are therefore uncoupled, but for each field we still need to solve a system of two equations.

1.3.1 Poisson equation

For the electric field, Eq. (1.19) implies that

$$\mathbf{E} = -\boldsymbol{\nabla} V \tag{1.21}$$

where V is a scalar field. Obviously, the solution is not unique, since adding any constant to V leaves the electric field unchanged. The field V has a simple physical interpretation: it gives the work required per unit positive charge to bring a particle to the position $\mathbf{r}$. It is therefore the electrostatic potential, or equivalently the potential energy per unit charge.

Inserting Eq. (1.21) into Eq. (1.17) gives

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0}. \tag{1.22}$$

This partial differential equation is called Poisson's equation. In regions free of charge, the corresponding homogeneous equation is Laplace's equation.

Several methods are available to obtain the electric field. We now review them.

1.3.2 Gauss's law

We begin with simple cases by exploiting the integral form of Eq. (1.17). Let V denote a closed region of Euclidean space $\mathcal{R}^3$, and let $S = \partial V$ be its boundary. Integrating both sides of Eq. (1.17) over the volume V, one obtains

$$\int_V d\mathbf{r}\, \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0} \int_V d\mathbf{r}\, \rho(\mathbf{r}) \tag{1.23}$$

The integral on the right-hand side defines the total charge contained in V,

$$Q = \int_V d\mathbf{r}\, \rho(\mathbf{r}). \tag{1.24}$$

For the left-hand side of Eq. (1.23), we use the divergence theorem, which transforms the volume integral into a surface integral:

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0} \tag{1.25}$$

The physical meaning of Eq. (1.25) is that the flux of the electric field through a closed surface is equal to the total charge enclosed divided by ϵ_0.

1.3.3 A single charge

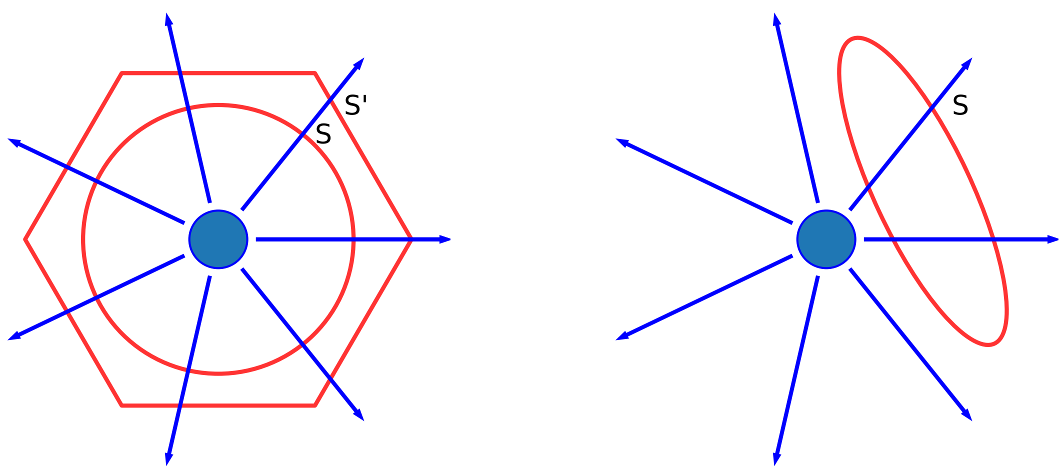

Figure 1.1: Left: the flux of the electric field through the two surfaces S and S' is identical. Right: the flux through the surface S is equal to 0.

Figure 1.1 illustrates the following situation. In the left panel, a charge Q lies inside two different closed surfaces: the circle S and the hexagonal contour S'. Gauss's theorem shows that the electric flux is the same through both surfaces. In the right panel, the closed surface S encloses no charge, and therefore the electric flux through S is zero.

If a single charge Q is isolated in Euclidean space, the electric field has spherical symmetry:

$$\mathbf{E}(\mathbf{r}) = E(r)\mathbf{e}_r,$$

where $\mathbf{e}_r$ is the radial unit vector in spherical coordinates. Using a sphere of radius R larger than the size of the charge distribution, Gauss's theorem gives

$$4\pi R^2 E(R) = \frac{Q}{\epsilon_0}, \tag{1.26}$$

so that

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{e}_r. \tag{1.27}$$

Note that Gauss's theorem cannot determine the full electric field if charges are present outside the chosen Gaussian surface.

1.3.4 Principle of superposition

When several charges are distributed in different regions of space, the principle of superposition allows us to obtain the total electric field.

Assume that the charge density can be decomposed as

$$\rho = \rho_1 + \rho_2. \tag{1.28}$$

If V_1 and V_2 satisfy

$$\begin{aligned} \boldsymbol{\nabla}^2 V_1 &= -\frac{\rho_1}{\epsilon_0}, \\ \boldsymbol{\nabla}^2 V_2 &= -\frac{\rho_2}{\epsilon_0}, \end{aligned} \tag{1.29}$$

then $V = V_1 + V_2$ is a solution of

$$\boldsymbol{\nabla}^2 V = -\frac{\rho_1 + \rho_2}{\epsilon_0}. \tag{1.30}$$

The generalization to a sum of N charge distributions is straightforward.

1.3.5 Dipole

A dipole consists of a pair of opposite charges q and $-q$ separated by a distance d (see Fig. 1.2). The potential is the sum of the potentials created by each charge:

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} - \frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|}. \tag{1.31}$$

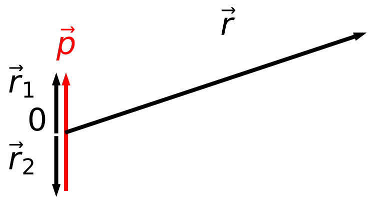

Figure 1.2: Dipole configuration.

At large distances, $r \gg r_1, r_2$, the first-order asymptotic expansion is

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}_i}{r^3} + O\left(\frac{1}{r^3}\right). \tag{1.32}$$

Since the total charge vanishes, the first term cancels and the potential becomes

$$V(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}, \tag{1.33}$$

where $\mathbf{p} = q(\mathbf{r}_1 - \mathbf{r}_2)$ is the dipole moment. Note that $\mathbf{p}$ is independent of the choice of origin.

The electric field is obtained from the gradient of the potential. For two opposite charges,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}-\mathbf{r}_1) - \frac{q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}-\mathbf{r}_2), \tag{1.34}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the charges q and $-q$, respectively.

At large distances, using either Eq. (1.34) or Eq. (1.33), one finds

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3} = \frac{3(\mathbf{p}\cdot\mathbf{e_r})\mathbf{e_r} - \mathbf{p}}{r^3}, \tag{1.35}$$

where $\mathbf{e_r} = \mathbf{r}/r$ is the radial unit vector. The second expression shows clearly that the dipolar electric field decays as $1/r^3$.

Using spherical coordinates, with $\mathbf{p}$ aligned along the z-axis, the field components are

$$\begin{aligned} E_r &= \mathbf{E}(\mathbf{r})\cdot\mathbf{e_r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \\ E_\phi &= \mathbf{E}(\mathbf{r})\cdot\mathbf{e}_\phi = 0, \\ E_\theta &= \mathbf{E}(\mathbf{r})\cdot\mathbf{e}_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}. \end{aligned} \tag{1.36}$$

Figure 1.3 illustrates the nonzero components of the electric field generated by two opposite charges separated by a distance d.

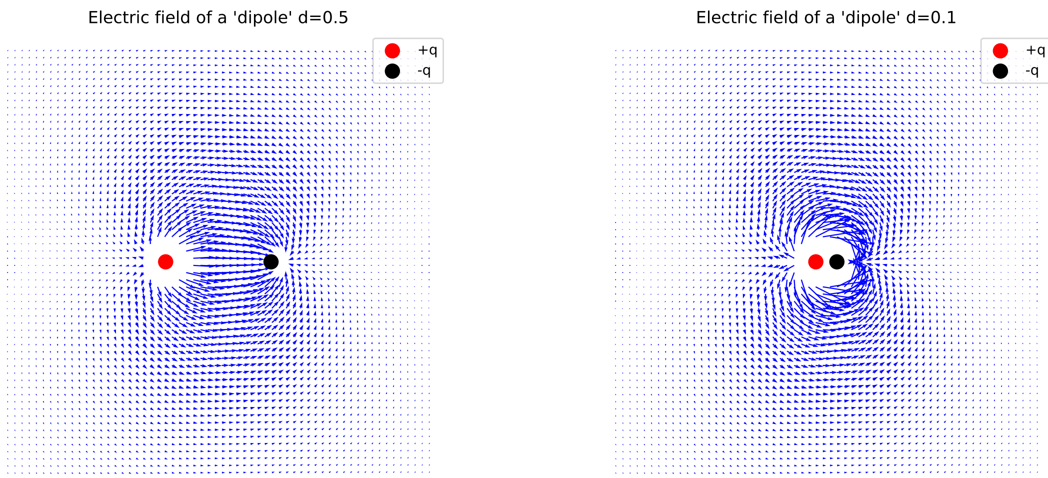

Figure 1.3: Electric field $\mathbf{E}$ produced by two opposite charges separated by a distance (left) $d = 0.5$ and (right) $d = 0.1$.

1.3.6 A charged sphere

Eq. (1.27) gives the electric field outside the region occupied by the charge. If the size of the charge distribution tends to zero, the field behaves like the inverse square of the distance everywhere. However, the field then diverges at short distance, which signals an oversimplification of the physical description.

A possible regularization is to model the charge as a uniformly charged sphere of radius R and density ρ. The symmetry of the electric field is unchanged, so Gauss's theorem can again be applied to a sphere of radius r centered at the origin.

Two cases must be distinguished:

- $r \geq R$. The Gaussian sphere encloses the total charge:

$$\begin{aligned} 4\pi r^2 E(r) &= \frac{4\pi\rho R^3}{3\epsilon_0} \\ &= \frac{Q}{\epsilon_0}, \end{aligned} \tag{1.37}$$

  where $Q = \frac{4\pi\rho R^3}{3}$. Therefore

$$\mathbf{E}(r) = \frac{Q}{4\pi r^2 \epsilon_0}\mathbf{e}_r. \tag{1.38}$$

- $r < R$. The Gaussian sphere encloses only part of the charge:

$$\begin{aligned} 4\pi r^2 E(r) &= \frac{4\pi\rho r^3}{3\epsilon_0}, \\ E(r) &= \frac{Qr}{4\pi\epsilon_0 R^3}, \end{aligned} \tag{1.39}$$

  so that

$$\mathbf{E}(r) = \frac{Qr}{4\pi\epsilon_0 R^3}\mathbf{e}_r. \tag{1.40}$$

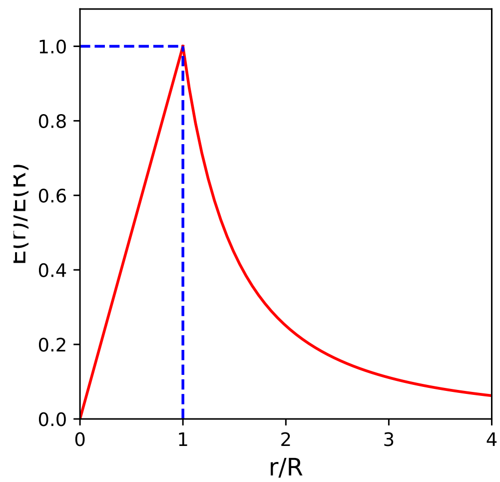

Figure 1.4: Dimensionless electric field $E(r)/E(R)$ versus the dimensionless distance r/R.

Figure 1.4 shows the dimensionless field $E(r)/E(R)$ as a function of r/R. Inside the sphere, the electric field increases linearly with r and reaches its maximum at $r = R$. Outside the sphere, it decreases as $1/r^2$.

Some remarks:

- The electric field remains continuous, but its derivative has a cusp at $r = R$.

- Introducing a microscopic length scale removes the divergence. However, for an electron the classical description is no longer valid at such scales, and no classical electron radius can be meaningfully assigned in this framework.

1.3.7 A charged line

To show that Gauss's theorem remains useful in other geometries with sufficient symmetry, we now consider an infinite straight line carrying a uniform linear charge density λ.

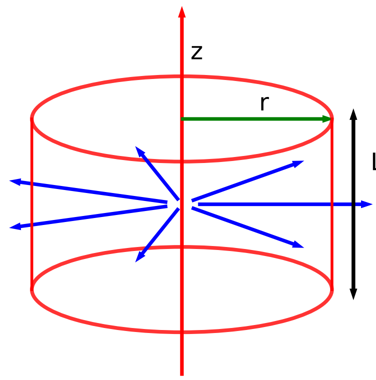

Figure 1.5: Infinite straight line with linear charge density λ. The electric field (blue arrows) has cylindrical symmetry.

By cylindrical symmetry, the electric field has only a radial component,

$$\mathbf{E} = E(r)\mathbf{e}_r,$$

where $\mathbf{e}_r$ is the radial unit vector. Using Gauss's theorem with a cylinder of height L and radius r, the flux through the top and bottom vanishes, while the charge enclosed is λL. Hence

$$2\pi r L E(r) = \frac{\lambda L}{\epsilon_0}, \tag{1.41}$$

which gives

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\,\mathbf{e}_r. \tag{1.42}$$

For a point charge, the electric field decays as r^{-2}, whereas for an infinite line it decays more slowly, as r^{-1}.

The potential of a charged line can be obtained by integrating the electric field. In cylindrical coordinates,

$$V(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r}{r_0}\right), \tag{1.43}$$

where r_0 is an arbitrary reference distance. Unlike the potential of a localized charge distribution, this potential diverges at large distance. Its value is therefore defined only relative to a finite reference radius r_0.

1.3.8 A charged plane

To conclude this series of basic examples, we consider an infinite plane, chosen for simplicity as the plane $z = 0$.

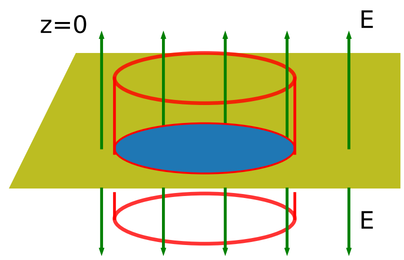

Figure 1.6: Charged plane with surface density σ. The electric field (blue arrows) has translational symmetry.

Translational symmetry along the x and y directions implies that the electric field must be of the form

$$\mathbf{E} = E(z)\mathbf{e}_z. \tag{1.44}$$

Moreover, symmetry with respect to reflection across the plane implies

$$E(-z) = -E(z). \tag{1.45}$$

Choosing a cylindrical Gaussian surface crossing the plane at $z = 0$, Gauss's theorem gives

$$\pi r^2(E(z) - E(-z)) = \frac{\pi r^2 \sigma}{\epsilon_0}. \tag{1.46}$$

Using Eq. (1.45), one obtains

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\mathbf{e}_z. \tag{1.47}$$

Thus, for an infinite uniformly charged plane, the electric field is independent of the distance from the plane. This is a consequence of the fact that the plane is infinite: the total charge contributing to the field grows without bound. Moreover, the electric field exhibits a finite discontinuity at the plane:

$$E(z \to 0^+) - E(z \to 0^-) = \frac{\sigma}{\epsilon_0}. \tag{1.48}$$

1.3.9 A charged slab

This discontinuity is a consequence of modeling the charge as an infinitely thin plane. A more physical model is a slab of finite thickness $2d$, with uniform volume charge density ρ. To compare with the charged plane, one has $\sigma = 2\rho d$. The symmetry is the same, but one must distinguish two regions: (i) $0 \leq z \leq d$ and (ii) $z > d$.

(i) Outside the slab, Gauss's theorem gives

$$\pi r^2(E(z) - E(-z)) = \frac{\pi r^2 \rho 2d}{\epsilon_0}, \tag{1.49}$$

which leads to

$$\mathbf{E}(z) = \frac{\sigma}{2\epsilon_0}\mathbf{e}_z. \tag{1.50}$$

(ii) Inside the slab, using a cylinder of height $2z$, one obtains

$$\pi r^2(E(z) - E(-z)) = \frac{\pi r^2 \rho 2z}{\epsilon_0}, \tag{1.51}$$

so that

$$\mathbf{E}(z) = \frac{\sigma z}{2d\epsilon_0}\mathbf{e}_z. \tag{1.52}$$

Figure 1.7 shows the electric field as a function of z for two slabs of widths $d = 1$ and $d = 0.5$. Outside the slab, the fields are identical. Inside the slab, the field varies linearly with z. In the limit $d \to 0$, one recovers the finite discontinuity at $z = 0$.

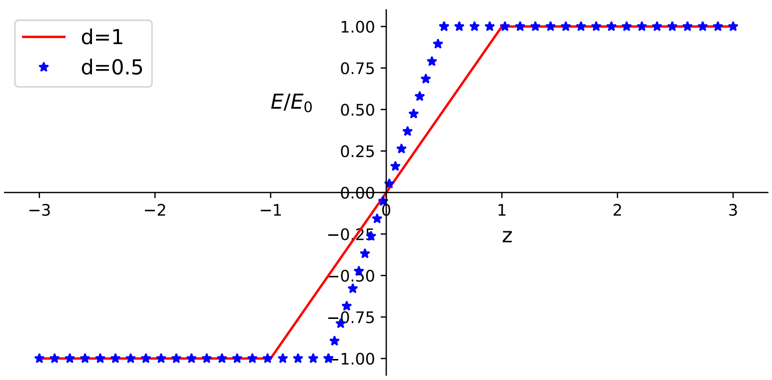

Figure 1.7: Electric field versus z for two slabs with $d = 0.5$ and $d = 1$.

1.3.10 General cases: Green's function

To go beyond situations in which symmetry arguments are sufficient, we first discuss the uniqueness of the electrostatic solution. We consider the electrostatic potential V, which satisfies Poisson's equation

$$\boldsymbol{\nabla}^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \tag{1.53}$$

A unique electric field

A first issue concerns uniqueness. For a given electric field associated with a potential V_1, any potential $V_1 + A$, where A is a constant, produces the same electric field. Now suppose that there are two solutions of Eq. (1.53), denoted V_1 and V_2, which are not related by a simple additive constant.

Consider a finite volume V bounded by a surface Σ. When the values of the potential are fixed on the boundary (Dirichlet conditions), or when its normal derivative is fixed there (Neumann conditions), one can show that the solution of Poisson's equation, when it exists, is unique up to an additive constant.

Assume that V_1 and V_2 satisfy the same boundary conditions. Then

$$W = V_1 - V_2$$

obeys Laplace's equation,

$$\boldsymbol{\nabla}^2 W = 0. \tag{1.54}$$

Using the identity

$$\boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} W) = \boldsymbol{\nabla}\alpha \cdot \boldsymbol{\nabla} W + \alpha \boldsymbol{\nabla}^2 W, \tag{1.55}$$

where α is a scalar function, and noting that the second term vanishes by Eq. (1.54), we choose $\alpha = W$ and compute

$$\begin{aligned} \mathcal{I} &= \iiint_V d\mathbf{r}^3 \, (\boldsymbol{\nabla} W)^2 \\ &= \iint_S W \, \boldsymbol{\nabla} W \cdot d\boldsymbol{\Sigma}. \end{aligned} \tag{1.56}$$

The second line follows from the divergence theorem. Because the two solutions satisfy the same Dirichlet or Neumann boundary conditions, the surface term vanishes, so $\mathcal{I} = 0$. Since the integrand is nonnegative, we conclude that

$$\boldsymbol{\nabla} W = 0 \tag{1.57}$$

everywhere in the volume. Therefore

$$V_1 = V_2 + A,$$

where A is a constant, and the electric field is unique.

Definition

The Green's function of the Poisson operator is defined by

$$\boldsymbol{\nabla}^2 G(\mathbf{r}, \mathbf{r}') = \delta^d(\mathbf{r} - \mathbf{r}'), \tag{1.58}$$

where δ^d is the Dirac delta distribution in d spatial dimensions.

The Green's function can be interpreted as the impulse response of a linear operator. Because the Laplacian is translation invariant, the Green's function is also translation invariant:

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}'). \tag{1.59}$$

Therefore, for the Poisson equation

$$\boldsymbol{\nabla}^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}, \tag{1.60}$$

the solution can be written as

$$V(\mathbf{r}) = -\iiint_V d\mathbf{r}'^d \, G(\mathbf{r} - \mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0}. \tag{1.61}$$

The potential is thus given by the convolution of the Green's function with the charge density.

When the domain is the full space $\mathcal{R}^d$, the Green's functions are known explicitly:

- In three dimensions ($d = 3$),
$$G(\mathbf{r} - \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{1.62}$$

- In two dimensions ($d = 2$),
$$G(\mathbf{r} - \mathbf{r}') = -\frac{1}{2\pi} \ln|\mathbf{r} - \mathbf{r}'|. \tag{1.63}$$

- In one dimension ($d = 1$),
$$G(\mathbf{r} - \mathbf{r}') = -\frac{1}{2}|\mathbf{r} - \mathbf{r}'|. \tag{1.64}$$

In one and two dimensions, the potential diverges at large distance. By contrast, in three dimensions the potential decays to zero as $r \to \infty$.

1.3.11 Multipole expansion of the potential

When the charge distribution is localized within a finite region, the potential is well defined and can be written as

$$V(\mathbf{r}) = \iiint_V d\mathbf{r}'^3 \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}. \tag{1.65}$$

It is then possible to obtain an asymptotic expansion of the potential at large distance.

In three dimensions, for $r > r'$, one can expand

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta), \tag{1.66}$$

where P_l is the Legendre polynomial of order l[1].

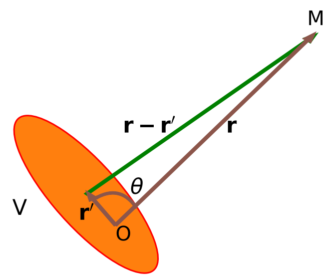

Figure 1.8: Charged volume V (orange region) and observation point M.

Inserting Eq. (1.66) into Eq. (1.65) gives

$$V(\mathbf{r}) = \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\iiint_V d\mathbf{r}'^3 \frac{(r')^l\rho(\mathbf{r}')}{4\pi\epsilon_0}P_l(\cos\theta). \tag{1.70}$$

- Zeroth order:

$$\begin{aligned} V_0(\mathbf{r}) &= \frac{1}{r}\iiint_V d\mathbf{r}'^3\frac{\rho(\mathbf{r}')}{4\pi\epsilon_0} \\ &= \frac{Q}{4\pi\epsilon_0 r}, \end{aligned} \tag{1.71}$$

  where Q is the total charge.

[1] The Legendre polynomials form a sequence of orthogonal polynomials. Their generating function is

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n. \tag{1.67}$$

Their normalization on the interval $[-1,1]$ is

$$\int_{-1}^{1}dx\,P_n(x)P_m(x) = \frac{2}{2n+1}\delta_{nm}. \tag{1.68}$$

The first few polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{1}{2}(3x^2-1). \tag{1.69}$$

- First order:

$$V(\mathbf{r}) = V_0(\mathbf{r}) + \frac{1}{r^2}\iiint_V d\mathbf{r}'^3 \frac{\rho(\mathbf{r}')r'\cos\theta}{4\pi\epsilon_0}. \tag{1.72}$$

  Since θ is the angle between $\mathbf{r}$ and $\mathbf{r}'$,

$$\cos\theta = \frac{\mathbf{r}\cdot\mathbf{r}'}{rr'}. \tag{1.73}$$

  Thus

$$V_1(\mathbf{r}) = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0\, r^3}, \tag{1.74}$$

  where the dipole moment is

$$\mathbf{p} = \iiint_V d\mathbf{r}'^3 \rho(\mathbf{r}')\mathbf{r}'. \tag{1.75}$$

  Note that when $Q \neq 0$, the dipole moment depends on the choice of origin. If the origin is shifted from O to O', then

$$\begin{aligned}\mathbf{p}' &= \iiint_V d\mathbf{r}''^3 \rho(\mathbf{r}'')\mathbf{r}'' \\ &= \iiint_V d\mathbf{r}'^3 \rho(\mathbf{r}')(\mathbf{r}' + \mathbf{O'O}) \\ &= \mathbf{p} + Q\mathbf{O'O}.\end{aligned} \tag{1.76}$$

  When $Q = 0$, the dipole moment is independent of the origin. In this case, it characterizes the separation between positive and negative charges. Many physical systems exhibit such a structure; diatomic molecules are a standard example.

- Second order: the potential contains a quadrupolar contribution,

$$\begin{aligned}V_2(\mathbf{r}) &= \frac{1}{r^3}\iiint_V d\mathbf{r}'^3 \frac{\rho(\mathbf{r}')r'^2(3\cos^2\theta - 1)}{8\pi\epsilon_0} \\ &= \frac{1}{r^5}\iiint_V d\mathbf{r}'^3 \frac{\rho(\mathbf{r}')\left[3(\mathbf{r}\cdot\mathbf{r}')^2 - r'^2r^2\right]}{8\pi\epsilon_0}.\end{aligned} \tag{1.77}$$

  The quantity in brackets can be written in matrix form as

$$3(\mathbf{r}\cdot\mathbf{r}')^2 - r'^2r^2 = \mathbf{r}\cdot Q\,\mathbf{r}, \tag{1.78}$$

  where Q is the quadrupole tensor with components

$$Q_{ij} = \iiint_V d\mathbf{r}'^3 \rho(\mathbf{r}')\left(3r'_ir'_j - r'^2\delta_{ij}\right). \tag{1.79}$$

  Here r'_i is the ith component of $\mathbf{r}'$. The tensor Q is symmetric and traceless. For it to be independent of the choice of origin, both the total charge and the dipole moment must vanish.

  Thus

$$V_2(\mathbf{r}) = \frac{\mathbf{r}\cdot Q\mathbf{r}}{8\pi\epsilon_0 r^5}. \tag{1.80}$$

The expansion can be continued to higher order, although the expressions become increasingly cumbersome. While it is common for atoms or molecules to be neutral and to have no permanent dipole moment, it is less common for the quadrupole moment to vanish as well. A typical example is the CO_2 molecule.

1.3.12 Multipole expansion of the field

Since we are ultimately interested in the electric field, combining Eqs. (1.21) and (1.61) gives

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= -\boldsymbol{\nabla} V(\mathbf{r}) \\
&= -\iiint_V d\mathbf{r}'^d \boldsymbol{\nabla}_{\mathbf{r}} G(\mathbf{r}-\mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0} \\
&= \iiint_V d\mathbf{r}'^3 \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0}.
\end{aligned}
\tag{1.81}
$$

This gives the electric field generated by an arbitrary charge density ρ.

The long-distance expansion of the field can also be obtained by differentiating the multipole expansion of the potential:

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \mathbf{E}_0(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}) + \cdots \\
&= \frac{Q}{4\pi\epsilon_0 r^3}\mathbf{r} + \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - r^2\mathbf{p}}{4\pi\epsilon_0 r^5} + \frac{5(\mathbf{r}\cdot Q\mathbf{r})\mathbf{r} - 2r^2 Q\mathbf{r}}{8\pi\epsilon_0 r^7} + \cdots
\end{aligned}
\tag{1.82}
$$

At large distance, the electric field decays as $1/r^2$ for the monopole contribution $\mathbf{E}_0$, as $1/r^3$ for the dipole contribution $\mathbf{E}_1$, and as $1/r^4$ for the quadrupole contribution $\mathbf{E}_2$.

1.3.13 Method of images

As we have already seen, several methods can be used to determine the electric field. In all the previous examples, the charge distribution was assumed to be fixed. For conductors, however, the charge distribution is not known a priori; it must satisfy additional constraints: the potential is constant on the surface, and the electric field vanishes inside the conductor. Gauss's law then implies that the volume charge density inside the conductor is zero.

A conductor may be neutral or charged, but in electrostatic equilibrium all excess charge lies on its surface.

A charged particle near a conducting plane

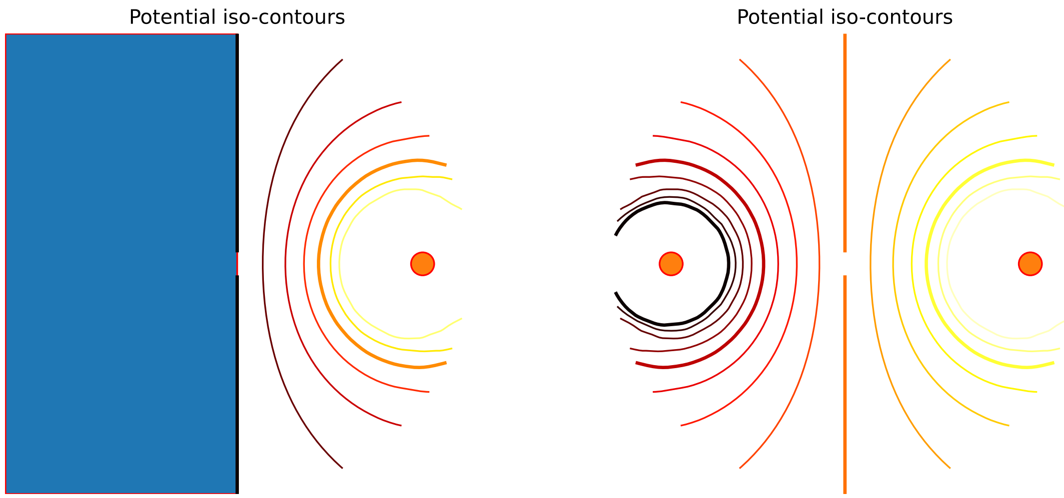

Figure 1.9: Equipotential contours for a positive charge near a conducting wall (left) and for a positive and a negative charge placed symmetrically with respect to a plane (right).

The left panel of Fig. 1.9 shows the equipotential contours of a positive charge near a conducting wall. The right panel shows the equipotentials produced by two opposite charges placed symmetrically with respect to a vertical plane. On the right-hand side of the plane, the two potentials are identical.

The method of images consists of removing the conducting region and replacing its effect by a fictitious charge distribution located in the region previously occupied by the conductor.

In the present case, the conducting plane is replaced by an image charge. The boundary $x = 0$ must be an equipotential surface, corresponding to the conducting plane. This is achieved by placing a charge $-q$ symmetrically with respect to the plane. The resulting potential is

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0\sqrt{(x-x_0)^2+y^2+z^2}} - \frac{q}{4\pi\epsilon_0\sqrt{(x+x_0)^2+y^2+z^2}}. \tag{1.83}$$

It is easy to verify that $V = 0$ when $x = 0$.

The electric field is obtained from $\mathbf{E} = -\boldsymbol{\nabla} V$. Its components are

$$\begin{aligned}
E_x &= \frac{q(x-x_0)}{4\pi\epsilon_0\left((x-x_0)^2+y^2+z^2\right)^{3/2}} - \frac{q(x+x_0)}{4\pi\epsilon_0\left((x+x_0)^2+y^2+z^2\right)^{3/2}},\\
E_y &= \frac{qy}{4\pi\epsilon_0\left((x-x_0)^2+y^2+z^2\right)^{3/2}} - \frac{qy}{4\pi\epsilon_0\left((x+x_0)^2+y^2+z^2\right)^{3/2}},\\
E_z &= \frac{qz}{4\pi\epsilon_0\left((x-x_0)^2+y^2+z^2\right)^{3/2}} - \frac{qz}{4\pi\epsilon_0\left((x+x_0)^2+y^2+z^2\right)^{3/2}}.
\end{aligned} \tag{1.84}$$

As expected, when $x = 0$, one has $E_y = 0$ and $E_z = 0$, so the electric field is normal to the plane. The normal component E_x is nonzero and therefore exhibits a finite discontinuity across the boundary.

The induced surface charge density is given by $\sigma = \epsilon_0 E_x|_{x=0^+}$, namely

$$\sigma = -\frac{qx_0}{2\pi(x_0^2+y^2+z^2)^{3/2}}. \tag{1.85}$$

It is maximum at the point closest to the charge, and at large distance it decays algebraically as $(y^2 + z^2)^{-3/2}$.

Integrating the surface density over the plane gives the total induced charge:

$$\begin{aligned} Q_{\mathrm{ind}} &= -\iint dy\, dz \frac{q x_0}{2\pi(x_0^2 + y^2 + z^2)^{3/2}} \\ &= -\int_0^{2\pi} d\theta \int_0^\infty dr \frac{r q x_0}{2\pi(x_0^2 + r^2)^{3/2}} \\ &= -\int_0^\infty dr \frac{r q x_0}{(x_0^2 + r^2)^{3/2}}. \end{aligned} \tag{1.86}$$

Using

$$\int_0^\infty dr \frac{r}{(1 + r^2)^{3/2}} = 1, \tag{1.87}$$

one finds that the total induced charge is $-q$. Thus the total charge induced on the plane compensates the charge in the region $x > 0$.

A charged particle near a conducting sphere

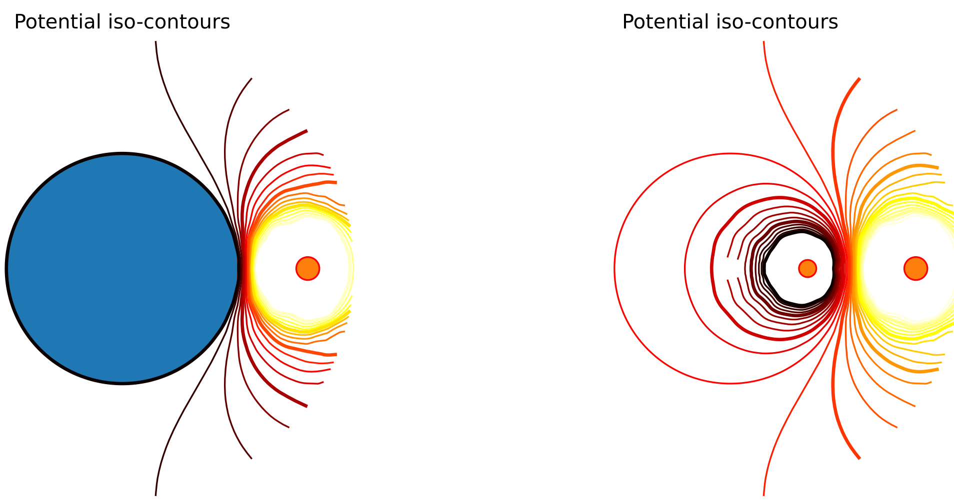

Figure 1.10: Equipotential contours for a charge near a conducting sphere (left) and for two image charges chosen to reproduce the same exterior potential (right).

The left panel of Fig. 1.10 shows the equipotential contours of a positive charge near a conducting sphere. The right panel shows the equipotentials produced by two suitably chosen charges. Outside the sphere, the two potentials coincide.

The potential produced by these two charges is

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0\sqrt{(x - x_0)^2 + y^2 + z^2}} + \frac{q'}{4\pi\epsilon_0\sqrt{(x - x_1)^2 + y^2 + z^2}}. \tag{1.88}$$

To produce an equipotential closed surface corresponding to the conducting sphere, one must have $q' < 0$ and $|q'| < q$. We therefore write $q' = -aq$ with $0 < a < 1$.

The boundary of the conductor is the sphere

$$x^2 + y^2 + z^2 = R^2.$$

Imposing $V = 0$ leads to

$$\left(1 - \frac{1}{a^2}\right)(x^2 + y^2 + z^2) + 2x\left(\frac{x_1}{a^2} - x_0\right) + x_0^2 - \frac{x_1^2}{a^2} = 0. \tag{1.89}$$

To recover the equation of a sphere centered at the origin, one must require

$$\frac{x_1}{a^2} - x_0 = 0, \tag{1.90}$$

which implies

$$a^2 = \frac{x_1}{x_0}.$$

Imposing the radius to be R then gives

$$R^2 = \frac{a^2 x_0^2 - x_1^2}{1 - a^2}, \tag{1.91}$$

from which one obtains

$$x_1 = \frac{R^2}{x_0}. \tag{1.92}$$

The surface charge density on the sphere can be obtained from the normal derivative of the potential, although the resulting expression is rather cumbersome. However, the total induced charge is easy to determine: by Gauss's law, it is equal to the image charge q' enclosed by the sphere.

1.4 Force and energy

1.4.1 A single charge in an electric field

So far, we have focused on the electric field itself. We now turn to its action on charged particles.

A particle of charge q placed in an electric field $\mathbf{E}(\mathbf{r})$ experiences the electrostatic force

$$\mathbf{F} = q\mathbf{E}. \tag{1.93}$$

Consider the work done by this force when the particle moves from $\mathbf{r}$ to $\mathbf{r}'$ along a trajectory C:

$$W = \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{F} \cdot d\mathbf{r}. \tag{1.94}$$

Since the electric field can be written as the gradient of the potential,

$$\mathbf{F} = -q\boldsymbol{\nabla} V(\mathbf{r}), \tag{1.95}$$

the electrostatic force is conservative. We may therefore introduce the potential energy

$$U(\mathbf{r}) = qV(\mathbf{r}), \tag{1.96}$$

and the work depends only on the initial and final points, not on the trajectory.

1.4.2 Two interacting charges

When the electric field is produced by a charge q_2 located at $\mathbf{r}_2$, and a second charge q_1 is located at $\mathbf{r}_1$, their interaction energy is

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|}. \tag{1.97}$$

1.4.3 N interacting charges

For three charges, the potential energy is the sum over the three distinct pairs:

$$U = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|} + \frac{q_1 q_3}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_3|} + \frac{q_2 q_3}{4\pi\epsilon_0 |\mathbf{r}_2 - \mathbf{r}_3|}. \tag{1.98}$$

The generalization to N charges is straightforward:

$$\begin{aligned} U &= \sum_{i<j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_j|} \\ &= \frac{1}{2} \sum_{i\neq j}^{N} \frac{q_i q_j}{4\pi\epsilon_0 |\mathbf{r}_i - \mathbf{r}_j|}. \end{aligned} \tag{1.99}$$

The second line sums over all ordered pairs rather than distinct pairs.

Note that if the total charge of the system grows proportionally to the number of particles, the slow decay of the Coulomb potential implies that the total electrostatic energy is not extensive, but super-extensive.

1.4.4 A dipole in an electric field

As an application of the previous results, we consider the potential energy of a dipole placed in an external electric field $\mathbf{E}(\mathbf{r})$.

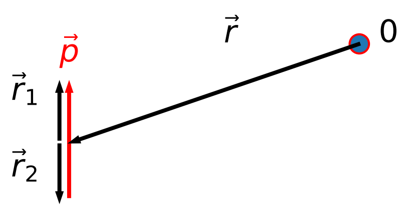

Figure 1.11: Dipole $\mathbf{p}$ in an electric field $\mathbf{E}$. The vectors indicate the displacements of the charges q and $-q$ from the reference position $\mathbf{r}$.

Consider two opposite charges located at $\mathbf{r} + \mathbf{r}_1$ and $\mathbf{r} + \mathbf{r}_2$ (see Fig. 1.11). Their potential energy is

$$U(\mathbf{r}) = qV(\mathbf{r} + \mathbf{r}_1) - qV(\mathbf{r} + \mathbf{r}_2). \tag{1.100}$$

Expanding the potential around $\mathbf{r}$ gives

$$\begin{aligned} V(\mathbf{r} + \mathbf{r}_i) &= V(\mathbf{r}) + \mathbf{r}_i \cdot \boldsymbol{\nabla} V(\mathbf{r}) + \cdots \\ &= V(\mathbf{r}) - \mathbf{r}_i \cdot \mathbf{E}(\mathbf{r}) + \cdots . \end{aligned} \tag{1.101}$$

The zeroth-order terms cancel, and at first order one finds

$$\begin{aligned} U(\mathbf{r}) &= -q(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{E}(\mathbf{r}) \\ &= -\mathbf{p} \cdot \mathbf{E}(\mathbf{r}), \end{aligned} \tag{1.102}$$

where $\mathbf{p} = q(\mathbf{r}_1 - \mathbf{r}_2)$ is the dipole moment. The potential energy is minimal when $\mathbf{p}$ and $\mathbf{E}$ are parallel and point in the same direction.

1.4.5 Interaction between two dipoles

Consider now the interaction energy between two dipoles: dipole $\mathbf{p}_1$ placed in the electric field created by dipole $\mathbf{p}_2$. The interaction energy is

$$U(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{r^3} - \frac{3(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})}{r^5}\right), \tag{1.103}$$

where $\mathbf{r}$ is the vector joining the two dipoles.

This expression is symmetric under exchange of the two dipoles. At fixed separation r, the energy is minimized when the two dipoles are aligned in the same direction.

The force exerted on dipole 1 by dipole 2 is obtained by taking minus the gradient of the potential energy:

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\boldsymbol{\nabla}\left(\frac{3(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})}{r^5} - \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{r^3}\right). \tag{1.104}$$

1.4.6 Interaction between charge distributions

Consider now a localized charge distribution $\rho(\mathbf{r})$. For a discrete set of point charges, Eq. (1.99) can be rewritten as

$$U = \frac{1}{2}\sum_{i=1}^{N} q_i V(\mathbf{r}_i), \tag{1.105}$$

where $V(\mathbf{r})$ is the potential created by all charges at the point $\mathbf{r}$. Passing to the continuous limit, one obtains

$$U = \frac{1}{2}\int_V d\mathbf{r}^3 \rho(\mathbf{r}) V(\mathbf{r}). \tag{1.106}$$

Using Gauss's law, this can be written as

$$U = \frac{\epsilon_0}{2}\int_V d\mathbf{r}^3 \left(\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r})\right) V(\mathbf{r}). \tag{1.107}$$

Using the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{E}(\mathbf{r})V(\mathbf{r})) = V(\mathbf{r})\boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}) + \boldsymbol{\nabla} V(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}), \tag{1.108}$$

we obtain

$$U = \frac{\epsilon_0}{2}\int_V d\mathbf{r}^3 \left[\boldsymbol{\nabla} \cdot (\mathbf{E}(\mathbf{r})V(\mathbf{r})) - \boldsymbol{\nabla} V(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\right]. \tag{1.109}$$

The first term is a total divergence. Since the charge distribution is localized, the product $\mathbf{E}V$ decays sufficiently rapidly at infinity, so its contribution vanishes. Using $\mathbf{E} = -\boldsymbol{\nabla} V$, one finally gets

$$U = \frac{\epsilon_0}{2}\int_V d\mathbf{r}^3 \mathbf{E}^2(\mathbf{r}). \tag{1.110}$$

Thus the electrostatic energy can be expressed in terms of the electric field alone, and one may define the local energy density

$$u(\mathbf{r}) = \frac{\epsilon_0}{2}\mathbf{E}^2(\mathbf{r}). \tag{1.111}$$

1.5 Conductors

1.5.1 Definition

Before discussing Maxwell's equations in matter, it is useful to consider the special case of conductors. We consider a region of space in which charges are free to move. Since we are dealing with electrostatics, only stationary situations are considered in this section.

Inside a conductor, the electric field is zero, which means that the potential is constant. Consequently, Gauss's law implies that the volume charge density is zero inside the conductor. A conductor may nevertheless carry a net charge, but this charge resides only on its surface. Since the surface is equipotential, the electric field is everywhere orthogonal to it.

1.5.2 Faraday cage

Consider a charge-free region surrounded by a conductor. Because the conductor is equipotential, its surface is at constant potential. Inside the cavity there are no charges, so the potential satisfies Laplace's equation. By uniqueness, the only solution in a finite region with constant boundary value is a constant potential everywhere inside. Therefore the electric field vanishes throughout the cavity.

1.5.3 Capacitors

Parallel-plate geometry

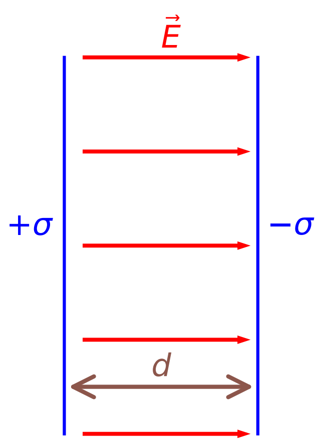

Figure 1.12: Two parallel plates separated by a distance d and carrying opposite surface charge densities $\pm\sigma$.

The left and right parts of space are occupied by conductors, so the electric field vanishes inside them. Between the two conductors, translational invariance parallel to the plates implies that the electric field has only one nonzero component:

$$\mathbf{E} = E(z)\mathbf{e}_z. \tag{1.112}$$

Gauss's theorem then gives

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\mathbf{e}_z. \tag{1.113}$$

The corresponding potential is obtained by integration:

$$V(z) = -\frac{\sigma}{\epsilon_0} z + c, \tag{1.114}$$

where c is a constant. If the potential difference between the plates is denoted by ΔV, then

$$\Delta V = \frac{\sigma d}{\epsilon_0}. \tag{1.115}$$

Thus the ratio between charge density and potential difference is independent of the actual values of the charge and potential. For this geometry, it defines the capacitance per unit area.

For a finite system in which the infinite plates are replaced by two parallel rectangles of area $L_1 \times L_2$, with $d \ll \min(L_1, L_2)$, the electric field is approximately uniform except near the edges. Since the total charge is $Q = \sigma L_1 L_2$, the capacitance is approximately

$$C = \frac{\epsilon_0 L_1 L_2}{d}. \tag{1.116}$$

Concentric-sphere geometry

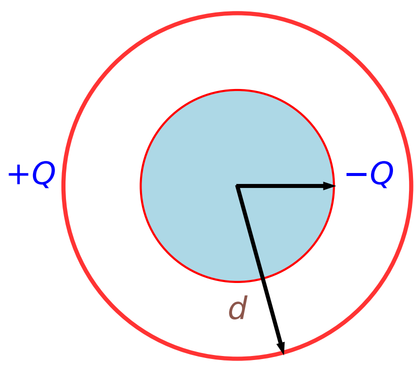

Figure 1.13: Two concentric spheres of radii R_1 and R_2 carrying charges $-Q$ and Q, respectively.

Consider two concentric conducting spheres of radii R_1 and R_2 carrying charges $-Q$ and Q (see Fig. 1.13). By spherical symmetry, the electric field is radial and depends only on r:

$$\mathbf{E}(r) = E(r)\mathbf{e_r}. \tag{1.117}$$

Gauss's theorem gives

$$\mathbf{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{e_r} \tag{1.118}$$

for $R_1 \leq r \leq R_2$.

The potential is then

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}. \tag{1.119}$$

The voltage between the two conductors is

$$\begin{aligned}\Delta V &= V(R_1) - V(R_2)\\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{R_2 - R_1}{R_2 R_1}\right).\end{aligned} \tag{1.120}$$

Since $Q = C\Delta V$, the capacitance is

$$C = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}. \tag{1.121}$$

1.6 Conclusion

In this first chapter, we have shown that several different methods can be used to determine the electric field in a wide variety of situations.

We have not addressed the question of the stability of systems of charges. For two point charges, whether they have the same or opposite signs, the interaction force is nonzero at all distances, which prevents stable static equilibrium. A similar argument applies to the interaction between a charge and a dipole, between two dipoles, and so on. In practice, however, charges are not always isolated in free space and their motion may be constrained. The existence of stable bound systems is ultimately explained by quantum mechanics.

Chapter 2

Magnetostatics

2.1 Introduction

Magnetostatics does not mean that charges are at rest. Rather, it means that the current density $\mathbf{J}$ is time independent, and consequently the magnetic field is also time independent. We are therefore in a situation where the electric and magnetic fields are decoupled.

Let us recall that Maxwell's equations reduce to

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{2.1}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2.2}$$

Equation (2.1) states that there are no magnetic monopoles, in contrast with the electric case. The second equation, known as Ampère's law, shows that the source of the magnetic field is the current density. Taking the divergence of Ampère's law gives

$$\boldsymbol{\nabla} \cdot \mathbf{J} = 0. \tag{2.3}$$

If one considers a finite volume V, the net flux of current through its surface is always zero. Hence, there is no accumulation of charge.

2.2 Integral form of Ampère's law

The differential equation (2.2) can be rewritten in integral form. Let S be an oriented surface in Euclidean space $\mathcal{R}^3$, and let its boundary be $C = \partial S$. Integrating both sides of Eq. (2.2) over the surface S, one obtains

$$\int_S d\mathbf{S} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) = \mu_0 \int_S d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}) \tag{2.4}$$

Using Stokes' theorem, Eq. (1.15), this becomes

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}). \tag{2.5}$$

To define these integrals properly, we must fix an orientation. Once a normal vector to the surface is chosen, the right-hand rule determines the orientation of the line integral along C (see Fig. 2.1).

Finally, the integral of the current density over the surface S is simply the total current I through S. The integral form of Ampère's law is therefore

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I. \tag{2.6}$$

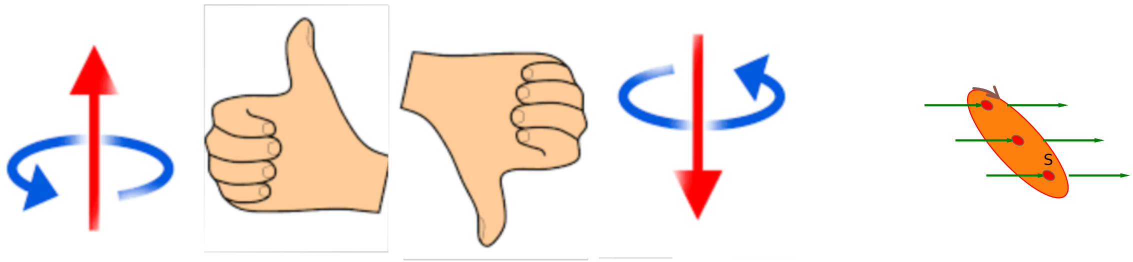

Figure 2.1: Left: with the right-hand rule, one first aligns the thumb with the chosen normal to the surface. Curling the fingers then fixes the orientation of the line integral. Right: flux of the magnetic field through the surface S and the associated oriented contour C.

For physical situations in which the current distribution has a high degree of symmetry, the magnetic field has a simple form, and Eq. (2.6) can be used to derive its analytical expression. Let us now review some basic examples.

2.2.1 A thin straight wire

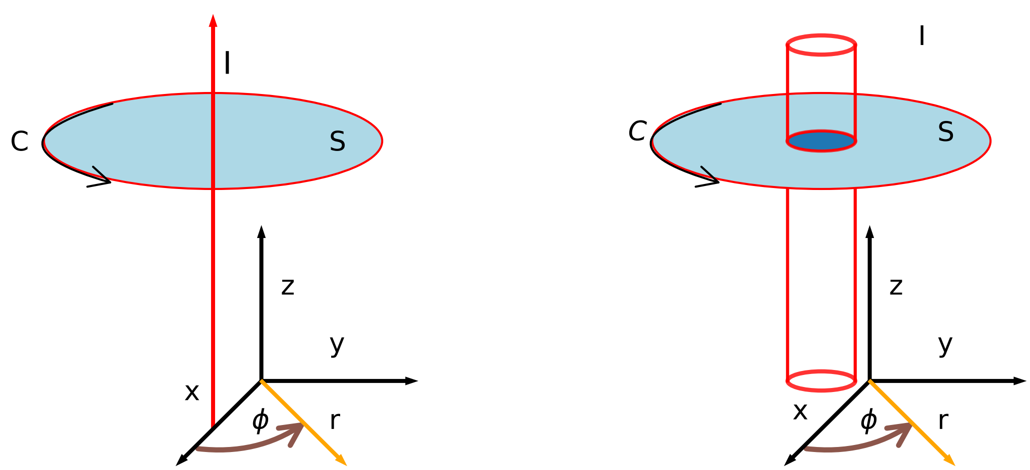

Figure 2.2: Left: infinitely thin straight wire carrying a total current I. S is a centered disk, and C is an oriented circle. Right: thick straight wire with uniform current density. All labels have the same meaning.

We consider an infinite straight wire carrying a current I along the vertical direction. The cylindrical symmetry of the system naturally leads us to use cylindrical coordinates (r, ϕ, z) (see the left panel of Fig. 2.2).

The magnetic field cannot depend on ϕ because of rotational symmetry around the wire, and it cannot depend on z because of translational symmetry along the wire. The z component of $\mathbf{B}$ is zero because the current flows along the z axis. Since the divergence of the magnetic field vanishes, the radial component must also be zero. Therefore,

$$\mathbf{B} = B(r)\mathbf{e}_\phi, \tag{2.7}$$

where $\mathbf{e}_\phi$ is the azimuthal unit vector.

Evaluating the line integral gives

$$2\pi r\, B(r) = \mu_0 I, \tag{2.8}$$

and therefore

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi. \tag{2.9}$$

The magnetic field decays algebraically with the same exponent as the electric field of a uniformly charged line. One also observes a divergence as $r \to 0$.

2.2.2 A thick straight wire

We now consider a cylindrical wire of radius R in which the current density is uniform, such that the total current is I:

$$\iint_{S_R} d\mathbf{S} \cdot \mathbf{j} = I,$$

where S_R denotes the disk of radius R. The symmetries are the same as for the thin wire, but the calculation must be split into two regions.

- For a disk of radius $r < R$, the enclosed current is

  $$2\pi r\, B(r) = \mu_0 j\, \pi r^2. \tag{2.10}$$

  Since

  $$j = \frac{I}{\pi R^2}, \tag{2.11}$$

  the magnetic field is

  $$\mathbf{B} = \frac{\mu_0 I r}{2\pi R^2}\mathbf{e}_\phi. \tag{2.12}$$

- For a disk of radius $r > R$, one recovers the thin-wire result:

  $$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi. \tag{2.13}$$

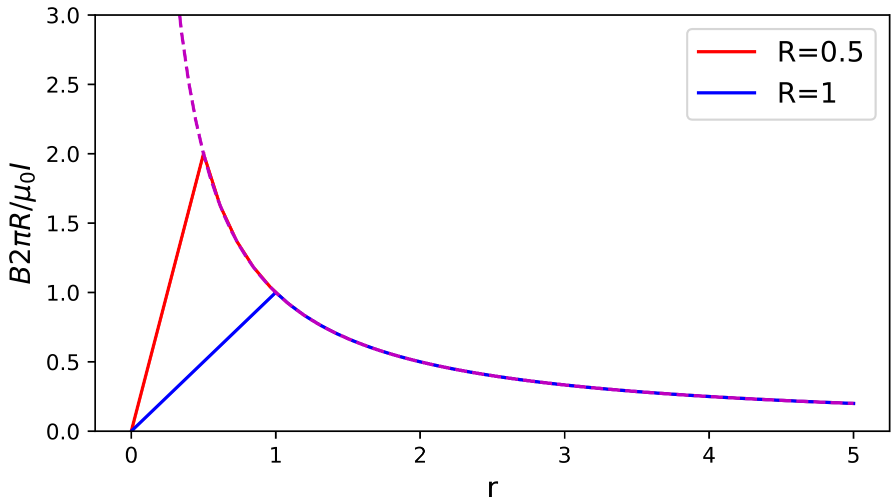

Figure 2.3: Magnitude of the magnetic field versus the distance r from the center of the wire, for two radii $R = 0.5, 1$ and for the thin wire (dashed curve).

Figure 2.3 shows that the divergence present for the thin wire is removed when the wire has a finite radius. Inside the wire, the field decreases linearly to zero at the center.

2.3 The vector potential

In order to express the solution directly in terms of the current density $\mathbf{J}$, we introduce the vector potential $\mathbf{A}$.

Since $\nabla \cdot \mathbf{B} = 0$, one may write

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.14}$$

The vector $\mathbf{A}$ is called the vector potential. As in electrostatics, $\mathbf{A}$ is not unique. If $\mathbf{A}'$ is another potential producing the same magnetic field, then

$$\nabla \times (\mathbf{A}' - \mathbf{A}) = 0. \tag{2.15}$$

Thus the difference between the two vector potentials is curl-free, which means that

$$\mathbf{A}' = \mathbf{A} + \nabla\phi \tag{2.16}$$

for some scalar field ϕ.

This gives a large class of equivalent vector potentials. Substituting $\mathbf{B} = \nabla \times \mathbf{A}$ into Ampère's law gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}. \tag{2.17}$$

Using the standard vector identity, this becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{2.18}$$

Because another solution of the form (2.16) gives the same magnetic field, we have

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2\phi. \tag{2.19}$$

To restrict the set of possible solutions, one imposes the Coulomb gauge:

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{2.20}$$

The vector potential then satisfies

$$\boldsymbol{\nabla}^2 \mathbf{A} = -\mu_0 \mathbf{J}. \tag{2.21}$$

This is a Poisson equation for each Cartesian component, although this statement is no longer true in curvilinear coordinates.

Explicitly,

$$\boldsymbol{\nabla}^2 A_i = -\mu_0 J_i, \tag{2.22}$$

where A_i and J_i are the Cartesian components of the vector potential and the current density, respectively.

2.3.1 Green's function

The Green's function is the same for each Cartesian component, so the solution is

$$A_i(\mathbf{r}) = -\mu_0 \iiint_V d\mathbf{r}'^d \, G(\mathbf{r} - \mathbf{r}') J_i(\mathbf{r}') \tag{2.23}$$

for each component i. This can be written more compactly as

$$\mathbf{A}(\mathbf{r}) = -\mu_0 \iiint_V d\mathbf{r}'^d \, G(\mathbf{r} - \mathbf{r}') \mathbf{J}(\mathbf{r}'). \tag{2.24}$$

In three dimensions, inserting the explicit Green's function gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3 \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.25}$$

This is a solution of Poisson's equation, but one should check that it also satisfies the Coulomb gauge, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. Taking the divergence gives

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3 \, \boldsymbol{\nabla}_{\mathbf{r}} \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) \\ &= \frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3 \, \mathbf{J}(\mathbf{r}') \cdot \boldsymbol{\nabla}_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \end{aligned} \tag{2.26}$$

The index $\mathbf{r}$ on $\boldsymbol{\nabla}$ indicates that the derivative is taken with respect to $\mathbf{r}$. Using the symmetry of $1/|\mathbf{r}-\mathbf{r}'|$, one has

$$\boldsymbol{\nabla}_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\boldsymbol{\nabla}_{\mathbf{r}'} \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.27}$$

Integrating by parts in Eq. (2.26), one obtains

$$\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3 \left[\boldsymbol{\nabla}_{\mathbf{r}'} \cdot \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{\boldsymbol{\nabla}_{\mathbf{r}'} \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right]. \tag{2.28}$$

For stationary currents, $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$, so the second term vanishes. Moreover, if the integration volume V contains the full current distribution, then $\mathbf{J}$ vanishes on the boundary. Hence the surface term is zero and the Coulomb gauge is satisfied.

2.3.2 Biot–Savart law

To obtain the magnetic field $\mathbf{B}$, one takes the curl of the vector potential:

$$\begin{aligned}\mathbf{B} &= \boldsymbol{\nabla} \times \frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3 \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3\, \boldsymbol{\nabla}_{\mathbf{r}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \times \mathbf{J}(\mathbf{r}') \\ &= \frac{\mu_0}{4\pi} \iiint_V d\mathbf{r}'^3\, \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3},\end{aligned} \tag{2.29}$$

where we used the identity

$$\boldsymbol{\nabla} \times (\phi \mathbf{C}) = \boldsymbol{\nabla}\phi \times \mathbf{C}$$

for a constant vector $\mathbf{C}$.

A useful variant When the current density $\mathbf{J}$ is confined to a thin wire, one may write the current through a small volume element as $\mathbf{J}\,\delta V = JA\,\delta x$, where A is the local cross-sectional area of the wire. Assuming that $JA = I$ is constant, the magnetic field becomes

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_C d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{2.30}$$

This formula gives the magnetic field generated by a line current.

To verify this result, let us recover the field of an infinite straight wire.

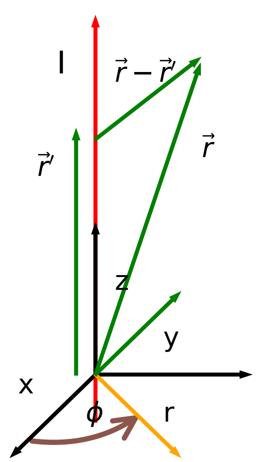

Figure 2.4: Infinitely thin straight wire carrying a total current I.

Using cylindrical coordinates with the z axis along the wire, one has

$$\mathbf{r}' = z\mathbf{e}_z, \qquad d\mathbf{r}' = dz\mathbf{e}_z.$$

The cross product is

$$d\mathbf{r}' \times (\mathbf{r}-\mathbf{r}') = r\,dz\,\mathbf{e}_\phi. \tag{2.31}$$

Hence the Biot–Savart law gives

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\mathbf{e}_\phi \int_{-\infty}^{+\infty} dz\, \frac{r}{(r^2+z^2)^{3/2}}. \tag{2.32}$$

Using the change of variable $z = rz'$, one gets

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\mathbf{e}_\phi \int_{0}^{+\infty} dz' \frac{1}{(1+z'^2)^{3/2}}. \tag{2.33}$$

The integral is equal to 1[1], and one recovers the result obtained previously.

2.3.3 Long-distance behavior

Using the vector potential (2.25), assume that at large distances the current is confined to a closed loop:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.35}$$

For $r \gg r'$, one may use the expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \cdots . \tag{2.36}$$

Substituting Eq. (2.36) into Eq. (2.35), the zeroth-order term vanishes because the integral of a constant along a closed loop is zero. This is consistent with the absence of magnetic monopoles.

For the first-order term, one may use the identity

$$\oint_C d\mathbf{r}'\,(\mathbf{r}\cdot\mathbf{r}') = \mathbf{S}\times\mathbf{r}, \tag{2.37}$$

where $\mathbf{S}$ is the vector area enclosed by the loop:

$$\mathbf{S} = \int_S d\mathbf{S}. \tag{2.38}$$

The vector potential then becomes

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\,\mathbf{m}\times\mathbf{r}, \tag{2.39}$$

where

$$\mathbf{m} = I\mathbf{S} \tag{2.40}$$

is the magnetic dipole moment.

Taking the curl gives the dipolar magnetic field

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\left(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right). \tag{2.41}$$

Comparing this with the electric dipole field, one sees that the two expressions are very similar. The quantity $\mathbf{m}$ therefore plays the same role in magnetostatics as the electric dipole moment does in electrostatics. The analogy is incomplete only because there is no magnetic monopole term.

[1] By integration by parts,

$$\begin{aligned}\int_0^{+\infty} dz \frac{1}{(1+z^2)^{3/2}} &= \left[\frac{z}{(1+z^2)^{3/2}}\right]_0^{\infty} + 3\int_0^{+\infty} dz \frac{z^2}{(1+z^2)^{5/2}} \\ &= 1.\end{aligned} \tag{2.34}$$

Generalized magnetic dipole moment The definition of the magnetic dipole moment can be generalized to an arbitrary three-dimensional current density. Starting from Eq. (2.25), one expands the vector potential at large distances:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r} \iiint_V d\mathbf{r}'^3\, \mathbf{J}(\mathbf{r}') + \frac{\mu_0}{4\pi r^3} \iiint_V d\mathbf{r}'^3\, \mathbf{J}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}'). \tag{2.42}$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{r}\,\mathbf{J}) = \mathbf{J} + (\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}, \tag{2.43}$$

and the fact that $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, one finds that the first integral vanishes for a localized current distribution.

For the second term, one uses

$$\mathbf{J}(\mathbf{r}'\mathbf{r}) = \frac{1}{2}\left(\mathbf{J}(\mathbf{r}\cdot\mathbf{r}') - (\mathbf{J}\cdot\mathbf{r})\mathbf{r}'\right) + \frac{1}{2}\left(\mathbf{J}(\mathbf{r}\cdot\mathbf{r}') + (\mathbf{J}\cdot\mathbf{r})\mathbf{r}'\right) \tag{2.44}$$

which consists in splitting the expression in an antisymmetric and a symmetric parts. The integral of the symmetric part does not contribute to the vector potential.

One then obtains that

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{8\pi} \iiint_V d\mathbf{r}'^3\, \frac{\mathbf{J}(\mathbf{r}')(\mathbf{r}\cdot\mathbf{r}') - (\mathbf{J}\cdot\mathbf{r})\mathbf{r}'}{r^3}. \tag{2.45}$$

Using the vector identity

$$\mathbf{r}\times(\mathbf{J}\times\mathbf{r}') = \mathbf{J}(\mathbf{r}\cdot\mathbf{r}') - (\mathbf{J}\cdot\mathbf{r})\mathbf{r}', \tag{2.46}$$

one finally obtains

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}, \tag{2.47}$$

where the magnetic dipole moment is

$$\mathbf{m} = \frac{1}{2}\iiint_V d\mathbf{r}'^3\, \mathbf{r}'\times\mathbf{J}(\mathbf{r}'). \tag{2.48}$$

2.3.4 Surface current

Consider an infinite horizontal plane carrying a surface current density $\mathbf{K}$ (see Fig. 2.5).

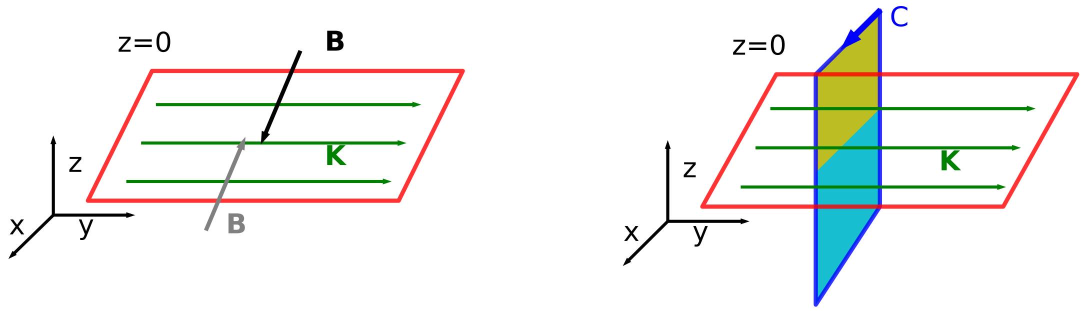

Figure 2.5: Plane carrying a surface current density $\mathbf{K}$. Left: magnetic field above and below the plane. Right: rectangular Amperian loop and oriented contour C.

Translational invariance along the x and y directions implies that the magnetic field has the form

$$\mathbf{B} = B(z)\mathbf{e}_x \tag{2.49}$$

with the antisymmetry property

$$B(-z) = -B(z). \tag{2.50}$$

Applying Ampère's law to the rectangular contour shown in Fig. 2.5, one obtains

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = L_1(B(z) - B(-z)) = \mu_0 K L_1. \tag{2.51}$$

Thus

$$B(z) = \frac{\mu_0 K}{2} \tag{2.52}$$

for $z > 0$.

This result should be compared with the electric field of a charged plane. First, both fields are independent of the distance from the plane. Second, both fields display a finite discontinuity across the plane, but for the electric field the discontinuity affects the normal component, whereas for the magnetic field it affects the tangential component. Third, the tangential component of the electric field and the normal component of the magnetic field remain continuous.

2.4 Magnetic force and energy

In the previous sections, we saw that a current generates a magnetic field. When a charge q moves with velocity $\mathbf{v}$ in a magnetic field, it experiences the Lorentz force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{2.53}$$

When a charged particle moves in a magnetic field, it contributes to a current. If the particle is constrained to move along a prescribed path, as in a conductor, the resulting current produces a magnetic field. Hence two current-carrying conductors exert magnetic forces on each other.

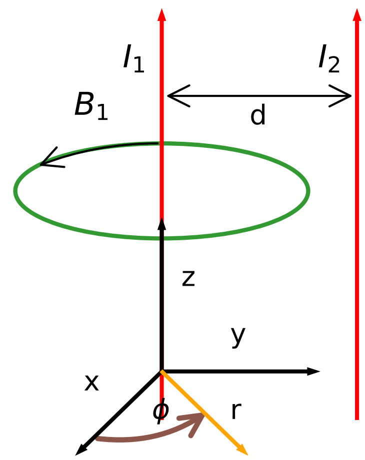

Figure 2.6: Two parallel thin wires separated by a distance d, carrying currents I_1 and I_2, respectively.

Consider two parallel wires separated by a distance d. The current I_1 in the first wire generates the magnetic field

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0 I_1}{2\pi r}\mathbf{e}_\phi. \tag{2.54}$$

For a charge moving along the second wire,

$$\mathbf{F}(\mathbf{r}) = q\mathbf{v} \times \mathbf{B}_1(\mathbf{r}), \tag{2.55}$$

with $\mathbf{v} = v\mathbf{e}_z$ and $r = d$. If the linear density of charges is λ, then the current is

$$I_2 = qv\lambda. \tag{2.56}$$

The force per unit length is

$$\begin{aligned} \mathbf{f}(\mathbf{r}) &= \lambda\mathbf{F}(\mathbf{r}) \\ &= \frac{\mu_0 I_1 \lambda q v}{2\pi r}\mathbf{e}_z \times \mathbf{e}_x \\ &= -\frac{\mu_0 I_1 I_2}{2\pi r}\mathbf{e}_y. \end{aligned} \tag{2.57}$$

Therefore, the force is attractive when $I_1 I_2 > 0$, that is, when the two currents flow in the same direction, and repulsive when the currents flow in opposite directions.

General currents Consider now a current loop C_1. The Biot–Savart law gives

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0 I_1}{4\pi}\oint_{C_1} \frac{d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}. \tag{2.58}$$

For a charge moving with velocity $\mathbf{v}$, the force is

$$\mathbf{F}(\mathbf{r}) = q\mathbf{v} \times \frac{\mu_0 I_1}{4\pi} \oint_{C_1} \frac{d\mathbf{r}_1 \times (\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3}. \tag{2.59}$$

Assuming a constant linear charge density along a second loop C_2, the total force exerted by loop 1 on loop 2 is

$$\mathbf{F}_{1\to 2} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_2} \oint_{C_1} d\mathbf{r}_2 \times \frac{d\mathbf{r}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{2.60}$$

Similarly,

$$\mathbf{F}_{2\to 1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} d\mathbf{r}_1 \times \frac{d\mathbf{r}_2 \times (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{2.61}$$

Using the double cross-product identity, one finds

$$\mathbf{F}_{2\to 1} = -\mathbf{F}_{1\to 2}, \tag{2.62}$$

as expected.

A dipole in a magnetic field Starting from the Lorentz force exerted on a current distribution $\mathbf{J}(\mathbf{r})$ contained in a volume V, one has

$$\mathbf{F} = \int_V d\mathbf{r}^3\, \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \tag{2.63}$$

If the magnetic field varies slowly over the volume V, it may be expanded as

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{B}(\mathbf{R}), \tag{2.64}$$

where $\mathbf{R}$ is a reference point in the volume.

Substituting Eq. (2.64) into Eq. (2.63), one obtains

$$\mathbf{F} = \left(\int_V d\mathbf{r}^3\, \mathbf{J}(\mathbf{r})\right) \times \mathbf{B}(\mathbf{R}) + \int_V d\mathbf{r}^3\, \mathbf{J}(\mathbf{r}) \times (\mathbf{r} \cdot \boldsymbol{\nabla})\mathbf{B}(\mathbf{r}')\big|_{\mathbf{r}'=\mathbf{R}}. \tag{2.65}$$

Since $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$, the first term vanishes.

After some algebra, the force can be written as

$$\mathbf{F} = \boldsymbol{\nabla}' \times \int_V d\mathbf{r}^3\, \left[(\mathbf{r} \cdot \mathbf{B}(\mathbf{r}'))\mathbf{J}(\mathbf{r})\right]\big|_{\mathbf{r}'=\mathbf{R}}. \tag{2.66}$$

Using the identity

$$\int_V d\mathbf{r}^3\, (\mathbf{B} \cdot \mathbf{r})\mathbf{J} = \frac{1}{2}\mathbf{B} \times \int_V d\mathbf{r}^3\, \mathbf{J} \times \mathbf{r} = -\mathbf{B} \times \mathbf{m}, \tag{2.67}$$

where $\mathbf{m}$ is the magnetic dipole moment, one gets

$$\mathbf{F} = -\boldsymbol{\nabla} \times (\mathbf{m} \times \mathbf{B}). \tag{2.68}$$

Since $\mathbf{m}$ is constant in space, and using $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, this simplifies to

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}). \tag{2.69}$$

Therefore, the force derives from the potential energy

$$U = -\mathbf{m} \cdot \mathbf{B}. \tag{2.70}$$

If the dipole is fixed in position but free to rotate, there is also a torque

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}. \tag{2.71}$$

This torque tends to align the magnetic moment with the magnetic field. When the two vectors are collinear, the energy is minimal and the torque vanishes.

2.4.1 Interaction between dipoles

The magnetic field created by a dipole moment $\mathbf{m}_2$ is given by Eq. (2.41):

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\left(3(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2\right). \tag{2.72}$$

Substituting this into Eq. (2.70), one obtains the interaction energy between two dipoles:

$$U = -\frac{\mu_0}{4\pi r^3}\left(3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) - \mathbf{m}_1 \cdot \mathbf{m}_2\right). \tag{2.73}$$

Chapter 3

Electrodynamics and Electromagnetism

3.1 Time-dependent fields

In the previous chapters, we considered situations in which the electric and magnetic fields were independent of time. In that limit, the two fields are uncoupled: electrostatics can be studied separately from magnetostatics. This separation disappears as soon as the fields vary in time.

Even in the absence of charges and currents, Faraday's law reveals a first coupling between the two fields:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In other words, a time-dependent magnetic field generates an electric field. This induced electric field can accelerate or decelerate charged particles. The phenomenon is called *electromagnetic induction*.

This coupling is one of the central ideas of electrodynamics: electric and magnetic fields are no longer independent objects, but two aspects of the same electromagnetic field.

3.2 Lenz–Faraday law

Immobile circuit

Let us consider a simple conducting circuit defined by a closed curve C (see Fig. 3.1).

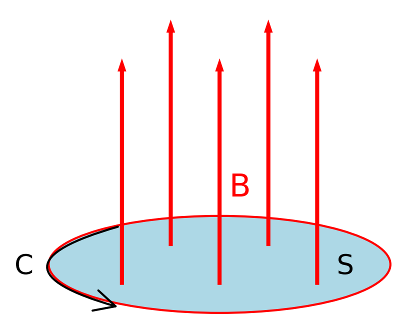

Figure 3.1: The curve C is a thin wire, and S denotes the oriented surface bounded by C.

Integrating Faraday's law over the surface S gives

$$\begin{aligned}\int_S d\mathbf{S}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) &= -\int_S d\mathbf{S}\cdot\frac{\partial\mathbf{B}}{\partial t}\\ &= -\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{B},\end{aligned} \tag{3.1}$$

where we have assumed that the surface S is fixed in time.

Using Stokes' theorem, the integral of the curl of $\mathbf{E}$ over the surface becomes a line integral along the contour:

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}. \tag{3.2}$$

The left-hand side of Eq. (3.2) is called the *electromotive force*, or *emf*:

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{r}. \tag{3.3}$$

Strictly speaking, this name is somewhat misleading: $\mathcal{E}$ is not a force but a voltage, that is, an energy per unit charge. Nevertheless, the traditional terminology has been retained.

If we denote by

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{S} \tag{3.4}$$

the magnetic flux through the surface S, then the integral form of Faraday's law becomes

$$\mathcal{E} = -\frac{d\Phi}{dt}. \tag{3.5}$$

The minus sign is essential. It expresses *Lenz's law*: the induced current always flows in such a direction that the magnetic field it creates opposes the variation of the magnetic flux. This is a direct consequence of energy conservation. Without this minus sign, the induced current would reinforce the change in flux and the system would become unstable.

Mobile circuit

We now assume that the circuit itself moves. In that case, the second line of Eq. (3.1) is no longer valid as written, because the surface bounded by the circuit changes with time. The correct expression is

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = \int_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S} - \oint_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r}, \tag{3.6}$$

where $\mathbf{v}$ is the local velocity of the wire.

Using Faraday's law, one obtains

$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r} = -\mathcal{E}. \tag{3.7}$$

Thus, in a moving circuit, the emf has two distinct contributions:

- the *transformer emf*, produced by a time-dependent magnetic field;
- the *motional emf*, produced by the motion of the circuit through a magnetic field.

This distinction is important in practice. In some devices, such as transformers, the first mechanism dominates. In others, such as generators, the second mechanism is the relevant one.

Laplace rail

To illustrate motional emf, consider the Laplace rail: the circuit is formed by a conducting rod sliding on two rails (see Fig. 3.2).

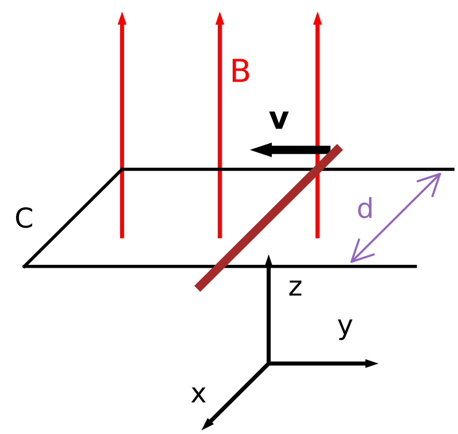

Figure 3.2: Laplace rail. The rod closes the circuit and moves in a uniform magnetic field.

The rod moves with velocity $\mathbf{v} = \dot{y}\mathbf{e}_y$. The magnetic field is constant and uniform, oriented along the z axis:

$$\mathbf{B} = B\mathbf{e}_z.$$

The distance between the two rails is constant and equal to d.

The magnetic flux through the circuit is proportional to the area enclosed by the rod and the rails. Since this area changes as the rod moves, one has

$$\frac{d\Phi}{dt} = Bd\,\dot{y}. \tag{3.8}$$

It is also possible to derive the emf directly from the magnetic force acting on a charge q inside the rod. There is no electric field in the rod, but the moving charge experiences the Lorentz force

$$q\mathbf{v} \times \mathbf{B}.$$

Since $\mathbf{v}$ is along the y direction and $\mathbf{B}$ is along the z direction, the magnetic force is directed positively along the rod. The emf, defined as the work per unit charge along the rod, is therefore. The minus sign comes from the scalar product between the curvilinear displacement and the magnetic force

$$\mathcal{E} = -Bd\,\dot{y}, \tag{3.9}$$

in agreement with Faraday's law.

This example is particularly important because it shows that induction does not require a time-dependent magnetic field: a constant magnetic field is sufficient if the circuit moves.

3.3 Magnetostatic energy

To construct the magnetic energy, consider a closed circuit C carrying a stationary current I. This current creates a magnetic field in space. As a consequence, the surface S bounded by the curve C is threaded by a magnetic flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}.$$

When the current changes, the magnetic flux changes as well. In the linear regime, the flux is proportional to the current:

$$\Phi = LI, \tag{3.10}$$

where L is the *inductance* of the circuit. It depends only on the geometry of the device and on the magnetic properties of the medium.

If one changes the current from I to $I + \delta I$, an emf appears according to Faraday's law and opposes the change, in agreement with Lenz's law. The infinitesimal work needed to change the current is

$$\delta W = \mathcal{E} I \, \delta t. \tag{3.11}$$

Using $\mathcal{E} = -L \, dI/dt$, one obtains

$$\frac{dW}{dt} = -L\frac{dI}{dt}I = -\frac{1}{2}L\frac{d(I^2)}{dt}. \tag{3.12}$$

Integrating over time gives the total work. Defining the magnetic energy by $U = -W$, one finds

$$U = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi. \tag{3.13}$$

This formula is the magnetic analogue of the electrostatic energy stored in a capacitor.

Expressing the flux in terms of the magnetic field and using Stokes' theorem, one obtains

$$\begin{aligned} U &= \frac{I}{2}\int_S \mathbf{B} \cdot d\mathbf{S} \\ &= \frac{I}{2}\int_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{S} \\ &= \frac{I}{2}\oint_C \mathbf{A} \cdot d\mathbf{r}. \end{aligned} \tag{3.14}$$

To obtain an energy density, one generalizes this result from a line current to a continuous current density $\mathbf{J}$. The magnetic energy becomes

$$U = \frac{1}{2}\int_V d\mathbf{r}^3 \, \mathbf{J} \cdot \mathbf{A}. \tag{3.15}$$

This should be compared with the electrostatic energy

$$U = \frac{1}{2}\int_V d\mathbf{r}^3 \, \rho \, \phi.$$

Using Ampère's law $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$ and the vector identity

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}), \tag{3.16}$$

and recalling that $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$, one gets

$$U = \frac{1}{2\mu_0}\int_V d\mathbf{r}^3 \left[\mathbf{B}^2 - \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B})\right]. \tag{3.17}$$

If the current distribution is localized, the surface term vanishes at infinity. Hence

$$U = \frac{1}{2\mu_0}\int_V d\mathbf{r}^3\,\mathbf{B}^2. \tag{3.18}$$

Combining this with the electrostatic energy, the total electromagnetic energy in static situations is

$$U = \frac{1}{2\mu_0}\int_V d\mathbf{r}^3\,\mathbf{B}^2 + \frac{\epsilon_0}{2}\int_V d\mathbf{r}^3\,\mathbf{E}^2. \tag{3.19}$$

Anticipating the general time-dependent case, the same energy density remains valid:

$$u = \frac{1}{2\mu_0}\mathbf{B}^2 + \frac{\epsilon_0}{2}\mathbf{E}^2 = \frac{\epsilon_0}{2}\left(\mathbf{E}^2 + \frac{\mathbf{B}^2}{c^2}\right). \tag{3.20}$$

3.4 Inductance

As shown above, the ratio of the magnetic flux to the current depends only on the geometry of the device. The inductance can also be defined from the magnetic energy:

$$L = \frac{2U}{I^2}. \tag{3.21}$$

Starting from Eq. (3.15) and using the expression of the vector potential in the Coulomb gauge, one obtains

$$U = \frac{\mu_0}{8\pi}\int_V d\mathbf{r}^3\int_V d\mathbf{r}'^3\,\frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.22}$$

In general, analytical expressions for the inductance are available only for a limited number of geometries.

If the system contains N circuits, the magnetic energy can be written as

$$U = \frac{1}{2}\sum_{i=1}^{N} L_{ii}I_i^2 + \sum_{i=1}^{N}\sum_{j\neq i}^{N} L_{ij}I_iI_j. \tag{3.23}$$

The matrix L is symmetric. The diagonal terms L_{ii} are the self-inductances:

$$L_{ii} = \frac{\mu_0}{4\pi I_i^2}\int_{V_i} d\mathbf{r}^3\int_{V_i} d\mathbf{r}'^3\,\frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.24}$$

while the off-diagonal terms L_{ij} are the mutual inductances:

$$L_{ij} = \frac{\mu_0}{4\pi I_iI_j}\int_{V_i} d\mathbf{r}^3\int_{V_j} d\mathbf{r}'^3\,\frac{\mathbf{J}(\mathbf{r})\cdot\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.25}$$

One readily checks that

$$L_{ij} = L_{ji}.$$

3.5 Resistance

3.5.1 Ohm's law

In conductors, moving charges interact with the lattice. As a result, a constitutive relation exists between the current density and the electric field:

$$\mathbf{J} = \sigma\mathbf{E}, \tag{3.26}$$

where σ is the conductivity of the material.

A microscopic derivation of this law requires quantum mechanics, but at the macroscopic level it is one of the most useful phenomenological relations in electromagnetism.

3.5.2 Laplace rail revisited

Let us return to the Laplace rail, now including resistance (see Fig. 3.3).

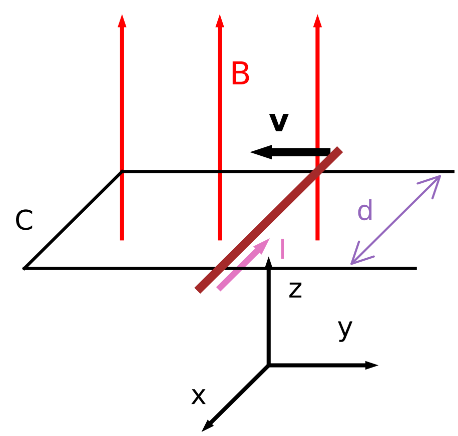

Figure 3.3: Laplace rail with a resistive circuit.

The magnetic force on the rod is

$$\mathbf{F} = IdB\,\mathbf{e}_y. \tag{3.27}$$

Newton's second law gives

$$m\ddot{y} = IBd. \tag{3.28}$$

We assume that the emf is entirely due to Faraday's law:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -Bd\dot{y}. \tag{3.29}$$

Using Ohm's law for the circuit,

$$\mathcal{E} = RI, \tag{3.30}$$

one obtains a closed equation for the motion of the rod:

$$m\ddot{y} = -\frac{B^2d^2}{R}\dot{y}. \tag{3.31}$$

Thus the electrical resistance acts as an effective viscous damping term. The solution is

$$\dot{y} = v_0 e^{-\frac{B^2d^2}{Rm}t}, \tag{3.32}$$

where v_0 is the initial velocity of the rod. The characteristic relaxation time is

$$\tau = \frac{Rm}{B^2d^2}. \tag{3.33}$$

This time increases when the resistance increases and decreases when the magnetic field becomes stronger.

The rod progressively slows down, and when $t \to \infty$, the mechanical energy loss is

$$\Delta E = E(\infty) - E(0) = -\frac{1}{2} m v_0^2.$$

Where does this energy go?

It is dissipated as Joule heat:

$$W = \int_0^\infty R I^2(t)\, dt. \tag{3.34}$$

Using Eqs. (3.29) and (3.30), the current is

$$I(t) = -\frac{Bd}{R} \dot{y}. \tag{3.35}$$

Hence

$$W = \int_0^\infty \frac{B^2 d^2}{R} v_0^2 e^{-\frac{2B^2d^2}{Rm} t}\, dt = \frac{1}{2} m v_0^2. \tag{3.36}$$

Thus, the initial mechanical energy is entirely converted into heat.

It is worth noting that this heat is independent of the magnetic field, the rod length, and the resistance. These quantities affect only the relaxation timescale.

One may also modify the circuit by inserting a capacitor of capacitance C in series with the resistance R. The first part of the analysis remains unchanged, but the circuit equation becomes

$$\mathcal{E} = \frac{Q}{C} + RI, \tag{3.37}$$

where Q is the charge stored in the capacitor.

Assuming $Q = 0$ when $y = 0$, integrating Newton's law gives

$$\dot{y} = v_0 + \frac{QBd}{m}. \tag{3.38}$$

Substituting into Eq. (3.29), one gets

$$\dot{Q} + \left(\frac{1}{RC} + \frac{B^2 d^2}{Rm} \right) Q = -\frac{Bd}{R} v_0. \tag{3.39}$$

Defining

$$\tau_1 = \frac{Rm}{B^2 d^2}, \qquad \tau = RC,$$

this becomes

$$\dot{Q} + \left(\frac{1}{\tau} + \frac{1}{\tau_1} \right) Q = -\frac{Bd}{R} v_0. \tag{3.40}$$

The solution is

$$Q(t) = Q(\infty) \left(1 - e^{-t/\tau_2} \right), \tag{3.41}$$

where

$$\begin{aligned} \tau_2 &= \frac{\tau \tau_1}{\tau + \tau_1}, \\ Q(\infty) &= -\frac{Bd}{R} v_0 \tau_2. \end{aligned} \tag{3.42}$$

In this case, the rod does not stop completely. Instead, its velocity tends to

$$\begin{aligned} v(\infty) &= v_0 \left(1 - \frac{\tau_2}{\tau_1} \right) \\ &= v_0 \frac{\tau}{\tau + \tau_1}. \end{aligned} \tag{3.43}$$

Thus, in the absence of the capacitor, one recovers $v(\infty) = 0$.

3.6 General solutions of Maxwell's equations

3.6.1 Introduction

To construct the general solutions of Maxwell's equations, it is natural to introduce scalar and vector potentials.

Starting from

$$\nabla \cdot \mathbf{B} = 0, \tag{3.44}$$

one introduces the vector potential $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{3.45}$$

Substituting this into Faraday's law gives

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{3.46}$$

Since the curl vanishes, the quantity in parentheses can be written as the gradient of a scalar potential:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V. \tag{3.47}$$

Therefore, the electric and magnetic fields may be expressed as

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A}, \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \nabla V. \end{aligned} \tag{3.48}$$

These expressions are completely general.

3.6.2 Gauge transformation and Lorenz gauge

Let us consider two pairs of potentials $(V, \mathbf{A})$ and $(V', \mathbf{A}')$ that produce the same fields. Since they lead to the same magnetic field,

$$\mathbf{A}' = \mathbf{A} + \nabla \phi \tag{3.49}$$

for some scalar field ϕ.

To preserve the electric field, one must then have

$$V' = V - \frac{\partial \phi}{\partial t}. \tag{3.50}$$

Thus the fields are unchanged under the gauge transformation

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla \phi, \\ V' &= V - \frac{\partial \phi}{\partial t}. \end{aligned} \tag{3.51}$$

Let us now write Maxwell's equations in terms of the potentials. Gauss's law becomes

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}, \tag{3.52}$$

and the Ampère–Maxwell equation gives

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla V \right). \tag{3.53}$$

Using the identity for the curl of a curl, one obtains

$$\boldsymbol{\nabla}^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{3.54}$$

The Coulomb gauge remains possible, but it does not decouple the scalar and vector potentials in a simple way for time-dependent fields. A more elegant choice is the *Lorenz gauge*:

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0. \tag{3.55}$$

Substituting this into Eqs. (3.52) and (3.53), one obtains

$$\begin{aligned} \boldsymbol{\nabla}^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho}{\epsilon_0}, \\ \boldsymbol{\nabla}^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} &= -\mu_0\mathbf{J}, \end{aligned} \tag{3.56}$$

where

$$c^2 = \frac{1}{\mu_0\epsilon_0}.$$

It is common to introduce the d'Alembertian operator

$$\Box \equiv \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{3.57}$$

The equations for the potentials then take the compact form

$$\Box V = -\frac{\rho}{\epsilon_0}, \qquad \Box\mathbf{A} = -\mu_0\mathbf{J}.$$

3.7 Green function for the wave equation

Consider the wave equation

$$\boldsymbol{\nabla}^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi f(\mathbf{r}, t). \tag{3.58}$$

If one knows the Green function G, the solution is obtained by superposition:

$$\phi(\mathbf{r}, t) = \int dt' \int d\mathbf{r}'^3\, G(\mathbf{r}-\mathbf{r}', t-t') f(\mathbf{r}', t'). \tag{3.59}$$

The Green function satisfies

$$\boldsymbol{\nabla}^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{3.60}$$

In an infinite homogeneous medium, the Green function depends only on the differences $\mathbf{r}-\mathbf{r}'$ and $t-t'$. Using Fourier transforms in time, one is led to the Helmholtz equation

$$\boldsymbol{\nabla}^2\tilde{G}(R,\omega) + k^2\tilde{G}(R,\omega) = -4\pi\delta(R), \tag{3.61}$$

where

$$R = |\mathbf{r}-\mathbf{r}'|, \qquad k = \frac{\omega}{c}.$$

For $R \neq 0$, one solves

$$\frac{1}{R}\frac{d^2(R\tilde{G})}{dR^2} + k^2\tilde{G} = 0, \tag{3.62}$$

whose general solution is

$$\tilde{G}(R,\omega) = \frac{1}{R}\left(Ae^{ikR} + Be^{-ikR}\right). \tag{3.63}$$

These correspond to outgoing and incoming spherical waves.

The inverse Fourier transform gives

$$G^{(\pm)}(R,\tau) = \frac{\delta(\tau \mp R/c)}{R}. \tag{3.64}$$

Thus

$$G^{(\pm)}(\mathbf{r},t;\mathbf{r}',t') = \frac{\delta\left(t' - \left[t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right]\right)}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.65}$$

The solution $G^{(+)}$ is the *retarded Green function*: it corresponds to causal propagation, where the field at time t depends on the source at an earlier time t'. The solution $G^{(-)}$ is the *advanced Green function*, which is mathematically valid but is usually not used in classical electrodynamics because it violates causality.

3.8 Jefimenko's equations

Using the retarded Green function, the potentials in the Lorenz gauge are

$$\begin{aligned} \phi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}', \\ \mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}', \end{aligned} \tag{3.66}$$

where

$$t_r = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

is the retarded time.

Substituting these expressions into Eq. (3.48), one obtains

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{R}}{R^3} \times \mathbf{J}(\mathbf{r}',t_r)\, d^3\mathbf{r}' + \frac{\mu_0}{4\pi c}\int \frac{\mathbf{R}}{R^2} \times \frac{\partial \mathbf{J}(\mathbf{r}',t_r)}{\partial t}\, d^3\mathbf{r}', \tag{3.67}$$

where

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \qquad R = |\mathbf{R}|.$$

This is the time-dependent generalization of the Biot–Savart law. The first term reduces to the usual magnetostatic contribution when the current is stationary, while the second term accounts for the time dependence of the current density.

For the electric field, one finds

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\Bigg[&\int \frac{\mathbf{R}}{R^3}\rho(\mathbf{r}',t_r)\, d^3\mathbf{r}' + \frac{1}{c}\int \frac{\mathbf{R}}{R^2}\frac{\partial \rho(\mathbf{r}',t_r)}{\partial t}\, d^3\mathbf{r}' \\ &- \frac{1}{c^2}\int \frac{1}{R}\frac{\partial \mathbf{J}(\mathbf{r}',t_r)}{\partial t}\, d^3\mathbf{r}' \Bigg]. \end{aligned} \tag{3.68}$$

These are *Jefimenko's equations*. They show explicitly that the electric and magnetic fields depend on the charge and current densities evaluated at the retarded time. They generalize the Coulomb and Biot–Savart laws to arbitrary time-dependent sources.

3.9 Plane waves

3.9.1 Derivation

In vacuum, Maxwell's equations are

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= 0, \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
\end{aligned} \tag{3.69}$$

Taking the curl of Faraday's law gives

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{3.70}$$

Using the identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 \mathbf{E},$$

and Gauss's law, one obtains

$$\boldsymbol{\nabla}^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{3.71}$$

or equivalently

$$\boldsymbol{\nabla}^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{3.72}$$

with

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

A similar derivation gives the same wave equation for the magnetic field:

$$\boldsymbol{\nabla}^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \tag{3.73}$$

Thus both fields propagate as waves in vacuum. In contrast with electrostatics and magnetostatics, the electric and magnetic fields are now intrinsically coupled.

A particularly important class of solutions is given by *plane waves*. For simplicity, let the propagation direction be the z axis.

Gauss's law then implies

$$\frac{\partial E_z}{\partial z} = 0. \tag{3.74}$$

Although a constant E_z would satisfy this equation, we set it to zero in order to describe transverse waves.

The transverse components satisfy

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_i = 0, \qquad i = x, y. \tag{3.75}$$

This may be factorized as

$$\left(\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) E_i = 0. \tag{3.76}$$

Introducing

$$X = z - ct, \qquad Y = z + ct,$$

the equation becomes

$$\frac{\partial^2 E_i}{\partial X \partial Y} = 0. \tag{3.77}$$

Its general solution is

$$E_i = f(X) + g(Y), \tag{3.78}$$

that is,

$$E_i = f(z - ct) + g(z + ct). \tag{3.79}$$

The first term describes a progressive wave moving in the $+z$ direction, and the second a regressive wave moving in the $-z$ direction.

Let us now consider a monochromatic progressive wave:

$$E_i = E_0 \delta_{i,x} \sin(kz - \omega t), \tag{3.80}$$

where

$$k = \frac{\omega}{c}.$$

The Kronecker symbol means that the electric field is polarized along the x direction.

To determine the magnetic field, use Faraday's law. Since $\mathbf{E}$ points along x and the wave propagates along z, one finds that $\mathbf{B}$ must be along y:

$$B_y = \frac{E_0}{c} \sin(kz - \omega t). \tag{3.81}$$

The electric and magnetic fields oscillate in phase, are perpendicular to each other, and are both perpendicular to the propagation direction. The triad $(\mathbf{E}, \mathbf{B}, \mathbf{k})$ is right-handed.

3.9.2 Polarization

Complex solutions

It is convenient to describe plane waves using complex notation:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \\ \mathbf{B} &= \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \end{aligned} \tag{3.82}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constant complex vectors.

In this notation, $\boldsymbol{\nabla}$ is replaced by $i\mathbf{k}$, and Maxwell's equations become

$$\begin{aligned} \mathbf{k} \cdot \mathbf{E}_0 &= 0, \\ \mathbf{k} \cdot \mathbf{B}_0 &= 0, \\ \mathbf{k} \times \mathbf{E}_0 &= \omega \mathbf{B}_0, \\ \mathbf{k} \times \mathbf{B}_0 &= -\omega \mu_0 \epsilon_0 \mathbf{E}_0. \end{aligned} \tag{3.83}$$

Thus, both fields are transverse, and the last two equations relate their amplitudes and orientations.

Linear polarization

If $\mathbf{E}_0$ and $\mathbf{B}_0$ are real vectors, the electric and magnetic fields oscillate along fixed directions. This corresponds to *linear polarization*. The electric field oscillates along a fixed axis, and so does the magnetic field, in a perpendicular direction (see Fig. 3.4).

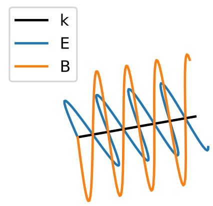

Figure 3.4: Sketch of the propagation of a linearly polarized electromagnetic wave.

Circular and elliptical polarizations

More general polarizations are obtained when $\mathbf{E}_0$ and $\mathbf{B}_0$ are complex. Write

$$\begin{aligned}\mathbf{E}_0 &= \mathbf{E}_{0,r} - i\mathbf{E}_{0,i},\\ \mathbf{B}_0 &= \mathbf{B}_{0,r} - i\mathbf{B}_{0,i}.\end{aligned} \tag{3.84}$$

The physical electric and magnetic fields are then

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_{0,r}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + \mathbf{E}_{0,i}\sin(\mathbf{k}\cdot\mathbf{r} - \omega t),\\ \mathbf{B} &= \mathbf{B}_{0,r}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + \mathbf{B}_{0,i}\sin(\mathbf{k}\cdot\mathbf{r} - \omega t).\end{aligned} \tag{3.85}$$

At each point in space, the tip of the electric field vector describes a curve in the plane perpendicular to $\mathbf{k}$. If the two orthogonal components have the same amplitude and are phase-shifted by $\pi/2$, the tip of $\mathbf{E}$ describes a circle: this is *circular polarization*. More generally, the trajectory is an ellipse: this is *elliptical polarization*.

Figure 3.5: Left- and right-handed circularly polarized electromagnetic waves.

3.10 Electromagnetic energy and Poynting vector

3.10.1 General case

The energy stored in the electromagnetic field is

$$U = \int_V d\mathbf{r}^3 \left(\frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 \right). \tag{3.86}$$

The integrand is the electromagnetic energy density:

$$u = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2.$$

Taking the time derivative of Eq. (3.86) gives

$$\frac{dU}{dt} = \int_V d\mathbf{r}^3 \left(\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \right). \tag{3.87}$$

Using Faraday's law and Ampère–Maxwell's law, one gets

$$\frac{dU}{dt} = \frac{1}{\mu_0} \int_V d\mathbf{r}^3 \left[\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \mu_0 \mathbf{E} \cdot \mathbf{J} + \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) \right]. \tag{3.88}$$

Now use the identity

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}). \tag{3.89}$$

This yields

$$\frac{dU}{dt} = -\int_V d\mathbf{r}^3 \, \mathbf{J} \cdot \mathbf{E} - \frac{1}{\mu_0} \int_V d\mathbf{r}^3 \, \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{B}). \tag{3.90}$$

We therefore obtain the local form of Poynting's theorem:

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}, \tag{3.91}$$

where

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \tag{3.92}$$

is the *Poynting vector*.

This equation expresses local energy conservation for the electromagnetic field. The term $\partial u/\partial t$ describes the local time variation of the electromagnetic energy density. The divergence term $\boldsymbol{\nabla} \cdot \mathbf{S}$ describes the net flux of electromagnetic energy leaving the volume. Finally, the term $\mathbf{J} \cdot \mathbf{E}$ is the power transferred from the electromagnetic field to matter.

For a finite volume V with boundary ∂V, the integral form is

$$\frac{d}{dt} \int_V d\mathbf{r}^3 \, u + \oint_{\partial V} \mathbf{S} \cdot d\mathbf{s} = -\int_V d\mathbf{r}^3 \, \mathbf{J} \cdot \mathbf{E}, \tag{3.93}$$

where $d\mathbf{s}$ is the outward surface element.

If $\mathbf{J} = \sigma \mathbf{E}$, the right-hand side is always negative, meaning that electromagnetic energy is irreversibly converted into heat by the Joule effect.

3.10.2 Energy in wave propagation

For a linearly polarized plane wave,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t), \\ \mathbf{B} &= \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t), \end{aligned} \tag{3.94}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$.

The Poynting vector is

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \\ &= \frac{1}{c\mu_0}\mathbf{E}_0 \times (\hat{\mathbf{k}} \times \mathbf{E}_0) \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t) \\ &= \frac{1}{c\mu_0}\mathbf{E}_0^2 \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t)\,\hat{\mathbf{k}} \\ &= c\epsilon_0 \mathbf{E}_0^2 \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t)\,\hat{\mathbf{k}}. \end{aligned} \tag{3.95}$$

The energy density is

$$\begin{aligned} u &= \frac{1}{2}\left[\epsilon_0 \mathbf{E}_0^2 + \frac{1}{\mu_0}\left(\hat{\mathbf{k}} \times \mathbf{E}_0\right)^2\right] \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t) \\ &= \epsilon_0 \mathbf{E}_0^2 \cos^2(\mathbf{k}\cdot\mathbf{r} - \omega t). \end{aligned} \tag{3.96}$$

Hence

$$\mathbf{S} = cu\,\hat{\mathbf{k}}. \tag{3.97}$$

This result has a clear physical interpretation:

- the energy flux is directed along the propagation direction;
- the magnitude of the energy flux is the energy density multiplied by the propagation speed c;
- the electric and magnetic fields contribute equally to the energy density.

Averaging over one period, one obtains

$$\overline{u} = \frac{1}{T}\int_0^T u\,dt = \frac{\epsilon_0}{2}\mathbf{E}_0^2, \tag{3.98}$$

and

$$\overline{\mathbf{S}} = \frac{c\epsilon_0}{2}\mathbf{E}_0^2\,\hat{\mathbf{k}}. \tag{3.99}$$

Thus, the time-averaged energy density and energy flux are uniform in space for a monochromatic plane wave.

Chapter 4

Source Radiation

We now consider several situations in which charges move through space and generate electromagnetic waves, thereby converting mechanical energy into electromagnetic energy.

4.1 Dipole radiation

4.1.1 Electric and magnetic fields

We have already obtained the time-dependent scalar and vector potentials for localized moving charges. Far from the source, these potentials can be simplified.

Starting from the retarded expression for the vector potential, we have

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_V d\mathbf{r}'^3 \frac{\mathbf{J}(\mathbf{r}',t_{\text{ret}})}{|\mathbf{r}-\mathbf{r}'|} \tag{4.1}$$

where

$$t_{\text{ret}} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \tag{4.2}$$

For $r \gg r'$, one may use the asymptotic expansion

$$|\mathbf{r}-\mathbf{r}'|^{\alpha} = r^{\alpha}\left(1-\alpha\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)+\cdots. \tag{4.3}$$

Keeping only the leading contribution in the integrand of Eq. (4.1), the current density becomes

$$\begin{aligned}\mathbf{J}(\mathbf{r}',t_{\text{ret}}) &= \mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\\ &= \mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}+\frac{\mathbf{r}\cdot\mathbf{r}'}{rc}+\cdots\right)\\ &= \mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right)+\frac{\partial\mathbf{J}}{\partial t}\left(\mathbf{r}',t-\frac{r}{c}\right)\frac{\mathbf{r}\cdot\mathbf{r}'}{rc}+\cdots\end{aligned} \tag{4.4}$$

Assuming that the typical size d of the source is much smaller than the distance $c\tau$ over which the electromagnetic wave varies appreciably, where

$$\tau \sim \frac{\mathbf{J}}{\partial\mathbf{J}/\partial t},$$

the vector potential is dominated by the zeroth-order term:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi r}\int_V d\mathbf{r}'^3\, \mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}\right). \tag{4.5}$$

To proceed further, we use charge conservation,

$$\boldsymbol{\nabla}\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0, \tag{4.6}$$

together with the identity

$$\boldsymbol{\nabla} \cdot [(\mathbf{J} \cdot \mathbf{r})] = (\boldsymbol{\nabla} \cdot \mathbf{J})\, \mathbf{r} + \mathbf{J}. \tag{4.7}$$

Equation (4.5) may then be rewritten as

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) = {} & \frac{\mu_0}{4\pi r} \int_V d\mathbf{r}'^3\, \boldsymbol{\nabla} \cdot \left[\mathbf{J}\left(\mathbf{r}', t - \frac{r}{c}\right) \cdot \mathbf{r}' \right] \\ & - \frac{\mu_0}{4\pi r} \int_V d\mathbf{r}'^3\, (\boldsymbol{\nabla} \cdot \mathbf{J})\left(\mathbf{r}', t - \frac{r}{c}\right) \mathbf{r}'. \end{aligned} \tag{4.8}$$

Using Gauss's theorem, the first term vanishes because the current density is localized. For the second term, charge conservation gives

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi r} \int_V d\mathbf{r}'^3\, \frac{\partial \rho(\mathbf{r}', t - r/c)}{\partial t}\, \mathbf{r}' \\ &= \frac{\mu_0}{4\pi r} \frac{d}{dt} \int_V d\mathbf{r}'^3\, \rho(\mathbf{r}', t - r/c)\, \mathbf{r}' \\ &= \frac{\mu_0}{4\pi r} \frac{d\mathbf{p}}{dt}\left(t - \frac{r}{c}\right) \\ &= \frac{\mu_0}{4\pi r} \dot{\mathbf{p}}\left(t - \frac{r}{c}\right), \end{aligned} \tag{4.9}$$

where $\mathbf{p}$ is the electric dipole moment of the charge distribution,

$$\mathbf{p}(t) = \int_V d\mathbf{r}^3\, \rho(\mathbf{r}, t)\, \mathbf{r}. \tag{4.10}$$

This is the *dipole approximation*, as expected.

The magnetic field is obtained from $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. It contains two contributions: one from differentiating the factor $1/r$ and one from differentiating the retarded time in $\dot{\mathbf{p}}$. One obtains

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r^3} \mathbf{r} \times \dot{\mathbf{p}}\left(t - \frac{r}{c}\right) - \frac{\mu_0}{4\pi c r^2} \mathbf{r} \times \ddot{\mathbf{p}}\left(t - \frac{r}{c}\right). \tag{4.11}$$

The first term decays as r^{-2}, whereas the second decays as r^{-1}. Assuming an oscillating dipole moment with angular frequency ω, the second term dominates when

$$r \gg \lambda = \frac{c}{\omega},$$

that is, in the *radiation zone*. The dominant far-field behavior therefore follows from the hierarchy of length scales

$$r \gg \lambda \gg d.$$

Because the magnetic field is time dependent, an electric field is also generated. In the radiation zone, the leading term gives

$$\boldsymbol{\nabla} \times \mathbf{B} \simeq \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \tag{4.12}$$

A straightforward calculation shows that

$$\mathbf{E}(\mathbf{r}, t) \simeq \frac{1}{4\pi\epsilon_0 c^2 r} \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \ddot{\mathbf{p}}\left(t - \frac{r}{c}\right) \right) \right], \tag{4.13}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$.

The electric and magnetic fields satisfy the relation

$$\mathbf{E} = -c\,\hat{\mathbf{r}} \times \mathbf{B}, \tag{4.14}$$

exactly as for a plane wave. This reflects the fact that the oscillating dipole emits electromagnetic waves that locally resemble plane waves in the radiation zone.

Finally, one should note that the radiative electric field decays as $1/r$, much more slowly than the electrostatic field.

4.1.2 Larmor formula

The existence of time-dependent electric and magnetic fields implies the emission of electromagnetic radiation. The relevant quantity for measuring the radiated power is the Poynting vector:

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \\ &= \frac{c}{\mu_0} \mathbf{B}^2 \hat{\mathbf{r}} \\ &= \frac{\mu_0}{16\pi^2 c r^2} \left| \hat{\mathbf{r}} \times \ddot{\mathbf{p}} \right|^2 \hat{\mathbf{r}}. \end{aligned} \tag{4.15}$$

Thus the Poynting vector is radial, as expected for outgoing spherical radiation, but its amplitude depends on the angle between the dipole and the observation direction.

Let θ be the angle between $\mathbf{p}$ and $\mathbf{r}$. Then

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mu_0}{16\pi^2 c r^2} \, \ddot{p}^2 \sin^2\theta \, \hat{\mathbf{r}}. \tag{4.16}$$

The total radiated power $\mathcal{P}$ is obtained by integrating the flux of the Poynting vector over a sphere:

$$\begin{aligned} \mathcal{P} &= \int_{S^2} d^2\mathbf{r} \cdot \mathbf{S} \\ &= \frac{\mu_0 \ddot{p}^2}{16\pi^2 c} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin^3\theta \\ &= \frac{\mu_0 \ddot{p}^2}{6\pi c}. \end{aligned} \tag{4.17}$$

The angular integral is elementary[1]. The total radiated power is independent of the distance from the source.

[1] We use

$$\int_0^{\pi} d\theta \, \sin^3\theta = \frac{4}{3}. \tag{4.18}$$

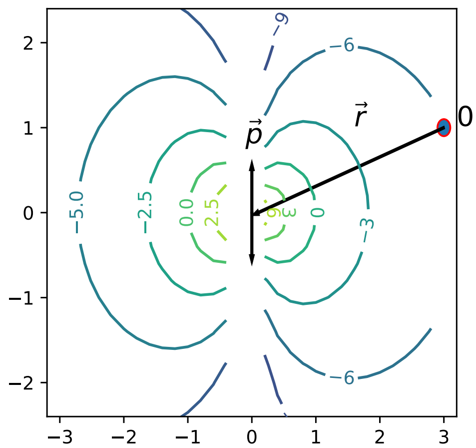

Figure 4.1: Oscillating electric dipole and contour plot of the radiated power.

As an example, consider a charge oscillating along the z-axis. The dipole moment is

$$\mathbf{p} = qd\cos(\omega t)\,\mathbf{e}_z, \tag{4.19}$$

so that

$$\ddot{\mathbf{p}} = -qd\omega^2\cos(\omega t)\,\mathbf{e}_z. \tag{4.20}$$

Averaging the radiated power over one period yields

$$\begin{aligned}\overline{\mathcal{P}} &= \frac{1}{T}\int_0^T dt\,\mathcal{P} \\ &= \frac{\mu_0\omega^4 p_0^2}{12\pi c},\end{aligned} \tag{4.21}$$

where $p_0 = qd$ is the amplitude of the dipole moment. The factor $1/2$ comes from the time average of $\cos^2(\omega t)$.

Using the acceleration amplitude $a = \omega^2 d$, one obtains

$$\overline{\mathcal{P}} = \frac{q^2a^2}{12\pi\epsilon_0 c^3}. \tag{4.22}$$

This expression is known as the *Larmor formula*.

4.1.3 Lifetime of a classical hydrogen atom

It is tempting to apply these results to the hydrogen atom. Because the proton is much heavier than the electron, we may treat the proton as fixed. In a classical picture, the electron moves around the proton under the action of the Coulomb force:

$$m_e\ddot{\mathbf{r}} = -\frac{e^2}{4\pi\epsilon_0 r^2}\,\mathbf{e}_r. \tag{4.23}$$

The electron-proton system therefore forms a time-dependent dipole. Using the Larmor formula,

$$\begin{aligned}\overline{\mathcal{P}} &= \frac{e^2}{12\pi\epsilon_0 c^3}\left(\frac{e^2}{4\pi\epsilon_0 m_e r^2}\right)^2 \\ &= \frac{1}{12\pi\epsilon_0 c^3}\left(\frac{e^3}{4\pi\epsilon_0 m_e r^2}\right)^2.\end{aligned} \tag{4.24}$$

For a circular orbit, the mechanical energy is

$$E = -\frac{e^2}{8\pi\epsilon_0 r}. \tag{4.25}$$

Therefore, in the presence of radiation, the mechanical energy decreases. Assuming, as a crude approximation, that the orbit remains circular, we have

$$\dot{E} = \frac{e^2}{8\pi\epsilon_0 r^2}\dot{r} = -\mathcal{P}. \tag{4.26}$$

Combining Eqs. (4.24) and (4.26), one obtains

$$\dot{r} = -\frac{\mu_0 e^4}{12\pi^2 m_e^2 c\,\epsilon_0\, r^2}. \tag{4.27}$$

This differential equation shows that the radius reaches zero in a finite time. Indeed,

$$\int_0^T dt = \int_0^{r_0} dr\,\frac{12\pi^2 m_e^2 c\,\epsilon_0\, r^2}{\mu_0 e^4}, \tag{4.28}$$

which gives

$$T = \frac{4\pi^2 m_e^2 c\,\epsilon_0\, r_0^3}{\mu_0 e^4}. \tag{4.29}$$

For a realistic initial radius $r_0 = 0.1\,\mathrm{nm}$, one finds

$$T \sim 10^{-11}\,\mathrm{s}.$$

This is clearly unacceptable: the hydrogen atom is stable. The resolution of this paradox is that atomic-scale systems cannot be described classically. The electron does not follow a classical orbit, and quantum mechanics replaces this picture by discrete energy levels, with a ground state that does not radiate.

4.2 Beyond dipole radiation

In the previous section, we assumed that the dipole moment is time dependent and nonzero. When the electric dipole contribution vanishes, one must continue the large-distance expansion to the next order.

Using Eqs. (4.1)–(4.4), the vector potential may be written

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi r^2 c}\int_V d\mathbf{r}'^3\,(\mathbf{r}\cdot\mathbf{r}')\,\dot{\mathbf{J}}\left(\mathbf{r}', t-\frac{r}{c}\right). \tag{4.30}$$

Using charge conservation and vector identities, this expression may be decomposed into two terms:

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}^{MD}(\mathbf{r},t) + \mathbf{A}^{EQ}(\mathbf{r},t), \tag{4.31}$$

where

$$\mathbf{A}^{MD}(\mathbf{r},t) = -\frac{\mu_0}{8\pi r^2 c}\,\mathbf{r}\times\int d\mathbf{r}'^3\,\mathbf{r}'\times\dot{\mathbf{J}}\left(\mathbf{r}', t-\frac{r}{c}\right) \tag{4.32}$$

is associated with magnetic dipole radiation, and

$$\mathbf{A}^{EQ}(\mathbf{r},t) = \frac{\mu_0}{8\pi r^2 c}\int d\mathbf{r}'^3\,(\mathbf{r}\cdot\mathbf{r}')\,\mathbf{r}'\,\ddot{\rho}\left(\mathbf{r}', t-\frac{r}{c}\right) \tag{4.33}$$

is associated with electric quadrupole radiation.

4.2.1 Magnetic dipole radiation

To see why Eq. (4.32) corresponds to a magnetic dipole contribution, recall that for a finite volume the magnetic dipole moment is

$$\mathbf{m}(t) = \frac{1}{2}\int_V d\mathbf{r}'^3\, \mathbf{r}' \times \mathbf{J}(\mathbf{r}', t). \tag{4.34}$$

Therefore,

$$\mathbf{A}^{MD}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi r^2 c}\,\mathbf{r} \times \dot{\mathbf{m}}\left(t - \frac{r}{c}\right). \tag{4.35}$$

Taking the curl and keeping only the leading far-field term gives

$$\mathbf{B}^{MD}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r c^2}\,\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \ddot{\mathbf{m}}\left(t - \frac{r}{c}\right)\right). \tag{4.36}$$

The corresponding electric field is obtained from Maxwell's equations:

$$\mathbf{E}^{MD}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r c}\,\hat{\mathbf{r}} \times \ddot{\mathbf{m}}\left(t - \frac{r}{c}\right). \tag{4.37}$$

These fields have a structure very similar to those produced by an electric dipole. The Poynting vector is then

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mu_0}{16\pi^2 r^2 c^3}\left|\hat{\mathbf{r}} \times \ddot{\mathbf{m}}\left(t - \frac{r}{c}\right)\right|^2 \hat{\mathbf{r}}. \tag{4.38}$$

If θ is the angle between $\mathbf{m}$ and $\hat{\mathbf{r}}$, then

$$\mathbf{S}(\mathbf{r}, t) = \frac{\mu_0}{16\pi^2 r^2 c^3}\ddot{m}^2 \sin^2\theta\, \hat{\mathbf{r}}. \tag{4.39}$$

The $1/r^2$ dependence ensures that the total flux through a sphere is independent of r, as required by energy conservation. The radiation pattern is anisotropic.

Averaging over one oscillation period gives the radiated power

$$\mathcal{P}^{MD} = \frac{\mu_0}{6\pi c^3}\,\ddot{m}^2. \tag{4.40}$$

It is interesting to compare this with the electric dipole result:

$$\frac{\mathcal{P}^{MD}}{\mathcal{P}^{ED}} = \frac{\ddot{m}^2}{c^2 \ddot{p}^2}. \tag{4.41}$$

Assuming a characteristic frequency ω, dimensional analysis gives

$$\ddot{p} \sim qd\,\omega^2, \qquad \ddot{m} \sim qd^2\,\omega^3,$$

so that

$$\frac{\mathcal{P}^{MD}}{\mathcal{P}^{ED}} \sim \frac{d^2\omega^2}{c^2}. \tag{4.42}$$

Since the derivation assumes that the source size is much smaller than the wavelength, this ratio is small. Because $d\omega$ is of the order of the particle velocity v, this corresponds precisely to the nonrelativistic limit $v \ll c$.

4.2.2 Electric quadrupole radiation

To obtain a tractable expression for the quadrupole contribution, we introduce the quadrupole tensor $\mathcal{Q}$:

$$\mathcal{Q}_{ij} = \int_V d\mathbf{r}'^3 \left(3r_i'r_j' - \delta_{ij}r'^2\right)\rho(\mathbf{r}', t). \tag{4.43}$$

Using Eq. (4.33), the components of the vector potential become

$$\begin{aligned} A_i^{EQ}(\mathbf{r}, t) &= \frac{\mu_0}{8\pi r^2 c}\int d\mathbf{r}'^3\,(\mathbf{r}\cdot\mathbf{r}')\,r_i'\,\ddot{\rho}\left(\mathbf{r}', t - \frac{r}{c}\right) \\ &= -\frac{\mu_0}{24\pi r^2 c}\left(x_j\ddot{\mathcal{Q}}_{ij}\left(t - \frac{r}{c}\right) + r_i\int d\mathbf{r}'^3\,r'^2\ddot{\rho}\left(\mathbf{r}', t - \frac{r}{c}\right)\right). \end{aligned} \tag{4.44}$$

The second term is purely radial and therefore does not contribute to the magnetic field, since $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Using the notation

$$(\mathbf{r}\cdot\ddot{\mathcal{Q}})_i = x_j\ddot{\mathcal{Q}}_{ij},$$

we may write

$$\mathbf{A}^{EQ}(\mathbf{r}, t) = -\frac{\mu_0}{24\pi r^2 c}\left(\mathbf{r}\cdot\ddot{\mathcal{Q}}\left(t - \frac{r}{c}\right)\right). \tag{4.45}$$

After some algebra, the far-field electric and magnetic fields are

$$\begin{aligned} \mathbf{B}^{EQ}(\mathbf{r}, t) &= \frac{\mu_0}{24\pi r c^2}\,\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\cdot\dddot{\mathcal{Q}}\left(t - \frac{r}{c}\right)\right), \\ \mathbf{E}^{EQ}(\mathbf{r}, t) &= \frac{\mu_0}{24\pi r c}\left[(\hat{\mathbf{r}}\cdot\dddot{\mathcal{Q}}\left(t - \frac{r}{c}\right)\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \dddot{\mathcal{Q}}\left(t - \frac{r}{c}\right)\cdot\hat{\mathbf{r}}\right]. \end{aligned} \tag{4.46}$$

The Poynting vector and the radiated power can also be computed, but in this case the angular distribution depends in a more intricate way on the geometry encoded in the quadrupole tensor.

4.3 Scattering

In the previous section, we saw that accelerated charges emit electromagnetic radiation. Conversely, when charged particles are placed in an incident electromagnetic wave, they are driven into motion and subsequently re-radiate. This is the phenomenon of scattering.

The particle motion depends on the physical situation. We consider two important cases: a free charged particle and a bound charged particle.

4.3.1 Thomson scattering

Assume that the incident electric field can be approximated by a plane wave:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{4.47}$$

The charged particle obeys the equation of motion

$$m\ddot{\mathbf{r}} = q\mathbf{E}. \tag{4.48}$$

Assuming that the oscillation amplitude is much smaller than the wavelength of the incident wave, one finds

$$\mathbf{r}(t) = -\frac{q}{m\omega^2}\mathbf{E}_0\sin(\omega t), \qquad \mathbf{v}(t) = \frac{q}{m\omega}\mathbf{E}_0\cos(\omega t). \tag{4.49}$$

The consistency condition is

$$\frac{qE_0}{m\omega^2} \ll \frac{c}{\omega}, \tag{4.50}$$

that is,

$$\frac{qE_0}{m\omega c} \ll 1. \tag{4.51}$$

This means that the maximum velocity remains much smaller than c. It is also the condition under which the Larmor formula applies.

Using the electric dipole formula, Eq. (4.22), the time-averaged radiated power is

$$\begin{aligned}
\overline{\mathcal{P}}_{\text{rad}} &= \frac{q^2 a^2}{12\pi\epsilon_0 c^3} \\
&= \frac{q^2}{12\pi\epsilon_0 c^3}\left(\frac{qE_0}{m}\right)^2 \\
&= \frac{q^4 E_0^2}{12\pi\epsilon_0 m^2 c^3}.
\end{aligned} \tag{4.52}$$

The power depends strongly on the charge, but not on the incident frequency.

To compare the emitted power with the incident power, we compute the time-averaged Poynting vector of the incoming wave (see Fig. 4.2).

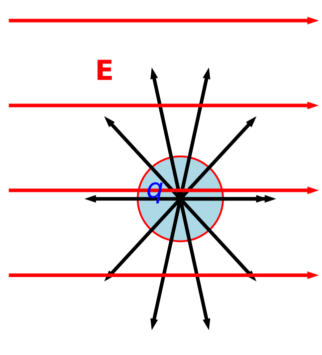

Figure 4.2: Incident electric field **E** driving a free charge, which then re-radiates as an oscillating electric dipole.

Using the Poynting vector for a plane wave, Eq. (3.95),

$$\begin{aligned}
\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\
&= \frac{1}{c\mu_0}\mathbf{E}_0 \times (\hat{\mathbf{k}}\times\mathbf{E}_0)\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) \\
&= c\epsilon_0 E_0^2 \cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)\,\hat{\mathbf{k}}.
\end{aligned} \tag{4.53}$$

Averaging over one period gives

$$\begin{aligned}\overline{\mathbf{S}} &= \overline{S}\,\hat{\mathbf{k}} \\ &= \frac{1}{2}c\epsilon_0 E_0^2\,\hat{\mathbf{k}}.\end{aligned} \tag{4.54}$$

The ratio of the emitted power to the incident flux defines an effective area, the scattering cross section:

$$\begin{aligned}\sigma &= \frac{\overline{\mathcal{P}}_{\text{rad}}}{\overline{S}} \\ &= \frac{q^4}{6\pi m^2 c^4 \epsilon_0^2}.\end{aligned} \tag{4.55}$$

Introducing the classical radius r_q of the particle by equating the electrostatic self-energy to the rest energy,

$$\frac{q^2}{4\pi\epsilon_0 r_q} = mc^2, \tag{4.56}$$

we obtain

$$r_q = \frac{q^2}{4\pi\epsilon_0 mc^2}, \tag{4.57}$$

and therefore

$$\sigma = \frac{8\pi}{3} r_q^2. \tag{4.58}$$

Some remarks:

- The cross section is independent of the frequency of the incident wave.
- Although the classical model of the charged particle is crude, it introduces a characteristic length scale, the classical radius.
- For the electron, one finds $r_e \simeq 2.8 \times 10^{-15}\,\text{m}$ and $\sigma \simeq 6 \times 10^{-29}\,\text{m}^2$.

4.3.2 Rayleigh scattering

We now consider the case in which the electron is not free, but bound inside an atom. For simplicity, the restoring force is modeled as harmonic. A damping term may also be included to allow the system to relax in the absence of external forcing (see Fig. 4.3).

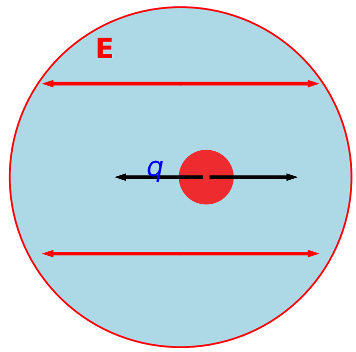

Figure 4.3: Incident electric field $\mathbf{E}$ driving a bound charge inside an atom.

The equation of motion is

$$m\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\mathbf{r} - m\gamma\dot{\mathbf{r}}. \tag{4.59}$$

Again, the displacement amplitude is assumed much smaller than the wavelength of the incoming light, so that the electric field is effectively uniform over the atomic scale.

Seeking a solution of the form $\mathbf{r} = \mathbf{r}_0 e^{-i\omega t}$, one obtains

$$\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\,\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{4.60}$$

In the regime $\omega \ll \omega_0$ and for weak damping, the restoring force dominates, so that

$$\mathbf{r}_0 \simeq \frac{q\mathbf{E}_0}{m\omega_0^2}. \tag{4.61}$$

The induced dipole moment is then

$$\mathbf{p} = q\mathbf{r} \simeq \frac{q^2\mathbf{E}_0}{m\omega_0^2}. \tag{4.62}$$

The time-averaged Poynting vector of the incident wave is the same as in Thomson scattering. Using the dipole radiation formula, the emitted power is

$$\begin{aligned} \overline{\mathcal{P}} &= \frac{\mu_0\omega^4 p^2}{12\pi c} \\ &= \frac{\mu_0 q^4 E_0^2}{12\pi c\, m^2\omega_0^4}\,\omega^4. \end{aligned} \tag{4.63}$$

The corresponding scattering cross section is therefore

$$\begin{aligned} \sigma &= \frac{\overline{\mathcal{P}}}{\overline{S}} \\ &= \frac{8\pi}{3} r_q^2 \left(\frac{\omega}{\omega_0}\right)^4. \end{aligned} \tag{4.64}$$

In Rayleigh scattering, the cross section depends strongly on the frequency. Since we assume $\omega < \omega_0$, the cross section is always smaller than in Thomson scattering. A major consequence is that short wavelengths are scattered more efficiently than long wavelengths. This effect explains, for example, the blue color of the sky.

4.4 Liénard–Wiechert potentials

4.4.1 A single particle

In the previous section, we used the multipole expansion to obtain tractable expressions for the electric and magnetic fields. For a point charge, however, one can derive exact expressions that remain valid even when the particle velocity is not negligible compared with the speed of light.

Consider a point charge q located at position $\mathbf{r}(t)$ and moving with velocity $\mathbf{v}(t)$. In this section, we denote the field point by $\mathbf{x}$.

The charge density is

$$\rho(\mathbf{x}, t) = q\,\delta(\mathbf{x} - \mathbf{r}(t)), \tag{4.65}$$

and the current density is

$$\mathbf{J}(\mathbf{x}, t) = q\,\mathbf{v}(t)\,\delta(\mathbf{x} - \mathbf{r}(t)). \tag{4.66}$$

4.4.2 Potentials

Using the retarded Green's function, the vector potential is

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int d\mathbf{x}' \frac{\mathbf{J}(\mathbf{x}', t_{\text{ret}})}{|\mathbf{x} - \mathbf{x}'|}, \tag{4.67}$$

while the scalar potential is

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{x}' \frac{\rho(\mathbf{x}', t_{\text{ret}})}{|\mathbf{x} - \mathbf{x}'|}, \tag{4.68}$$

where

$$t_{\text{ret}} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}. \tag{4.69}$$

Substituting the delta-function form of the charge density into the scalar potential gives

$$\begin{aligned} \phi(\mathbf{x}, t) &= \frac{q}{4\pi\epsilon_0} \int d\mathbf{x}' \frac{\delta(\mathbf{x}' - \mathbf{r}(t_{\text{ret}}))}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{q}{4\pi\epsilon_0} \int dt' \int d\mathbf{x}' \frac{\delta(\mathbf{x}' - \mathbf{r}(t'))}{|\mathbf{x} - \mathbf{x}'|} \delta(t' - t_{\text{ret}})\,. \end{aligned} \tag{4.70}$$

Performing the spatial integral yields

$$\begin{aligned} \phi(\mathbf{x}, t) &= \frac{q}{4\pi\epsilon_0} \int dt' \frac{1}{|\mathbf{x} - \mathbf{r}(t')|} \delta(t' - t_{\text{ret}}) \\ &= \frac{q}{4\pi\epsilon_0} \int dt' \frac{1}{|\mathbf{x} - \mathbf{r}(t')|} \delta\left(t - t' - \frac{|\mathbf{x} - \mathbf{r}(t')|}{c}\right). \end{aligned} \tag{4.71}$$

To perform the remaining integral, we use

$$\delta(f(t')) = \sum_i \frac{\delta(t' - t_i)}{|f'(t_i)|}, \tag{4.72}$$

where the t_i are the roots of $f(t') = 0$. Here,

$$f(t') = t - t' - \frac{|\mathbf{x} - \mathbf{r}(t')|}{c}, \tag{4.73}$$

whose derivative is

$$f'(t') = -1 + \frac{(\mathbf{x} - \mathbf{r}) \cdot \mathbf{v}(t')}{c|\mathbf{x} - \mathbf{r}|}. \tag{4.74}$$

Introducing the unit vector

$$\mathbf{n} = \frac{\mathbf{x} - \mathbf{r}(\mathbf{t})}{|\mathbf{x} - \mathbf{r}(\mathbf{t})|}, \tag{4.75}$$

one finally obtains

$$\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)} \right]_{t=t_{\text{ret}}}, \tag{4.76}$$

where

$$R = |\mathbf{x} - \mathbf{r}(t_{\text{ret}})|.$$

The same procedure applied to the vector potential gives

$$\mathbf{A}(\mathbf{x}, t) = \frac{q\mu_0}{4\pi} \left[\frac{\mathbf{v}}{R\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)} \right]_{t=t_{\text{ret}}}. \tag{4.77}$$

These two expressions are the *Liénard–Wiechert potentials*. They show explicitly that the fields measured at time t depend on the motion of the charge at the earlier retarded time t_{ret}.

The additional factor

$$\left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)^{-1}$$

encodes relativistic effects and is closely related to Lorentz invariance.

The electric and magnetic fields follow from differentiating the potentials. The calculation is lengthy, so we only quote the final result:

$$\mathbf{E}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0} \left[\frac{\mathbf{n} - \frac{\mathbf{v}}{c}}{\gamma^2 R^2 \left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)^3} + \frac{1}{R} \frac{\mathbf{n} \times \left[\left(\mathbf{n} - \frac{\mathbf{v}}{c}\right) \times \mathbf{a}\right]}{c^2 \left(1 - \frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)^3} \right]_{t=t_{\text{ret}}}, \tag{4.78}$$

where $\mathbf{a}$ is the particle acceleration and

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{4.79}$$

Two comments are in order. The first term decays as $1/R^2$ and is the relativistic generalization of the Coulomb field. The second term decays as $1/R$ and represents the radiative field associated with acceleration. Its angular dependence becomes strongly anisotropic when the particle velocity approaches the speed of light.

The magnetic field may be written compactly as

$$\mathbf{B}(\mathbf{x}, t) = \left[\frac{1}{c} \mathbf{n} \times \mathbf{E} \right]_{t=t_{\text{ret}}}. \tag{4.80}$$

Equivalent expressions for the fields generated by moving charges also exist, such as the Panofsky–Phillips formulas.

Chapter 5

Electromagnetism with Matter

5.1 Introduction

In previous chapters, we have considered charged systems in vacuum. This may correspond to an academic situation because we do not specify how charges can move and/or can stay in fixed positions, because we know that free charges do not have stable positions due the presence of the Coulomb force. A first workaround is given by the existence of conductor in which some electrons can be considered as free, but constrained to stay inside the conductor. However, we have shown that by inserting a relationship between current density and electric field, the rearrangement of the electron positions consist to move electrons on the boundary of the conductor in order to have a conductor with a constant potential.

It is now time to consider the electric and magnetic fields inside a system. A first view of the systems with a macroscopic number of atoms or molecules seems very complicated. A possible approach consists to start from a microscopic model in order to figure out the changes we have to include in the first description. Second, a coarse graining of the matter is necessary, because the local variation of the electric and magnetic are intense at the atomic level, but irrelevant at large scales.

To give an idea of what happens in matter, we first consider the situation where an atom is sensitive to a external electric field **E**. A very simple picture of an atom is given by a nucleus given by a positive charges particle and the electron which moves around, is represented by a negative cloud. Quantum mechanics learned us in the 20th century that this model is too crude, but it does not matter, because we only want to capture the basic phenomenon in dielectrics (or isolators). The framework we expect to build encompass different microscopic behaviors, where the Maxwell equations in matter are able to consider.

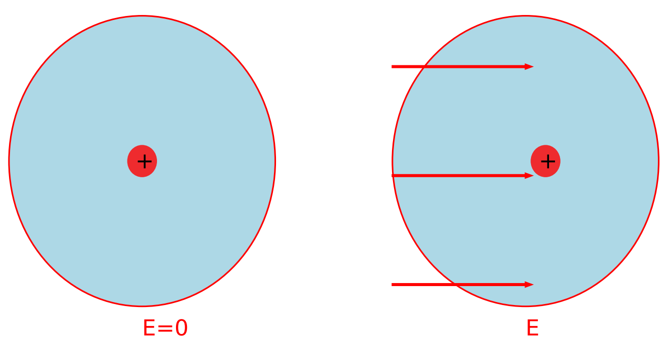

Figure 5.1

Fig.(5.1) gives a picture of the neutral atom without electric field (left) and in the presence of an electric field (right). On the left, the positions of the two charges coincide, namely the dipole of the neutral atom is equal to zero. On the right, the electric field shifts the position of the nucleus on the right, whereas the barycenter of the cloud is shifted on the left. Let us denote the distance d between the negative and positive charge, it appears a dipole moment $\mathbf{p} = q\mathbf{d}$, where $\mathbf{d}$ is vector in the same direction of the electric field. It comes that

$$\mathbf{p} = \alpha \mathbf{E} \tag{5.1}$$

which expresses that the response of the dielectric in the presence of an external field is the appearance of a dipole moment.α is named the *atomic polarizability.* This relation between these quantities is present in many systems. Additional effects can occur in a presence of a electric field. Keeping the same model, the cloud can be distorted and take a ellipsoidal shape instead of the initial spherical shape. In this case, it appears a quadrupolar moment due the spherical symmetry breaking. For the sake of simplicity, we keep the same model.

It is possible to estimate to atomic polarizability α within the model. Assuming that cloud is spherical and keeping the origin at the center of the cloud, the nucleus is subjected at two forces: the external electric field and the electric field due to the cloud. At equilibrium, the two forces cancel. In the chapter Electrostatic, we have obtained the electric field of a uniform charged sphere

$$q\mathbf{E} - \frac{q^2}{4a^3\pi\epsilon_0}\mathbf{r} = \mathbf{0} \tag{5.2}$$

where a is the sphere radius. Knowing that the dipole moment $\mathbf{p} = q\mathbf{r}$ one obtains that

$$\mathbf{p} = 4\pi a^3 \epsilon_0 \mathbf{E} \tag{5.3}$$

which gives

$$\alpha = 4\pi a^3 \epsilon_0 \tag{5.4}$$

Expressing the polarizability by dividing by $4\pi\epsilon$, α scales as the volume of the atom. The crude model captures the trend observed in the periodic table where the polarizabilty increases with the size of the atom. Moreover, considering the Hydrogen atom $\alpha/(4\pi\epsilon_0) = 0.7 \times 10^{-30} m^3$ and the radius a of the atom is $\sim 10^{-10} m$. A simple model, but a fair estimate!

5.2 Polarization

In the presence of an external electric field, many atoms acquire un dipole moment and defining the density of atoms by volume unit ϱ, one defines the polarization of the material as

$$\mathbf{P} = \varrho \mathbf{p} \tag{5.5}$$

This definition is a little bit tricky in the sense that the individual dipoles are not exactly the same at the different positions of the infinitesimal volume. The polarization can be understood as a local spatial average over a small volume which allows us to have a quantity which varies at the coarse-graining scale and not at the atomic scale.

The polarization is also present in systems where molecules carry a dipole moment, like water. In liquid phase, the thermal effects lead to the polarization equal to zero in the absence of electric field. Conversely, the response to an electric field can be proportional to the electric field, but non linear terms may contribute and dominate the the polarization when the intensity of the electric field increases.

A simple, but significant class of materials, called linear dielectrics satisfies

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{5.6}$$

where χ_e is called the electric susceptibility and is always positive. Compared to the simple model given in this chapter, this gives $\epsilon_0 \chi_e = \varrho\alpha$.

To better understand the meaning of the polarization and then the charge distribution of bounded electrons, one can compute the electrostatic potential associated with the dipole distribution $\mathbf{P}(\mathbf{r})$ Starting from the potential given by a single dipole

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}.\mathbf{r}}{r^3} \tag{5.7}$$

Summing over all dipoles of the material, one obtains

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V d\mathbf{r}'^3 \frac{\mathbf{P}(\mathbf{r}').(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5.8}$$

By using the identities

$$\boldsymbol{\nabla}'.\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5.9}$$

and

$$\boldsymbol{\nabla}'.\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \boldsymbol{\nabla}'\mathbf{P}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \mathbf{P}(\mathbf{r}').\boldsymbol{\nabla}'.\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \tag{5.10}$$

the potential can be written as

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V d\mathbf{r}'^3 \boldsymbol{\nabla}'\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi\epsilon_0}\int_V d\mathbf{r}'^3 \boldsymbol{\nabla}'\mathbf{P}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{5.11}$$

Using the Gauss's theorem, this gives

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_s d\mathbf{S}\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi\epsilon_0}\int_V d\mathbf{r}'^3 \boldsymbol{\nabla}'\mathbf{P}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{1}{4\pi\epsilon_0}\int_s dS\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\epsilon_0}\int_V d\mathbf{r}'^3 \frac{\rho_b}{|\mathbf{r}-\mathbf{r}'|}\end{aligned} \tag{5.12}$$

where

$$\sigma_b = \mathbf{P}(\mathbf{r}').\mathbf{n} \tag{5.13}$$

and

$$\rho_b = -\boldsymbol{\nabla}\mathbf{P}(\mathbf{r}) \tag{5.14}$$

are a charge surface density localized at the boundaries of the volume and a charge volume density in the volume. $\mathbf{n}$ is the unit vector normal to the surface.

Fig.5.2 illustrates the two situations: on the left panel, a small lattice is plotted with individual nucleus shifted to the right due to the presence of an electric field. No volume charges are present and excess charges are localized on the boundaries of the lattice: the two sides perpendicular to the direction of the field. The second situation is a priori less intuitive. Indeed the charge of a dipole is equal to zero and how to imagine a situation corresponding to the existence of a charge in the volume. To solve this apparent paradox, one first recalls that the polarization is a field (even if the elementary charge is discrete, the moment can be continuous). In the right panel, a configuration of non uniform $P(\mathbf{r})$ is plotted, where it appears a nonzero flux at the boundaries of the sphere.

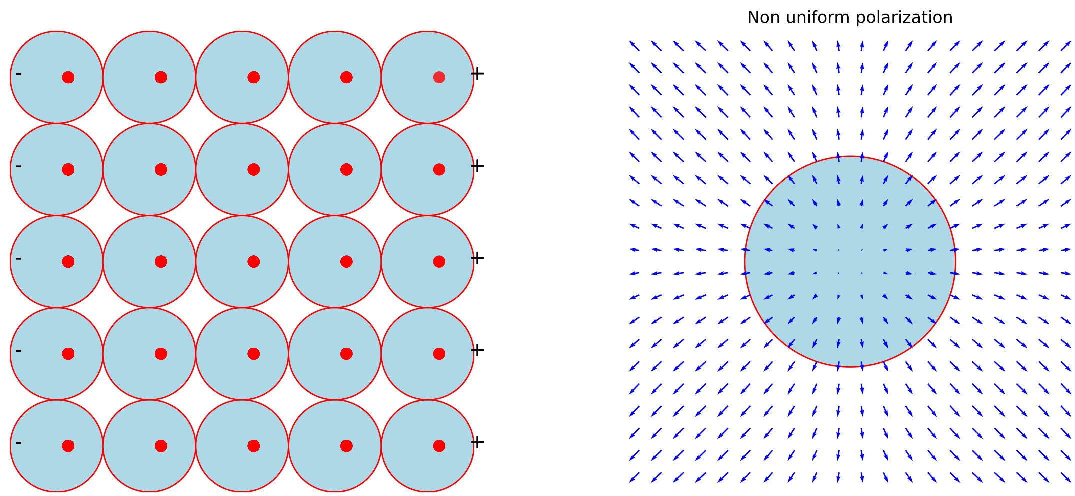

Figure 5.2: Uniform polarization leading to surface charges (left) and non uniform polarization leading to volume charges (right)

5.3 Gauss' law revisited

The Gauss' law is not modified if you consider vacuum or in the bulk of a material. However, we have seen that when the polarization is non uniform, it appears an additional charge density ρ_b. which gives

$$\boldsymbol{\nabla}.\mathbf{E} = \frac{\rho_f + \rho_b}{\epsilon_0} \tag{5.15}$$

where we add a subscript to the charge density where the charges are free to move. By using Eq.(5.14), the Gauss' law becomes

$$\boldsymbol{\nabla}.(\mathbf{E} + \epsilon_0 \mathbf{P}) = \frac{\rho_f}{\epsilon_0} \tag{5.16}$$

Let us define $\mathbf{D}$, the *electric displacement* field

$$\mathbf{D} = \mathbf{E} + \epsilon_0 \mathbf{P} \tag{5.17}$$

the Gauss's law is finally given as

$$\boldsymbol{\nabla}.\mathbf{D} = \frac{\rho_f}{\epsilon_0} \tag{5.18}$$

At this stage, this rewriting seems to be a mere sleight of hand. However, this is not entirely true. We have indeed defined a new quantity replacing the electric field, but we have absorbed the bound charge density, whose expression is not often known, in favor of polarization, which is more or less simply related to the electric field.

5.3.1 Linear dielectrics

In order to illustrate the interest of this approach, let us start with the linear dielectrics The electric displacement is proportional to the electric field

$$\mathbf{D} = \epsilon \mathbf{E} \tag{5.19}$$

where ϵ is called the permittivity of the material By using Eq.(5.6), one obtains

$$\epsilon = (1 + \chi_e)\epsilon_0 \tag{5.20}$$

Because χ_e is positive, the permittivity in the matter is larger than the permittivity in vacuum. The relative permittivity is defined as

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e \tag{5.21}$$

5.3.2 Clausius-Mosotti relation

To have a bridge between the macroscopic parameters and the microscopic parameters is important for understanding the characteristics of linear dielectrics. A mean-field approach can be done as follows:

At the microscopic level the dipole moment is given by

$$\mathbf{p} = \alpha(\mathbf{E} + \mathbf{E}_{dip}) \tag{5.22}$$

The relationship between $\mathbf{P}$ and $\mathbf{E}$ is given by Eq.(5.6)

To evaluate the electric field created by the dipoles, one split the volume in two parts: a spherical sphere and the rest of the space where the dipoles are described the field of the polarization.

At the surface of the sphere, it appears a surface surface which is given by $\sigma_b = P\cos(\theta)$ where θ is the angle between the polarization and the normal to surface of the sphere at a given position. After some calculation, the electric field coming from σ_b give an electric fields equal to $\mathbf{E}_P = \frac{\mathbf{P}}{3\epsilon}$. Now, we consider the fields to due to dipoles inside the cavity, but for a dense system where an short range order exists, the electric field due to the particles in the volume V cancels

By using the definition of the polarization, Eq.(5.5)

$$\mathbf{P} = \varrho\epsilon_0\gamma(\mathbf{E} + \frac{\mathbf{P}}{3\epsilon_0}) = \epsilon_0\chi_e\mathbf{E} \tag{5.23}$$

which gives

$$\chi_e = \frac{\varrho\gamma}{1 - \frac{1}{3}\varrho\gamma} \tag{5.24}$$

which gives a relationship between the susceptibility (macroscopic) and polarizability (microscopic) Inversing this relation and introducing the relative dielectric constant, one obtains

$$\gamma = \frac{3}{\varrho}\frac{\epsilon_r - 1}{\epsilon_r + 2} \tag{5.25}$$

a relationship known as the Clausius-Mosotti relation.

5.3.3 Dielectric sphere

This example illustrates the role of the dielectrics compared to the vaccum. We consider the following system: a point of charge q is placed at the origin and surrounded by a dielectric sphere of radius R with a permittivity ϵ The Gauss's law for the electric displacement inside the dielectrics can be obtained easily by the Gauss's la.

$$\mathbf{D} = \frac{q}{4\pi r^3}\mathbf{r} \tag{5.26}$$

From the relation betwen the electric displacement and the electric fields, one infers

$$\mathbf{E} = \frac{q}{4\epsilon\pi r^3}\mathbf{r} \tag{5.27}$$

This corresponds to a similar expression obtained for a charge in vacuum $\frac{q}{\epsilon_r}$

5.3.4 Energy in Matter

In order to calculate the work to bring both the free charges in the position defined at the final state (assuming they are at infinity initially) and the polarization of the system coming from the bound charges.

Fig.5.3 illustrates the situation with a finite volume of a dielectric with three conductors inside the volume

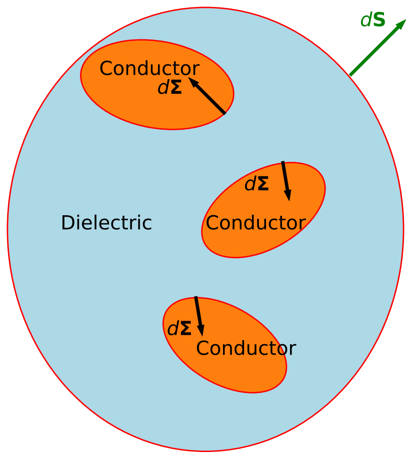

Figure 5.3: Dielectric with three conductors inside the volume. $d\mathbf{S}$ denotes the vector of the interface between the dielectric and the conductor and $d\mathbf{\Sigma}$ the vector of the interface between the dielectric and the vacuum

The power to be perfomed for moving the free charges is due to the power done by the electric field implying the free charges and the bound charges due to the polarization. Considering an infinitesimal work due to an infinitesimal change of the free charges density gives

$$\frac{dW}{dt} = \int_V d\mathbf{r}^3 \frac{\partial \rho_f}{\partial t}(\mathbf{r})V(\mathbf{r}) + \int_\Sigma d\Sigma \frac{\partial \sigma_f}{\partial t}(\mathbf{r})V(\mathbf{r}) \tag{5.28}$$

The second term is the contribution of the surface charges which are present in conductors.

Using the Gauss's theorem for the displacement field $\mathbf{D}$, one obtains

$$\sigma_f(\mathbf{r}) = -\mathbf{D}.\mathbf{n} \tag{5.29}$$

for the surface charges and

$$\rho_f(\mathbf{r}) = \mathbf{\nabla D} \tag{5.30}$$

The power becomes

$$\frac{dW}{dt} = \int_V d\mathbf{r}^3 V(\mathbf{r})\mathbf{\nabla}\frac{\partial \mathbf{D}}{\partial t} - \int_\Sigma V(\mathbf{r})\frac{\partial \mathbf{D}}{\partial t}.d\mathbf{\Sigma} \tag{5.31}$$

The Gauss's identity can be written as

$$\int_V d\mathbf{r}^3 \mathbf{\nabla}.(V\frac{\partial \mathbf{D}}{\partial t}) = \int_\Sigma d\mathbf{\Sigma}(V\frac{\partial \mathbf{D}}{\partial t}) + \int_V d\mathbf{r}^3 V\frac{\partial \mathbf{D}}{\partial t} + \int_V \nabla.V\frac{\partial \mathbf{D}}{\partial t}d\Sigma \tag{5.32}$$

Inserting the above equation in the power The infinitesimal work is simply expressed as

$$\frac{dW}{dt}W = \int_V d\mathbf{r}^3 \mathbf{E}(\mathbf{r}) \frac{\partial \mathbf{D}}{\partial t}(\mathbf{r}) \tag{5.33}$$

In order to define a potential energy, it is necessary to know the relationship between the electric field and the displacement. If $\mathbf{D}$ depends on the electric field not only at the position, but on the successive positions, the work can not be expressed as a potential energy. Conversely, if the the relationship is linear the differential

$$\mathbf{E}.\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{2}\frac{\partial(\mathbf{E}.\mathbf{D})}{\partial t} \tag{5.34}$$

and the power is given by

$$\frac{dW}{dt} = \frac{1}{2}\int_V d\mathbf{r}^3 \frac{\partial(\mathbf{E}(\mathbf{r})\mathbf{D}(\mathbf{r}))}{\partial t} \tag{5.35}$$

Because the electric field cancels insite conductors, the electrostatic eneregy is stored in the dielectric. A last remark: when $\epsilon \equiv \epsilon_0$ one recovers the result of the electric field in a vacuum.

5.4 Two laws unchanged!

In order to reassure the reader about the increasing complexity of transforming Maxwell's equations in the presence of matter, Maxwell's two laws in the absence of sources (charge and current) remain unchanged, of course.

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla}.\mathbf{B} &= 0 \end{aligned}$$

We still need to consider the presence of currents in order to rewrite Maxwell's last equation in matter.

5.5 Magnetism in Matter

To create a magnetic field, one needs a current which can be characterized by magnetic dipole moment $\mathbf{m}$. The existence of this current allows us to obtain the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3} \tag{5.36}$$

which leads to the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} \tag{5.37}$$

What is the origin of a current in a material? Electrons are mobile more or less in a material. In a dielectrics, most of electrons move around a nucleus and the current is associated to the orbital momentum (even if the motion of electrons require the description via quantum mechanics). The second reason has a pure quantum mechanic reason, the electron has a spin which contributes to the magnetic moment.

To build magnetism in matter, we follow a similar reasoning adopted for the electric field. First, we consider the average of the dipole moment on an infinitesimal volume which defined the magnetization per volume unit

$$\mathbf{M} = n\overline{\mathbf{m}} \tag{5.38}$$

where n is the density of dipole moment per volume unit and the overline an average over the coarse graining volume (typically several atomic scale). Whatever the origin of the magnetism, in many materials, the direction of the atomic moment are in all directions in the absence of a magnetic field. An important exception occurs for material, for instance ferromagnetic material. But this situation is not still present, the existence of a non zero magnetization can be destroyed when the thermal activity is increased sufficiently. Above a given temperature, all materials recover a zero magnetization in the absence of external field. For this reason, the ferromagnetism is a phase of a material for a finite range of temperature. It is not a property at all temperatures.

When the mean magnetization is equal to zero in the absence of an external magnetic field, the response is often linear when $\mathbf{B}$ is switch on. For convenience the relationship between the magnetization and the external field is written as

$$\mathbf{M} = \frac{1}{\mu_0} \frac{\chi_m}{1+\chi_m} \mathbf{B} \tag{5.39}$$

where χ_m is called the magnetic susceptibility. It is a dimensionless quantity.

Several cases should be considered

- $-1 < \chi_m < 0$ is negative, the material is then *diamagnetic*. The magnetization tends to be align in the opposite direction of the magnetic field. For many materials including water the magnetic susceptibility is of order $10^{-5} - 10^{-6}$ which is quite small. This means that living matter is weakly sensitive to an external field. Anyway, an experiment was realized with a living frog in an intense magnetic field and permits to observe the levitation (see Fig.5.4) This experiment deserved the Ig Nobel prize in 2000 to Andre Geim and Michal Berry. I recall that Ig means Ignoble. But Geim was awarded by the Nobel Prize in 2010 for graphene with Konstantin Novoselov!

Figure 5.4: Picture of a frog in levitation

- $\chi_m = -1$ This corresponds to Superconductors, where the magnetic field created by the electrons cancel the external field. This can viewed as a perfect diamagnetism.

- $\chi_m > 0$ is positive, the material is then *paramagnetic*. The magnetization is then in the same direction of the magnetic field.

Compared to the electrostatics, the susceptibilty has a larger range of possibilities, which depends on the microscopic interaction at the atomic level. For our description of the electromagnetism at the coarse-graining level, it is not necessary to know the details of the material and we restart the derivation of the Maxwell equations in matter.

Summing over all dipoles over the matter volume by using Eq. (5.36), one obtains

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V d\mathbf{r}'^3 \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{5.40}$$

Using the usual identities, one can express the vector potential as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V d\mathbf{r}'^3 \times \boldsymbol{\nabla}' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= -\frac{\mu_0}{4\pi}\int_S d\mathbf{S}' \times \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mu_0}{4\pi}\int_V d\mathbf{r}'^3 \frac{\boldsymbol{\nabla}' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\end{aligned} \tag{5.41}$$

These two integrals in Eq.(5.41) can be interpreted as follows: Let us define $\mathbf{K}_b$ a surface current as

$$\mathbf{K}_b = \mathbf{M} \times \mathbf{n} \tag{5.42}$$

where $\mathbf{n} = \frac{\mathbf{S}}{S}$ is the unit vector normal to the surface.

Second, the bound current $\mathbf{J}$ is given by

$$\mathbf{J}_b = \boldsymbol{\nabla}_{\times}\mathbf{M} \tag{5.43}$$

which gives a Biot-Savart formula of the second contribution of the vector potential

$$\mathbf{A}_2(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V d\mathbf{r}'^3 \frac{\mathbf{J}_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{5.44}$$

Because we are working in magnetostatic conditions, the current conservation is observed, because

$$\boldsymbol{\nabla}.\mathbf{J}_b = 0 \tag{5.45}$$

5.6 Ampère's law revisited

It is now time to add both contribution of the currents in the Ampere's law In order to distinguish the current coming from free moving charges to bound current, the free current has a supplemental index (temporarily)

One then writes

$$\boldsymbol{\nabla}_{\times}\mathbf{B} = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \tag{5.46}$$

By using Eq.(5.43), one gets

$$\boldsymbol{\nabla}_{\times}\mathbf{B} = \mu_0(\mathbf{J}_f + \boldsymbol{\nabla}_{\times}\mathbf{M}) \tag{5.47}$$

Let us define the *magnetizing* field $\mathbf{H}$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{5.48}$$

The Ampere's law becomes

$$\boldsymbol{\nabla}_{\times}\mathbf{H} = \mathbf{J}_f \tag{5.49}$$

Once again, this rewriting allows us to get rid of the bound current as the field displacement gives a Gauss's law without bound charges.

For diamagnetic and paramagnetic materials, we have a linear relationship between $\mathbf{H}$ and $\mathbf{B}$

$$\mathbf{B} = \mu\mathbf{H} \tag{5.50}$$

where μ is called the *permeability*. It is easy to obtain

$$\mu = \mu_0(1 + \chi_m) \tag{5.51}$$

It is worth noting that for diamagnetic matter the difference between μ and μ_0 is quite small.

5.7 The final cut: the Maxwell-Ampere law

For electrostatics and magnetostatics in matter, we have considered the charge density and current density is independent of time. When the bound current density $\mathbf{J}_b$ depends on time, the charge conservation becomes

$$\boldsymbol{\nabla}.\mathbf{J}_b + \frac{\partial \rho_b}{\partial t} = 0 \tag{5.52}$$

From electrostatics, we know that

$$\rho_b = -\boldsymbol{\nabla}.\mathbf{P} \tag{5.53}$$

where $\mathbf{P}$ is the polarization. Inserting Eq.(5.53) in Eq.(5.52) gives

$$\boldsymbol{\nabla}.\left(\mathbf{J}_b + \frac{\partial \mathbf{P}}{\partial t}\right) = 0 \tag{5.54}$$

The solution of this equation consists in modifying the expression of the bound current density

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} - \frac{\partial \mathbf{P}}{\partial t} \tag{5.55}$$

We have now completed the current density and the Maxwell-Amoère law can be expressed as

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \mu_0 \mathbf{J}_f + \mu_0 \mathbf{J}_b \\ &= \mu_0 \mathbf{J}_f + \mu_0 \boldsymbol{\nabla} \times \mathbf{M} - \frac{\partial \mathbf{P}}{\partial t} \end{aligned} \tag{5.56}$$

By using the definitions of the displacement filed $\mathbf{D}$ and of the magnetizing field $\mathbf{H}$, one finally obtains the Maxwell-Ampère equation in matter

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{5.57}$$

Once again, the definition of a second field for the magnetism allows to express electromagnetism with sources coming from free charges. The elegant formulation do not hide the fact that the relationships between the electric field, the magnetic field, the expressions of the displacement field and the magnetizing field are pending, but necessary for solving the Maxwell equations in matter

5.8 Eddy currents (Foucault's currents)

Eddy currents are induced in any conducting medium in which the magnetic field varies with time. Thus, eddy currents occur in the stator and rotor cores of electric motors and generators, in transformers, metallic recording heads (used in magnetic recording), and the like. These eddy currents can produce undesirable effects such as power loss, heating, and-in magnetic recording-degradation in performance.

Let us recall the physical meaning of the Maxwell-Faraday's law or the Lenz's law: "the emf induced in an electric circuit always acts in such a direction that the current it drives around a closed circuit produces a magnetic field which opposes the change in magnetic flux."

Using the Ohm's $= \mathbf{J} = \sigma \mathbf{E}$ where σ is the conductivity and Maxwell equations, and assuming that the frequency is small and $\mathbf{B} = \mu \mathbf{H}$ (μ is the permability) one easily obtains

$$\boldsymbol{\nabla}^2 \mathbf{H} = \sigma \mu \frac{\partial \mathbf{H}}{\partial t} \tag{5.58}$$

The same equation are obtained for the electric field and the current density We consider the simple model od slab of thickness d ($d > x > 0$) The translational invariance is two dimensions transforms Eq.(5.58) in

$$\frac{\partial^2 H_z}{\partial x^2} = \sigma\mu \frac{\partial H_z}{\partial t} \tag{5.59}$$

The external field at $x = 0$ is $H(t) = H_0 \sin(\omega t)$. Using complex solution, the stationary solution of H_z is given by

$$H_z(x,t) = H_0 e^{-x/\delta} e^{i(\omega t - x/\delta)} \tag{5.60}$$

where δ is the skin length.

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{5.61}$$

We are working in the limit where the thickness d is much smaller than the skin depth δ. The decay of the quantities are neglected

Using the Faraday's equation, one infers that

$$\dot{B}_z = -\sigma \frac{\partial j_y}{\partial x} \tag{5.62}$$

which gives

$$j_y = -\sigma \dot{B}_z x \tag{5.63}$$

The dissipated power per volume unit is given by

$$P = \frac{j_y^2}{\sigma} \tag{5.64}$$

Averaging over the time average over a period and over x, one obtains the mean power per surface unit

$$\overline{P} = \frac{1}{dT\sigma} \int_0^T dt \int_0^d dx j_y^2 \tag{5.65}$$

Moreover assuming ll_delta, the power is given by The dissipated power is proportional to the square of the frequency and of the intensity of the magnetic field.

$$\overline{P} = \frac{\sigma H_0^2 \omega^2 d^2}{6} \tag{5.66}$$

5.9 Wave propagation in matter

5.9.1 Linear dielectrics

For investigating the wave propagation in dielectrics, we first consider the simplest case of a linear dielectric. For simplicity, one first studies the case of the wave propagation The Maxwell equations in matter are given by

$$\begin{aligned} \boldsymbol{\nabla}\mathbf{D} = 0, &\qquad \boldsymbol{\nabla}\mathbf{B} = 0 \\ \boldsymbol{\nabla}\times\mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, &\qquad \boldsymbol{\nabla}\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

and the constitutive relations are

$$\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} \\ \mathbf{B} &= \mu \mathbf{H} \end{aligned} \tag{5.67}$$

We have now a set of closed equations and by using the constitutive relations, one obtains

$$\mu\epsilon\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\mathbf{C} = 0 \tag{5.68}$$

where $\mathbf{C}$ can be a vector field $\mathbf{E}$, $\mathbf{B}$. , $\mathbf{D}$, or $\mathbf{H}$. It looks similar to the situation in vacuum, but a first difference, the velocity of the wave propagation is now given by

$$v = \frac{1}{\sqrt{\epsilon\mu}} \tag{5.69}$$

Because $\epsilon > \epsilon_0$ and $\mu > \mu_0$, the velocity is always less than the velocity in vacuum.

One then defines the index of refraction as

$$n = \frac{c}{v} > 1 \tag{5.70}$$

which is also larger than 1.

For many materials the magnetic susceptibility is very small and $\mu \simeq \mu_0$ and the index of refraction can be approximated by

$$n \simeq \sqrt{\epsilon_r} \tag{5.71}$$

Consider the case of a plane wave

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \\ \mathbf{B} &= \mathbf{B}_0 e^{i(\mathbf{k}.\mathbf{r}-\omega t)} \end{aligned} \tag{5.72}$$

Insering these solutions in the equation of the wave propagation, one obtains the simple relation of dispersion

$$\omega^2 = v^2 k^2 \tag{5.73}$$

The two first Maxwell equations give that

$$\begin{aligned} \mathbf{k}.\mathbf{E}_0 &= 0 \\ \mathbf{k}.\mathbf{B}_0 &= 0 \end{aligned} \tag{5.74}$$

which expresses that the electric and magnetic field are perpendicular to the direction of propagation $\mathbf{k}$. The Faraday's law gives

$$\begin{aligned} \mathbf{B}_0 &= \frac{\mathbf{k}\times\mathbf{E}}{\omega} \\ &= \frac{\hat{\mathbf{k}}\times\mathbf{E}}{v} \end{aligned} \tag{5.75}$$

where $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$ is a unit vector of the direction of propagation. This last equation shows that the three vectors form a direct basis for the space in the same way as in the vacuum. The amplitude of the magnetic field is smaller because the velocity in the dielectric is smaller than in vacuum.

5.9.2 Interface of dielectrics

Except the difference of velocity, the wave propagation in a linear dielectric is very similar to this in vacuum. We now consider the space filled by two different linear dielectrics and we consider the wave propagation in full space.

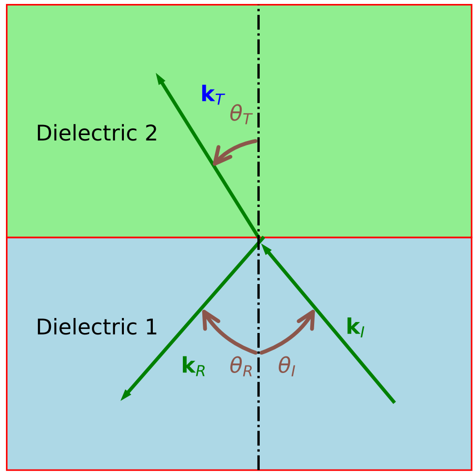

Figure 5.5: Interface between two dielectrics: $\vec{k}_I$, $\vec{k}_I$ and $\vec{k}_I$ denote the wave vectors of the incident, reflected and transmitted waves, respectively

Each linear dielectric has a couple of parameters characterizing the material (ϵ_i, μ_i) with $i = 1, 2$ By using the Maxwell equations with a volume containing both sides of the interface, we obtains ot a surface perpendicular to the interface, one obtains the following equations

$$\begin{aligned}
\hat{\mathbf{n}}.(\mathbf{D}_2 - \mathbf{D}_1) &= \sigma \\
\hat{\mathbf{n}}.(\mathbf{B}_2 - \mathbf{B}_1) &= 0 \\
\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) &= 0 \\
\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{K}
\end{aligned} \tag{5.76}$$

where $\hat{\mathbf{n}}$ is a unit normal vector to the surface, σ is the free surface charge density and $\mathbf{K}$ the free surface current at the interface.

For linear dielectrics, $\sigma = 0$ and $\mathbf{K} = \mathbf{0}$ Starting from Fig. 5.5, and by using cartésian coordinates, we have the electric field given by

$$\mathbf{E}_I = \mathbf{E}_i e^{i(\mathbf{k}_I.\mathbf{r} - \omega_I t)} \tag{5.77}$$

with awave vector of the incoming wave $\mathbf{k}_I$

$$\mathbf{k}_I = k_I(-\sin(\theta_I)\hat{\mathbf{x}} + \cos(\theta_I)\hat{\mathbf{z}}) \tag{5.78}$$

At the interface, the incoming plane wave is split in two wave planes: a reflected part with a wave vector $\mathbf{k}_R$ and a transmitted part with a wave vector $\mathbf{k}_T$.

Following the previous notations, the two wave vectors are

$$\begin{aligned}
\mathbf{k}_R &= k_I(-\sin(\theta_R)\hat{\mathbf{x}} - \cos(\theta_R)\hat{\mathbf{z}}) \\
\mathbf{k}_T &= k_I(-\sin(\theta_T)\hat{\mathbf{x}} + \cos(\theta_T)\hat{\mathbf{z}})
\end{aligned} \tag{5.79}$$

and the associated electric fields are given

$$\begin{aligned} \mathbf{E}_R &= \mathbf{E}_r e^{i(\mathbf{k}_R.\mathbf{r}-\omega_R t)} \\ \mathbf{E}_T &= \mathbf{E}_t e^{i(\mathbf{k}_T.\mathbf{r}-\omega_T t)} \end{aligned} \tag{5.80}$$

In order to know if this proposed solution satisfies the Maxwell equations (which is obvious in each dielectric, because the plane waves are solutions of the d'Alembertian equation), one needs to satisfy the boundary conditions, Eqs.(5.76) in the absence of surface charges and surface currents.

At the interface ($z = 0$), the time dependence of the electric field must be equal which gives

$$\omega_I = \omega_R = \omega_T = \omega \tag{5.81}$$

And the phases of all waves obey

$$\mathbf{k}_I.\mathbf{r} = \mathbf{k}_R.\mathbf{r} = \mathbf{k}_T.\mathbf{r} \tag{5.82}$$

which gives

$$k_I \sin(\theta_I) = k_R \sin(\theta_R) = k_T \sin(\theta_R) \tag{5.83}$$

Last, the module of wave vectors are given by $k = \omega n/c$ where n is the index of refraction of the dielectric. Consequently, $k_I = k_R$ and this gives

$$\theta_I = \theta_R \tag{5.84}$$

and because $k_I/n_1 = k_T/n_2$, one infers

$$n_1 \sin(\theta_I) = n_2 \sin(\theta_T) \tag{5.85}$$

These equations are known as the Snell-Descartes law.

5.9.3 Polarized wave planes

We have shown that the previous solution was suitable for describing the propagation of plane waves into dielectrics, but we skip the amplitudes of the electric fields. It is possible to determine theses quantities, but they depend on the direction of electric field that we know is orthogonal to the direction of the wave propagation. To solve this problem, we consider the two different polarizations

Normal polarization

In this first case, the electric field is normal to the plane defined by (Oxz) (see Fig.**??**)

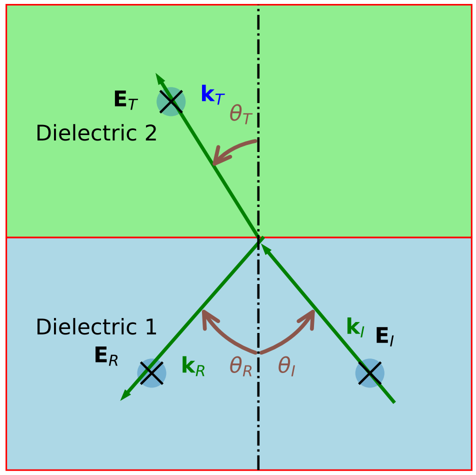

Figure 5.6: Interface between two dielectrics: $\vec{k}_I$, $\vec{k}_I$ and $\vec{k}_I$ denote the wave vectors of the incident, with a normal polarization.

It means that

$$\mathbf{E}_I = E_I \hat{\mathbf{y}} \tag{5.86}$$

The boundary conditions leads to keep the same direction for the reflected and transmitted electric field

$$\mathbf{E}_R = E_R \hat{\mathbf{y}} \quad , \quad \mathbf{E}_T = E_T \hat{\mathbf{y}} \tag{5.87}$$

which gives

$$E_I + E_R = E_T \tag{5.88}$$

In each dielectric, the magnetic field is closely related to the electric field by

$$\mathbf{B} = n \frac{\hat{\mathbf{k}} \times \mathbf{E}}{c} \tag{5.89}$$

This give for the magnetic field

$$B_I \cos(\theta_I) - B_R \cos(\theta_R) = B_T \cos(\theta_T) \tag{5.90}$$

The ratio of the module of magnetic field and of electric field are $\frac{B}{E} = \frac{n}{c}$ which gives

$$n_1(E_I \cos(\theta_I) - E_R \cos(\theta_R)) = n_2 E_T \cos(\theta_T) \tag{5.91}$$

Combining Eqs.(5.86) and (5.91), one obtains the two ratios

$$\frac{E_R}{E_I} = \frac{n_1 \cos(\theta_I) - n_2 \cos(\theta_T)}{n_1 \cos(\theta_I) + n_2 \cos(\theta_T)} \tag{5.92}$$

and

$$\frac{E_T}{E_I} = \frac{2 n_1 \cos(\theta_I)}{n_1 \cos(\theta_I) + n_2 \cos(\theta_T)} \tag{5.93}$$

which are known as the *Fresnel equations*.

Parallel polarization

In this first case, the electric field is parallel to the plane defined by (Oxz) (see Fig.\ref{eq:interfaces}) which also means that the magnetic field in perpendicular to the plane

$$\mathbf{B}_I = B\hat{\mathbf{y}} \tag{5.94}$$

After reflection the direction of the magnetic filed is reversed which gives

$$B_I - B_R = B_T \tag{5.95}$$

The components of the electric field here gives

$$E_I \cos(\theta_I) + E_R \cos(\theta_r) = E_T \cos(\theta_T) \tag{5.96}$$

After some calculation, one also obtains the Fresnel equations for a parallel polarization.

$$\frac{E_R}{E_I} = \frac{n_1 \cos(\theta_T) - n_2 \cos(\theta_I)}{n_1 \cos(\theta_T) + n_2 \cos(\theta_I)} \tag{5.97}$$

and

$$\frac{E_T}{E_I} = \frac{2n_1 \cos(\theta_I)}{n_1 \cos(\theta_T) + n_2 \cos(\theta_I)} \tag{5.98}$$

Obsviouly the two equations give the same result when the direction of the propagation is perpendicular to the interface, due to the rotation symmetry.

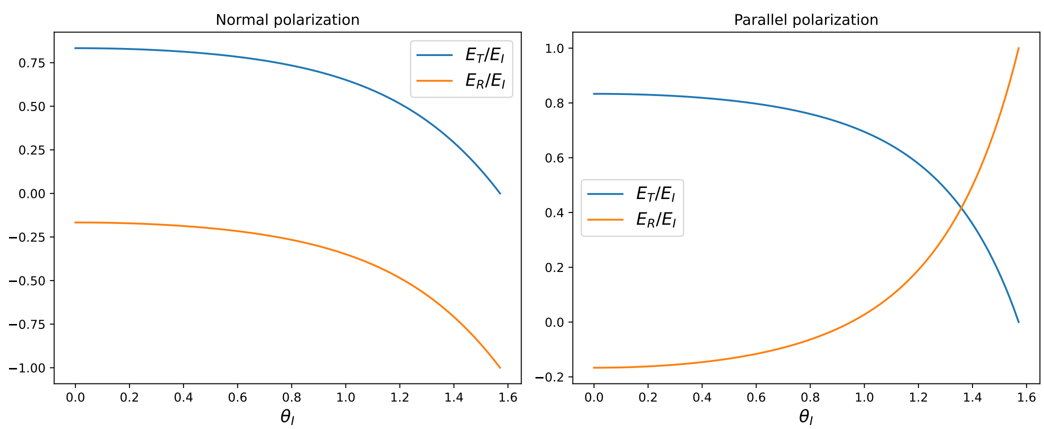

Figure 5.7: Evolution of the relative intensity of the reflected and transmitted wave versus θ_I. (Left Panel) Normal Polarization (Right panel) Parallel polarization.

Fig.5.7 shows the evolution of the reflected and transmitted electric amplitude versus the incidence angle when the indices de refraction are $n_1 = 1$ and $n_2 = 1.5$ These values are close the indices for the air and for a glass, respectively.

Brewster's angle

In the case of parallel polarization, Fig.5.7 reveals that, it exists an angle where the reflected amplitude of the electric field cancels. This corresponds to

$$n_1 \cos(\theta_T) = n_2 \cos(\theta_I) \tag{5.99}$$

In the same time, the Snell-Descartes law imposes that

$$n_1 \sin(\theta_I) = n_2 \sin(\theta_T) \tag{5.100}$$

which gives

$$\sin(\theta_I)\cos(\theta_I) = \sin(\theta_T)\cos(\theta_T) \tag{5.101}$$

For $0 \leq \theta \leq \frac{\pi}{2}$, the non trivial solution is

$$\theta_I + \theta_T = \frac{\pi}{2} \tag{5.102}$$

Inserting this solution in Eq.(5.100) one gets

$$\tan(\theta_I) = \frac{n_2}{n_1} \tag{5.103}$$

For a glass where the index of refraction $n_2 = 1.54$, this gives an angle $\theta_I = 57°$.

5.10 A simple Model of Polarization

We know that the origin of the polarization has a microscopic origin, where the electrons moves around the nucleus, but in the presence of an external electric field an dipole moment occurs locally. Even if the classical description of the motion of the electons is too crude for a quantitative prediction, we follow this way in order to capture some main characteristics that we have to include in the dielectric description.

We consider an electron submitted to an external field which depends on time, an harmonic force which captures the oscillating behavior around equilibrium and a viscous force which captures dissipative forces due to collisions with the neighboring atoms The second law of Newton of the model is given by

$$m\ddot{\mathbf{r}} = -q\mathbf{E}(t) - m\omega_0^2\mathbf{r} - \frac{m}{\tau}\dot{\mathbf{r}} \tag{5.104}$$

where ω is the pulsation of the oscillations, and τ the characteristic time of the relaxation.

Assuming that a monochromatic plane wave $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}.\mathbf{k}-\omega t)}$ is at the origin of the excitation, the scalar product $\mathbf{k}.\mathbf{r}$ can be considered as very small because the size of the atom is much smaller than the inverse of wave length. Thus, the electric field is then given by $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$

The stationary solution of the differential equation can be found by inserting the guess $\mathbf{r} = \mathbf{r}_0 e^{-i\omega t}$ and the solution is given by

$$\mathbf{r}_0 = -\frac{q\mathbf{E}_0}{m}\frac{1}{-\omega^2 + \omega_0^2 - i\frac{\omega}{\tau}} \tag{5.105}$$

Because the microscopic polarization is given by $\mathbf{p} = \alpha\mathbf{E}$ and $\mathbf{p} = -q\mathbf{r}$, one obtains

$$\alpha(\omega) = \frac{q^2}{m}\frac{1}{-\omega^2 + \omega_0^2 - i\frac{\omega}{\tau}} \tag{5.106}$$

Assuming that the bound electrons are independent, the permittivity of the dielectric can be written as

$$\epsilon(\omega) = \epsilon_0 + \varrho\alpha(\omega) \tag{5.107}$$

Note that when $\omega \to \infty$ the contribution of bound electrons goes to zero.

In the limit of small frequency, α goes to the finite real value, like in the previous model with

$$\omega_0^2 = \frac{q^2}{4\pi\epsilon_0 m a^3} \tag{5.108}$$

Conversely, one obtains that α is a complex function of the pulsation ω, a consequence of the fact that the response of the atom is not instantaneous with an amplitude dependent on ω, and complex which means that a part of the response leads to dissipation.

5.11 Wave propagation in more realistic dielectrics

The simple model shows the the polarization is linear with the electric field but in the time Fourier space. The first issue to solve is if you need to rebuild a complete strategy for solving the wave propagation described above. Hopefully, the answer is no! We just need to adapt previous results.

Previously we have considered monochromatic plane wave for the propagation. In time Fourier space the relationship between the displacement field and the electric field is also linear

$$\mathbf{D} = \epsilon(\omega)\mathbf{E} \tag{5.109}$$

The ansatz of plane wave solutions are

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0(\omega)e^{i(\mathbf{k}.\mathbf{k}-\omega t)}\\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}_0(\omega)e^{i(\mathbf{k}.\mathbf{k}-\omega t)}\\ \mathbf{D}(\mathbf{r},t) &= \mathbf{D}_0(\omega)e^{i(\mathbf{k}.\mathbf{k}-\omega t)}\\ \mathbf{H}(\mathbf{r},t) &= \mathbf{H}_0(\omega)e^{i(\mathbf{k}.\mathbf{k}-\omega t)}\end{aligned} \tag{5.110}$$

with $\mathbf{B}(\mathbf{r},t) = \mu_0\mathbf{H}(\mathbf{r},t)$

The Maxwell equations becomes

$$\begin{aligned}\epsilon(\omega)\mathbf{k}.\mathbf{E}_0(\omega) &= 0\\ \mathbf{k}.\mathbf{B}_0(\omega) &= 0\\ \mathbf{k}\times\mathbf{B}_0(\omega) &= -\mu\epsilon(\omega)\mathbf{E}_0(\omega)\\ \mathbf{k}\times\mathbf{E}_0(\omega) &= \mathbf{B}_0(\omega)\end{aligned} \tag{5.111}$$

By taking the cross product of the Ampere's equation, one obtains

$$(\mathbf{k}.\mathbf{k} - \mu\epsilon(\omega))\mathbf{B}_0(\omega) = 0 \tag{5.112}$$

The same equation for the electric field can be derived by taking the cross-product of the Faraday equation.

The geometrical configurations of the electric and magnetic fields are the same than for a dielectric with a constant permittivity ϵ, namely $\mathbf{E}$ and $\mathbf{B}$ are perpendicular between them and both perpendicular to the direction of propagation.

Finally, the change appears in the dispersion relation

$$\mathbf{k}.\mathbf{k} = \mu\epsilon(\omega) \tag{5.113}$$

where the wave number now $\mathbf{k}$ now depends on ω in a non linear manner and because the the polarizability is now complex, the solution of the dispersion relation is also complex.

The complex wavenumber can be decomposed in real and imaginary parts

$$\mathbf{k} = (k_r + ik_i)\hat{\mathbf{k}} \tag{5.114}$$

The electric field can be expressed

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\omega)e^{-k_i\hat{\mathbf{k}}.\mathbf{r}}e^{i(k_r\hat{\mathbf{k}}.\mathbf{r}-\omega t)} \tag{5.115}$$

which means that the amplitude of the wave decays exponentially with the distance. This is called *attenuation* of the wave propagation. The physical electric is obtained by taking the real part of this equation

$$\mathbf{E}(\mathbf{r},,t) = \mathbf{E}(\omega)e^{-k_i\hat{\mathbf{k}}.\mathbf{r}}\cos(k_r\hat{\mathbf{k}}.\mathbf{r}-\omega t) \tag{5.116}$$

The magnetic field is then given by

$$\mathbf{B}(\mathbf{r},,t) = \frac{k}{\omega}\hat{\mathbf{k}} \times \mathbf{E} \tag{5.117}$$

Using that $k = |k|e^{i\phi}$ with $\phi = \arctan\left(\frac{k_i}{k_r}\right)$ the magnetic field becomes

$$\mathbf{B}(\mathbf{r},,t) = \frac{\sqrt{k_i^2 + k_r^2}e^{i\phi}}{\omega}(\hat{\mathbf{k}} \times \mathbf{E}(\omega))e^{-k_i\hat{\mathbf{k}}.\mathbf{r}}e^{i(k_r\hat{\mathbf{k}}.\mathbf{r}-\omega t)} \tag{5.118}$$

Going back in real space, one obtains

$$\mathbf{B}(\mathbf{r},,t) = \frac{\sqrt{k_i^2 + k_r^2}}{\omega}(\hat{\mathbf{k}} \times \mathbf{E}(\omega))e^{-k_i\hat{\mathbf{k}}.\mathbf{r}}\cos(i(k_r\hat{\mathbf{k}}.\mathbf{r} - \omega t + \phi)) \tag{5.119}$$

Therefore, the magnetic field decays also in a similar way with the distance, but a phase shift between two fields are now present and increases with the attenuation.

By using our simple model, we now identify the origin of the attenuation: Expressing the permittivity as $\epsilon(\omega) = \epsilon_r + i\epsilon_i$

Because we consider material which are at least a little bit transparent, one can assume that $\epsilon_r \geq \epsilon_i$. at the first order one obtains

$$\begin{aligned} k_r &= \omega\sqrt{\mu\epsilon_r} \\ k_i &= \omega\frac{\epsilon_i}{2\sqrt{\epsilon_r}} \end{aligned} \tag{5.120}$$

Because $\mu \simeq \mu_0$ and $\epsilon = \epsilon_0(1 + \varrho\alpha)$, assuming also that $\omega_0 >> \omega$ and denoting $\epsilon_{r,0} = \lim_{\omega\to 0}(1 + \varrho\alpha(\omega))$

$$\begin{aligned} k_r &= \frac{\omega\sqrt{\epsilon_r}}{c} \\ k_i &= \frac{\omega^2}{\tau c} \end{aligned} \tag{5.121}$$

One notes that the existence of the attenuation is clearly associated to the existence of the irreversible viscous force. If in the simple model, it is defined as a phenomenological force, the spontaneous emission for a excited electron is the microscopic origin.

In summary, by introducing a frequency dependence of the permittivity and by keeping a linear relation between the displacement field and the electric field in time Fourier space, we keep the ability to obtain analytical solutions and we have shown that many properties of the wave propagation obtained in vacuum are conserved. However, it appears an attenuation effect whose typical length depends on the pulsation.

Phase and group velocity

When a monochromatic wave plane travels in a dielectric, the *phase* velocity is given by

$$v = \frac{\omega}{k} \tag{5.122}$$

Now, in many situations, the wave is a sum of different waves with many wavenumbers $\mathbf{k}$. Expressing the electric field as

$$\mathbf{E}(x,t) = \int \frac{d\hat{\mathbf{k}}}{2\pi}\mathbf{E}(k)e^{i(k\hat{\mathbf{k}}.\mathbf{r}-\omega(k)t)} \tag{5.123}$$

where $\mathbf{E}(k)$ has a bell shape centered at k_0. Expanding around k_0,

$$\begin{aligned} k\hat{\mathbf{k}}.\mathbf{r} - \omega(k)t &= k\hat{\mathbf{k}}.\mathbf{r} - \omega(k_0)t - \frac{d\omega}{dk}|_{k=k_0}(k-k_0)t \\ &= k\left(\hat{\mathbf{k}}.\mathbf{r} - \frac{d\omega}{dk}|_{k=k_0}t\right) - (\omega(k_0) - \frac{d\omega}{dk}|_{k=k_0})t \end{aligned} \tag{5.124}$$

The second term of the right-hand side of Eq.(5.124) is only a time oscillating function. The first term defines a propagating expression $\hat{\mathbf{k}}.\mathbf{r} - v_g t$ with v_g is the group velocity defined by

$$v_g = \frac{d\omega}{dk}|_{k=k_0} \tag{5.125}$$

which corresponds to the motion of the peak of the distribution.

Because the phase velocity depends on the pulsation, one also define an index of refraction

$$n(\omega) = \frac{c}{v_p(\omega)} \tag{5.126}$$

Therefore

$$\frac{1}{v_g} = \frac{dk}{d\omega} = \frac{d\left(\frac{n\omega}{c}\right)}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c}\frac{dn}{d\omega} \tag{5.127}$$

If $\frac{dn}{d\omega} > 0$, one says that we have a normal dispersion, whereas if $\frac{dn}{d\omega} < 0$, a anomalous dispersion. Note that because ω evolves as the inverse of the wavelength, $\frac{dn}{d\lambda} < 0$ corresponds to the normal dispersion, whereas $\frac{dn}{d\lambda} > 0$ to the anomalous dispersion.

Using Eq.(5.127), one easily infers that $v_g < v_p$ for a normal dispersion and $v_g > v_p$ for a anomalous dispersion.

For light rays and for air or glass, the dispersion is normal.

A little mnemonic device to remember this rule is to think of a famous album cover (see Fig.5.8)

Figure 5.8: Dependence of the angle deviation in a prism

5.12 Conductors

5.12.1 The Drude model

We have considered conductors in electrostatics where the charges are free to move in order to maitain an electric field inside the conductor. In order to gain a clearer understanding, we will introduce a microscopic dynamic approach while remaining at the level of a classical description. The model is known as the Drude model. A free charged particle is submitted to a external electric field but with the displacement of the particle, it appears also interactions with the rest of the lattice (phonons in a quantum mechanical approach) where the charged particle is submitted to a viscous force. Dynamics is less described by the Newton equation

$$m\ddot{\mathbf{r}}(\mathbf{t}) = q\mathbf{E}(r,t) - \frac{m}{\tau}\dot{\mathbf{r}}(t) \tag{5.128}$$

Compared to the model of dielectrics given above, there is no force between a nucleus and the electron, which explains the idea of free particle. However, one forgets the idea that no electron can be escape from the conductor easily. It is not implemented to this model.

Even free, the displacement of the electron between two collisions is much smaller than the wavelength of the propagating wave, and one can assume that the spatial dependence of the electric field can be neglected.

The electric field is then given by $\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t}$. The stationary solution of the equation of motion is

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-i\omega t} \tag{5.129}$$

which gives

$$\left(-i\omega + \frac{1}{\tau}\right)\mathbf{v}_0 = \frac{q}{m}\mathbf{E}_0 \tag{5.130}$$

The current density of the electrons is given by

$$\mathbf{J} = \varrho q \mathbf{v} \tag{5.131}$$

where ϱ is the density of free charges.

Inserting Eq.(5.130) in Eq.(5.131) gives

$$\mathbf{J}(\omega) = \sigma(\omega)\mathbf{E} \tag{5.132}$$

with

$$\sigma(\omega) = \frac{\varrho q^2 \tau}{m}\frac{1}{1 - i\omega\tau} \tag{5.133}$$

In the limit of $\omega = 0$, one recovers the standard Ohm' law the DC conductivity is given by

$$\sigma_{DC} = \frac{\varrho q^2 \tau}{m} \tag{5.134}$$

for time-dependent electric field, the conductivity, also called the optical conductivity depends on the frequency and is complex. The real and imaginary part of the conductivity give

$$\begin{aligned} Re(\sigma) &= \frac{\sigma_{DC}}{1 + (\omega\tau)^2} \\ Im(\sigma) &= \frac{\sigma_{DC}\omega\tau}{1 + (\omega\tau)^2} \end{aligned} \tag{5.135}$$

Fig.5.9 illustrated the variation of the real and imaginary parts of the conductivity. Whereas the real conductivity decays monotonously with the frequency, the imaginary part displays a maximum for $\omega = 1/\tau$, which is called the *Drude peak*.

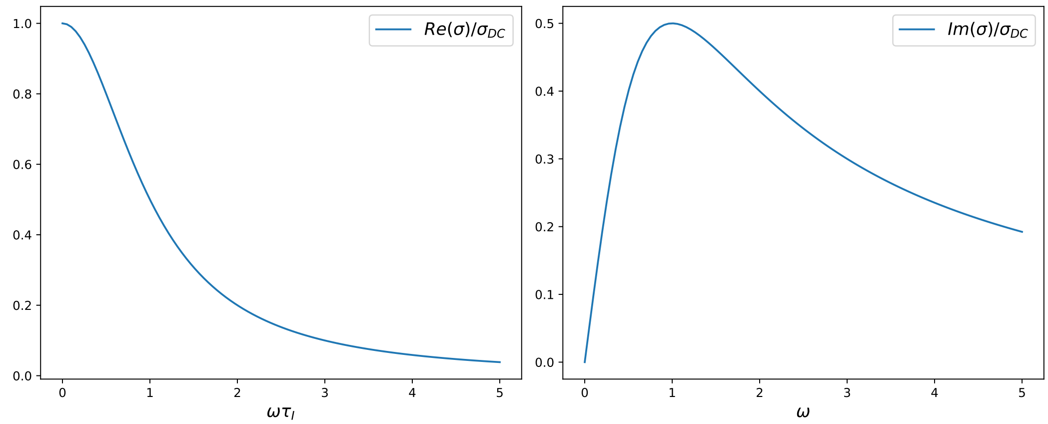

Figure 5.9: Evolution of the reals and imaginary part of the conductivity σ/σ_{DC} as a function of the dimensionless pulsation $\omega\tau$

It is worth noting that the very simple model is able to capture the behavior of many conductors, even if the classical description is not appropriate at the atomic scale where quantum effects are dominant. Anyway, this phenomenological approach allows us to go beyond, in particular to describe the wave propagation in conductors.

The physical interpretation of these behaviors can be given as follows: at low frequency, the relation time of the damping is smaller than the period of the oscillations of the electric field. Therefore the electron looses the energy gained by the electric field and the dissipation dominates the dynamics. Conversely the period of the oscillations are so intense that the viscous force appears weak compared to the force exerted by the electric field.

5.12.2 Wave propagation

Starting from the charge conservation $\mathbf{\nabla}.\mathbf{J} + \frac{\partial \rho}{\partial t} = 0$, we have in Fourier space by using the constitutive equation, Eq.(5.132), one gets

$$\sigma(\omega)\mathbf{k}.\mathbf{E}(\omega)e^{i(\mathbf{k}.\mathbf{r}-\omega t)} = \omega\rho(\omega) \tag{5.136}$$

This equation concerns the "free" charges and we know that in conductors a finite fraction of electrons are able to move, the other ones are bound to atoms. Going back to the Maxwell equations in which we now have a constitutive relation between the electric field and the displacement field, due to the bound charges.

$$\mathbf{D}(\omega) = \sigma(\omega)\mathbf{E}(\omega) \tag{5.137}$$

which is also linear in time Fourier space

We also remain with a linear constitutive relation (in real space) for the magnetic and magnetizing field, $\mathbf{B} = \mu\mathbf{H}$. Collecting all equations, we have a closed set of equations for all fields and this gives in Fourier space

$$\begin{aligned}
\left(\epsilon(\omega) + i\frac{\sigma(\omega)}{\omega}\right)\mathbf{k}.\mathbf{E}_0(\omega) &= 0 \\
\mathbf{k}.\mathbf{B}_0(\omega) &= 0 \\
\mathbf{k}\times\mathbf{B}_0(\omega) &= -\mu\left(\epsilon(\omega) + i\frac{\sigma(\omega)}{\omega}\right)\mathbf{E}_0(\omega) \\
\mathbf{k}\times\mathbf{E}_0(\omega) &= \mathbf{B}_0(\omega)
\end{aligned} \tag{5.138}$$

These equations looks like very similar to the Maxwell equations in dielectrics with a dispersion relation which can be obtained in the same way

$$\mathbf{k}.\mathbf{k} = \mu\left(\epsilon(\omega) + i\frac{\sigma(\omega)}{\omega}\right)\omega^2 = \mu\epsilon_{eff}(\omega)\omega^2 \tag{5.139}$$

Low frequency

In this regime, for many conductors, the contribution coming from free electrons dominates the contribution coming from bound electrons. In other words, the minority dictates the behavior of the system!

Writing $\mathbf{k} = (k_r + ik_i)\hat{\mathbf{h}}$, the dispersion relation becomes

$$k_r + ik_i = \sqrt{i\mu\sigma_{DC}\omega} = (1+i)\sqrt{\frac{\mu\sigma_{DC}\omega}{2}} \tag{5.140}$$

Let us denote

$$\delta = \frac{1}{k_r} = \sqrt{\frac{2}{\mu\sigma_{DC}\omega}} \tag{5.141}$$

and going back in real space, the electric field is given by the expression

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\omega)e^{-(\frac{\hat{\mathbf{k}}.\mathbf{r}}{\delta})}e^{i(\frac{\hat{\mathbf{k}}.\mathbf{r}}{\delta} - \omega t)} \tag{5.142}$$

The wave amplitude decays exponentially in the conductor with a typical length given by δ, called the *skin* length. In a conductor, the wave only penetrates in a conductor in the finite depth. In addition, the magnetic field is given by the ratio

$$\frac{|\mathbf{B}|}{|\mathbf{E}|} = \frac{k}{\omega} = \sqrt{\frac{\mu\sigma_{DC}}{2\omega}} \tag{5.143}$$

This ratio diverges when $\omega \to 0$, which means that the electric field becomes marginal compared to the magnetic field. Last, because the imaginary and real part of $\mathbf{k}$ are equal, the dephasing between the magnetic and electric fields is equal to $\pi/4$.

High frequency

In this regime the contributions of free and bound electrons goes to zero. We assume that $\omega >> \omega_0$ which gives $\epsilon \simeq \epsilon_0$. In this regime, electrons cease to have any influence on wave propagation and the permitivity can be written as

$$\epsilon_{eff}(\omega) = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{5.144}$$

where

$$\omega_p = q\sqrt{\frac{\varrho}{m\epsilon_0}} \tag{5.145}$$

is the plasma frequency.

Two cases must be considered:

- $\omega \gg \omega_p$. In this regime, the permittivity is real and the medium is transparent to the wave propagation
- $\omega \ll \omega_p$. In this regime, the permittivity is real negative and the wavevector is totally imaginary, which leads to a total reflection of the wave on the medium.

The frequency at which the crossover between the two behaviors appears is $\omega = \omega_p$. For many conductors $\omega = 10^{16}s^{-1}$, which gives a wave number $k \simeq 3.10^{-10}m$, which corresponds to the ultraviolet waves.

5.13 Electromagnetic Energy and Poynting theorem in matter

A single charged particle has an interaction with the electric and magnetic fields through the Lorentz force. The power brought to the particle is given by the scalar product of the Lorenz force and the velocity $q\mathbf{v}.\mathbf{E}$. For a continous distribution of charges and current in a finite volume, one obtains

$$\frac{dW}{dt} = \int_V d\mathbf{r}^3 \mathbf{J}.\mathbf{E} \tag{5.146}$$

This energy rate can be transformed either in a mechanical energy or heating energy if the systems has a constitutive relation between current and electric field like in the Ohm's law. Anyway, this power is obsviouly extracted from the electromagnetic energy, which is the translation of the conservation of energy

By using the Maxwell equations, one gets

$$\int_V d\mathbf{r}^3 \mathbf{J}.\mathbf{E} = \int_V d\mathbf{r}^3 \left(\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{dt}\right).\mathbf{E} \tag{5.147}$$

By using the identity

$$\boldsymbol{\nabla}.\left(\mathbf{E} \times \mathbf{H}\right) = -\mathbf{E}.\boldsymbol{\nabla} \times \mathbf{H} + \mathbf{H}.\boldsymbol{\nabla} \times \mathbf{E} \tag{5.148}$$

Inserting the Maxwell-Faraday equation, one obtains one obtains

$$\int_V d\mathbf{r}^3 \mathbf{J}.\mathbf{E} = \int_V d\mathbf{r}^3 \left(-\boldsymbol{\nabla}.\mathbf{E} \times \mathbf{H} - \frac{\partial \mathbf{B}}{dt}.\mathbf{H} - \frac{\partial \mathbf{D}}{dt}.\mathbf{E}\right) \tag{5.149}$$

To go further, it is necessary to assume an linearity between the displacement and electric field as well as the the magnetizing and magnetic fields.

Defining the volume density of electromagnetism energy as

$$u = \frac{1}{2}\left(\mathbf{B}.\mathbf{H} + \mathbf{D}.\mathbf{E}\right) \tag{5.150}$$

and the Poynting vector $\mathbf{S}$ as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{5.151}$$

one obtains

$$-\int_V d\mathbf{r}^3 \mathbf{J}.\mathbf{E} = \int_V d\mathbf{r}^3 \left(\frac{\partial u}{\partial t} + \boldsymbol{\nabla}.\mathbf{S}\right) \tag{5.152}$$

The integral form can be transformed in a local equation

$$-\mathbf{J}.\mathbf{E} = \frac{\partial u}{\partial t} + \boldsymbol{\nabla}.\mathbf{S} \tag{5.153}$$

The physical interpretation of this equation is the following: the rate of work coming the Lorentz force is the opposite of the global change of electromagnetism energy in a volume, by two mechanisms: a rate of the density of electromagnetism energy ($\frac{\partial u}{\partial t}$) and the flux of the electromagnetism at the boundary given by the Poynting's vector.

5.14 Conclusion

The resolution of these two simple models should not obscure the complexity of physical situations where the relationship between the displacement field and the electric field is not only linear in frequency.

We saw that by having a linear relationship in frequency between these two fields, we were able to continue using previously established methods. More generally, the system's response can remain linear in wave vector instead of being linear in position, which reflects a non-local response that is very often present. Another situation that appears is a linear relationship between the two fields, but with the permittivity becoming a tensor instead of a simple scalar, which reflects the breaking of the rotational invariance of the system's response: birefringent media. Once again, the conservation of linearity allows us to use the previous methods to determine the fields. It is of course possible to go further with responses that include nonlinear responses, but often with numerical resolution or approximate analytical methods.

Relativity

6.1 Michelson and Morley experiment

Because Maxwell's equations lead to a wave equation with the same propagation speed in every inertial frame, one early hypothesis was that there exists a *special* frame, the aether, in which light propagates at speed c, while in all other frames the usual Galilean velocity-addition law should apply. In an interference experiment, Michelson and Morley tried to measure the motion of the Earth relative to this luminiferous aether. In 1887, their experiment compared the speed of light in perpendicular directions in an attempt to detect the motion of matter, including their laboratory, through the aether.

For the sake of simplicity, let us consider the situation in which the device moves with velocity V in the aether frame (see Fig. 6.1).

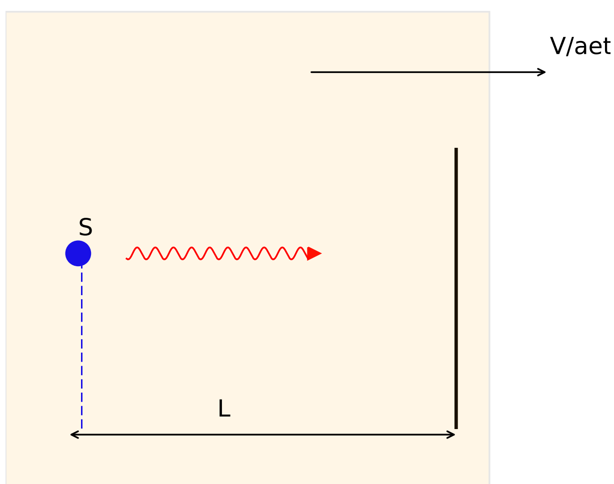

Figure 6.1: Sketch of the Michelson–Morley experiment in the case where the aether wind is parallel to one arm of the interferometer.

The time for a round trip of the light beam can be obtained as follows.

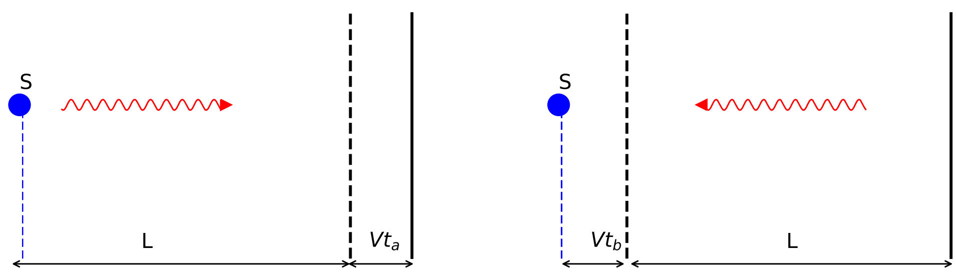

Figure 6.2: Sketch of the Michelson–Morley experiment for the two perpendicular arms.

For the propagation from the source to the mirror along the direction of motion, the travel time t_a is given by

$$ct_a = L + Vt_a. \tag{6.1}$$

For the return propagation, the travel time t_r is given by

$$ct_r = L - Vt_r. \tag{6.2}$$

The total duration T_x is therefore

$$\begin{aligned} T_x &= t_a + t_r \\ &= \frac{2L}{c}\frac{1}{1-\frac{V^2}{c^2}}. \end{aligned} \tag{6.3}$$

In the perpendicular direction, the Pythagorean theorem gives for the round-trip time T_y

$$\frac{cT_y}{2} = \sqrt{L^2 + V^2\left(\frac{T_y}{2}\right)^2}, \tag{6.4}$$

which yields

$$T_y = \frac{2L}{\sqrt{c^2 - V^2}}. \tag{6.5}$$

Assuming that $c \gg V$—which is indeed the relevant case, since the Earth's velocity relative to the hypothetical aether was expected to be comparable to its orbital velocity around the Sun, about 20 km/s—the time difference is

$$\Delta T = T_x - T_y = L\frac{V^2}{c^3}. \tag{6.6}$$

Taking $L = 10\,\mathrm{m}$ gives the estimate

$$\Delta T \simeq 10^{-15}\,\mathrm{s}. \tag{6.7}$$

The Michelson–Morley apparatus was sufficiently sensitive to detect a fringe shift in the interference pattern. The fringe shift is obtained by calculating the difference in optical path between the two orientations and then dividing by the wavelength. By rotating the device by $\pi/2$, the effect should be doubled:

$$n \simeq L\frac{V^2}{\lambda c^2}. \tag{6.8}$$

This gives approximately $n \simeq 0.44$. Despite numerous attempts, no such shift was observed. Michelson was deeply disappointed by this null result, but he nevertheless received the Nobel Prize for his work.

6.2 Lorentz transformations

Without yet discussing the full physical implications of the invariance of the speed of light, let us first study the wave equation of electromagnetism in search of a transformation that leaves it invariant.

For the sake of simplicity, consider a scalar quantity ϕ (for instance, one component of the electric or magnetic field) obeying

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0. \tag{6.9}$$

It is always possible to choose the propagation direction along one Cartesian axis, say Ox. Then the equation becomes

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0. \tag{6.10}$$

Using the identity $a^2 - b^2 = (a+b)(a-b)$, the wave equation can be factorized as

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\phi = 0. \tag{6.11}$$

Let us introduce the variables

$$X = x - ct, \qquad Y = x + ct. \tag{6.12}$$

Then

$$\frac{\partial}{\partial X} = \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial t}{\partial X}\frac{\partial}{\partial t}. \tag{6.13}$$

It is easy to check that

$$\begin{aligned} \frac{\partial x}{\partial X} &= 1, \\ \frac{\partial t}{\partial X} &= -\frac{1}{c}. \end{aligned} \tag{6.14}$$

Using a similar calculation for Y, the d'Alembert equation becomes

$$\frac{\partial}{\partial X}\frac{\partial}{\partial Y}\phi = 0. \tag{6.15}$$

It is now possible to integrate with respect to X:

$$\frac{\partial \phi}{\partial Y} = A(Y), \tag{6.16}$$

where $A(Y)$ is an arbitrary function. Integrating once more with respect to Y gives

$$\phi = \int^{Y} A(t)\,dt + B(X), \tag{6.17}$$

where $B(X)$ is another arbitrary function. Therefore, the solution is the sum of two functions depending on X and Y, respectively. Physically, this corresponds to the superposition of a progressive wave and a regressive wave.

What happens if a Galilean transformation is performed?

The progressive wave then moves in the new frame with phase $x-(c-v)t$, while the regressive wave has phase $x+(c+v)t$. Starting from Eq. (6.15), this leads to

$$\left(\frac{\partial}{\partial x}-\frac{1}{c-v}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}+\frac{1}{c+v}\frac{\partial}{\partial t}\right)\phi=0. \tag{6.18}$$

As expected, the structure of the d'Alembert equation is lost.

The problem is therefore to find a change of variables that keeps the speed of light invariant in all inertial frames and preserves the form of Maxwell's equations.

Let us define two one-dimensional frames, $\mathcal{R}$ and $\mathcal{R}'$, with coordinates (x,t) and (x',t'). For a progressive light wave, one has in the first frame

$$x=ct, \tag{6.19}$$

and in the second frame

$$x'=ct'. \tag{6.20}$$

This suggests the relation

$$x'-ct'=\lambda(x-ct), \tag{6.21}$$

where λ is a constant.

For a regressive wave, one similarly obtains

$$x'+ct'=\mu(x+ct), \tag{6.22}$$

where μ is another constant.

Both equations can be rewritten as

$$\begin{aligned} 2x'&=(\lambda+\mu)x-(\lambda-\mu)ct,\\ 2ct'&=(\lambda+\mu)ct-(\lambda-\mu)x. \end{aligned} \tag{6.23}$$

Let us define

$$\begin{aligned} a&=\frac{\lambda+\mu}{2},\\ b&=\frac{\lambda-\mu}{2}. \end{aligned} \tag{6.24}$$

Then we have

$$\begin{aligned} x'&=ax-bct,\\ ct'&=act-bx. \end{aligned} \tag{6.25}$$

At the origin of $\mathcal{R}'$, one has $x'=0$, hence

$$x=\frac{bc}{a}t. \tag{6.26}$$

Therefore, the velocity of O' with respect to $\mathcal{R}$ is

$$V_{O'/\mathcal{R}}=\frac{bc}{a}. \tag{6.27}$$

Conversely, at the origin of $\mathcal{R}$, one has

$$\begin{aligned} x'&=-bct,\\ ct'&=act, \end{aligned} \tag{6.28}$$

which gives

$$x' = -\frac{bc}{a}t'. \tag{6.29}$$

Therefore,

$$V_{O/\mathcal{R}'} = -\frac{bc}{a} = -V_{O'/\mathcal{R}}. \tag{6.30}$$

Let us denote $\beta = \frac{v}{c}$. The transformation can then be written as

$$\begin{aligned} ct' &= a(ct - \beta x), \\ x' &= a(x - \beta ct). \end{aligned} \tag{6.31}$$

or, in matrix form,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = a \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{6.32}$$

The inverse transformation is obtained by changing v into $-v$, which gives

$$\begin{pmatrix} ct \\ x \end{pmatrix} = a \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}. \tag{6.33}$$

Combining the two previous equations, one obtains

$$\begin{pmatrix} ct \\ x \end{pmatrix} = a^2 \begin{pmatrix} 1-\beta^2 & 0 \\ 0 & 1-\beta^2 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{6.34}$$

Because this matrix must be equal to the identity, one obtains the condition

$$a^2 = \frac{1}{1-\beta^2}. \tag{6.35}$$

The physical solution requires $a > 0$, and therefore

$$a = \frac{1}{\sqrt{1-\beta^2}} = \gamma. \tag{6.36}$$

This change of variables, called the Lorentz transformation, leaves Maxwell's equations invariant under a change of inertial frame.

In three-dimensional space, the Lorentz transformation reads

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{6.37}$$

Defining the matrix L as

$$L = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.38}$$

and an event as

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{6.39}$$

the transformation can be rewritten as

$$X' = LX. \tag{6.40}$$

Its inverse is

$$X = L^{-1}X', \tag{6.41}$$

where the inverse matrix is obtained by replacing β with $-\beta$.

Note that any linear combination of four-vectors is also a four-vector. In particular, the difference between two events, $X(t_2) - X(t_1)$, is also a four-vector.

To obtain the velocity transformation, we start from the differential form of the Lorentz transformation:

$$\begin{aligned} cdt' &= \gamma(cdt - \beta dx), \\ dx' &= \gamma(dx - \beta cdt), \\ dy' &= dy, \\ dz' &= dz. \end{aligned} \tag{6.42}$$

Dividing the last three lines of Eq. (6.42) by the first one, one obtains

$$\begin{aligned} \frac{dx'}{dt'} &= \frac{\left(\frac{dx}{dt} - V\right)}{1 - \frac{V\frac{dx}{dt}}{c^2}}, \\ \frac{dy'}{dt'} &= \frac{v_y}{\gamma\left(1 - \frac{V\frac{dx}{dt}}{c^2}\right)}, \\ \frac{dz'}{dt'} &= \frac{v_z}{\gamma\left(1 - \frac{V\frac{dx}{dt}}{c^2}\right)}. \end{aligned} \tag{6.43}$$

Finally,

$$\begin{aligned} v'_x &= \frac{v_x - V}{1 - \frac{Vv_x}{c^2}}, \\ v'_y &= \frac{v_y}{\gamma\left(1 - \frac{Vv_x}{c^2}\right)}, \\ v'_z &= \frac{v_z}{\gamma\left(1 - \frac{Vv_x}{c^2}\right)}. \end{aligned} \tag{6.44}$$

It is remarkable that (i) the velocity can never exceed c, and (ii) the Galilean transformation is recovered in the formal limit $c \to \infty$. Relativistic effects become important when velocities are comparable to, but still smaller than, the speed of light.

The inverse transformation is similarly obtained:

$$\begin{aligned} v_x &= \frac{v'_x + V}{1 + \frac{Vv'_x}{c^2}}, \\ v_y &= \frac{v'_y}{\gamma\left(1 + \frac{Vv'_x}{c^2}\right)}, \\ v_z &= \frac{v'_z}{\gamma\left(1 + \frac{Vv'_x}{c^2}\right)}. \end{aligned} \tag{6.45}$$

6.3 Length contraction

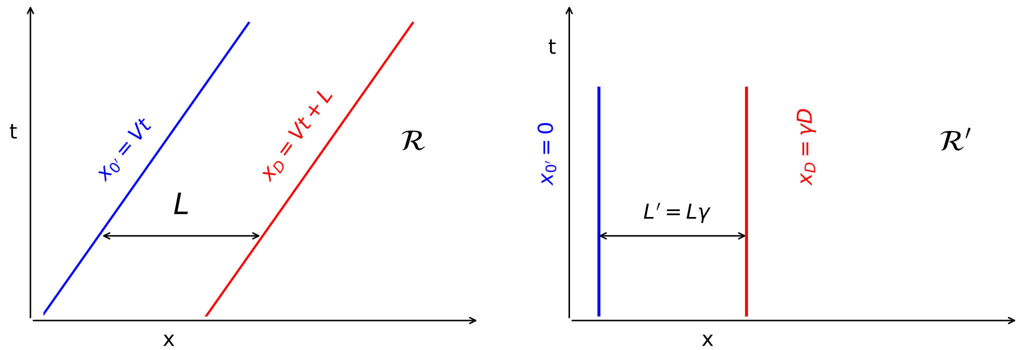

Figure 6.3: Minkowski diagrams: (left) ruler in the frame $\mathcal{R}$ and (right) in the frame $\mathcal{R}'$.

Consider a ruler at rest in $\mathcal{R}'$. Its left extremity is at the origin O', and D denotes its right extremity. In $\mathcal{R}$, these points follow the trajectories

$$x_{O'} = Vt, \qquad x_D = Vt + L, \tag{6.46}$$

where L is the length measured in $\mathcal{R}$. In $\mathcal{R}'$, one has

$$x'_{O'} = 0, \qquad x'_D = \gamma L. \tag{6.47}$$

Since the ruler is at rest in $\mathcal{R}'$, there is no time dependence in that frame.

Therefore the length L' measured in $\mathcal{R}'$ is

$$L' = \gamma L. \tag{6.48}$$

Because $\gamma > 1$, the length measured in $\mathcal{R}$ is smaller than the length in the frame $\mathcal{R}'$ where the ruler is at rest. This is the phenomenon of *length contraction*.

6.4 Time dilation

Conversely, consider a particle located at O' at all times. For a time interval t' measured in $\mathcal{R}'$, the inverse Lorentz transformation gives

$$t = \gamma t'. \tag{6.49}$$

Thus, time passes more slowly in the frame in which the particle is moving. Equivalently, the proper time measured in the rest frame is shorter than the corresponding time interval measured in another frame. This phenomenon is called *time dilation*.

6.5 Four-vector approach

Let us denote an event in the frame $\mathcal{R}$ by X and the corresponding event in the frame $\mathcal{R}'$ by X'. The two vectors are related by the Lorentz transformation

$$X' = LX. \tag{6.50}$$

Using the Minkowski metric $g_{ii} = (1, -1, -1, -1)$, the squared norm of a four-vector is Lorentz invariant:

$$\begin{aligned}(ct')^2 - x'^2 - y'^2 - z'^2 &= \gamma^2 \left[(ct - \beta x)^2 - (x - \beta ct)^2\right] - y^2 - z^2 \\ &= c^2t^2 - x^2 - y^2 - z^2.\end{aligned} \tag{6.51}$$

The next step is to define the four-velocity. A naive approach would be to take the derivative $\frac{dX}{dt}$. Unfortunately, this does not lead to a genuine four-vector, because time depends on the frame and therefore cannot play the same role in all inertial frames. To overcome this difficulty, one introduces the proper time τ.

The derivative of $X(\tau)$ is then

$$U = \frac{dX}{d\tau} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{dx}{dt}\frac{dt}{d\tau} \\ \frac{dy}{dt}\frac{dt}{d\tau} \\ \frac{dz}{dt}\frac{dt}{d\tau} \end{pmatrix}. \tag{6.52}$$

By choosing τ as the time measured by a clock in the comoving inertial frame, that is, the frame in which the particle is instantaneously at rest, one has

$$\frac{dt}{d\tau} = \gamma_u, \tag{6.53}$$

where

$$\gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{6.54}$$

Therefore, the four-velocity is

$$U = \frac{dX}{d\tau} = \gamma_u \begin{pmatrix} c \\ \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}. \tag{6.55}$$

The squared norm of U is equal to c^2 and is Lorentz invariant. Although this vector may appear somewhat artificial, it provides a systematic way to construct other physically useful four-vectors.

In the comoving inertial frame, the four-velocity reduces to

$$U = \frac{dX}{d\tau} = \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6.56}$$

Multiplying the four-velocity by the rest mass m_0 defines the four-momentum $P = m_0 U$. This vector can be written as

$$P = m_0\frac{dX}{d\tau} = \begin{pmatrix} \frac{E_{\text{tot}}}{c} \\ m_0\gamma_u\frac{dx}{dt} \\ m_0\gamma_u\frac{dy}{dt} \\ m_0\gamma_u\frac{dz}{dt} \end{pmatrix}, \tag{6.57}$$

where

$$E_{\text{tot}} = \gamma_u m_0 c^2 \tag{6.58}$$

is the total energy. The three-dimensional momentum is

$$\vec{p} = m_0\gamma_u\vec{u}. \tag{6.59}$$

The four-force $\mathcal{F}$ is defined as the derivative of the four-momentum with respect to proper time:

$$\begin{aligned}\mathcal{F} &= \frac{dP}{d\tau} \\ &= \frac{dP}{dt}\frac{dt}{d\tau} \\ &= \gamma_u \frac{dP}{dt}.\end{aligned} \tag{6.60}$$

This gives explicitly

$$\mathcal{F} = \gamma_u \begin{pmatrix} \frac{1}{c}\frac{dE_{\text{tot}}}{dt} \\ \frac{dp_x}{dt} \\ \frac{dp_y}{dt} \\ \frac{dp_z}{dt} \end{pmatrix}. \tag{6.61}$$

The three-force is defined by

$$\vec{f} = \frac{d\vec{p}}{dt}, \tag{6.62}$$

and the rate of change of the total energy is

$$\frac{dE_{\text{tot}}}{dt} = \vec{f} \cdot \vec{u}. \tag{6.63}$$

Therefore the four-force can be written as

$$\mathcal{F} = \gamma_u \begin{pmatrix} \frac{1}{c}\vec{f} \cdot \vec{u} \\ f_x \\ f_y \\ f_z \end{pmatrix}. \tag{6.64}$$

Another useful four-vector is the four-acceleration A:

$$\begin{aligned}A &= \frac{dU}{d\tau} \\ &= \frac{dU}{dt}\frac{dt}{d\tau} \\ &= \gamma_u \frac{dU}{dt}.\end{aligned} \tag{6.65}$$

The time derivative of the four-velocity requires the derivative of γ_u, which varies with time:

$$\begin{aligned}\frac{d\gamma_u}{dt} &= \frac{d}{dt}\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \\ &= \frac{1}{2}\left(1 - \frac{u^2}{c^2}\right)^{-3/2} \frac{2\vec{u} \cdot \vec{a}}{c^2} \\ &= \frac{\gamma_u^3}{c^2}\vec{u} \cdot \vec{a},\end{aligned} \tag{6.66}$$

where

$$\vec{a} = \frac{d\vec{u}}{dt} \tag{6.67}$$

is the ordinary three-acceleration.

Inserting this result into the expression of the four-acceleration, one obtains

$$A = \begin{pmatrix} \frac{\gamma_u^4}{c}\vec{u}\cdot\vec{a} \\ \frac{\gamma_u^4}{c^2}(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u^2\vec{a} \end{pmatrix}. \tag{6.68}$$

By definition, $\mathcal{F} = m_0 A$, and using Eq. (6.61), one obtains

$$\begin{aligned} \frac{dE_{\text{tot}}}{dt} &= \vec{f}\cdot\vec{u} = \gamma_u^3 m_0 \vec{u}\cdot\vec{a}, \\ \vec{f} &= m_0\left(\frac{\gamma_u^3}{c^2}(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u\vec{a}\right). \end{aligned} \tag{6.69}$$

Consequently, the force and the acceleration are not parallel in general.

However, when the velocity and acceleration are parallel, the force becomes

$$\begin{aligned} \vec{f} &= m_0\left(\frac{\gamma_u^3}{c^2}u^2\vec{a} + \gamma_u\vec{a}\right) \\ &= m_0\gamma_u^3\vec{a}. \end{aligned} \tag{6.70}$$

The quantity $m_0\gamma_u^3$ is sometimes called the longitudinal mass.

Conversely, when the velocity and acceleration are perpendicular, the force reduces to

$$\vec{f} = m_0\gamma_u\vec{a}, \tag{6.71}$$

and $m_0\gamma_u$ is called the transverse mass.

6.6 Properties of four-vectors

6.6.1 Inner product

For two four-vectors

$$X = (x_0, x_1, x_2, x_3)^T, \qquad Y = (y_0, y_1, y_2, y_3)^T, \tag{6.72}$$

the inner (or scalar) product is defined as

$$X\cdot Y = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3. \tag{6.73}$$

This definition differs from the usual Euclidean dot product by the minus signs in the spatial components, reflecting the Minkowski geometry of spacetime.

This inner product is symmetric,

$$X\cdot Y = Y\cdot X,$$

and, most importantly, it is invariant under Lorentz transformations. This means that its value is the same in all inertial reference frames, making it a fundamental quantity in special relativity.

For the four-vectors introduced in Sec. 6.5, several useful relations follow:

1. $U\cdot U = c^2$, This expresses the normalization of the four-velocity.
2. $P\cdot U = m_0 U\cdot U = m_0c^2 = E_0$,
   which shows that the projection of the four-momentum onto the four-velocity gives the rest energy.
3. $P\cdot P = m_0^2c^2$,
   a Lorentz-invariant quantity related to the invariant mass.

4. $A \cdot U = A' \cdot U' = 0$,
   which means that the four-acceleration is orthogonal to the four-velocity in any frame.

Here, A' and U' denote the acceleration and velocity in the comoving (instantaneous rest) frame. Note that this orthogonality applies to four-vectors and does *not* imply that the ordinary three-dimensional velocity and acceleration are perpendicular.

6.6.2 Conserved quantity

The conservation of four-momentum is expressed as

$$\sum_i P_i = \sum_j P_j, \tag{6.74}$$

where the index i runs over all particles before the interaction, and j runs over all particles after the interaction.

For example, in a process involving two incoming and two outgoing particles,

$$P_1 + P_2 = P_3 + P_4. \tag{6.75}$$

This single four-vector equation encodes both energy and momentum conservation.

From this relation, one can derive additional identities. For instance,

$$P_1 \cdot P_4 + P_2 \cdot P_4 = P_3 \cdot P_4 + P_4 \cdot P_4, \tag{6.76}$$

and also

$$(P_1 + P_2)^2 = (P_3 + P_4)^2. \tag{6.77}$$

Such expressions are useful because scalar products of four-vectors are Lorentz invariant, allowing calculations to be performed in any convenient reference frame.

6.7 Contravariant and covariant coordinates

The inner product can be written in a compact form using the metric tensor, represented by the matrix

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6.78}$$

This matrix defines the Minkowski spacetime geometry.

We define the contravariant components of the four-position as

$$X^\mu = (ct, x, y, z), \tag{6.79}$$

and the covariant components of the same vector as

$$X_\mu = \eta_{\mu\alpha} X^\alpha = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix}. \tag{6.80}$$

Here, we use Einstein summation convention: repeated indices, one upper and one lower, are implicitly summed over.

The inner product of a four-vector with itself can then be written as

$$X_\mu X^\mu = \eta_{\mu\nu} X^\mu X^\nu. \tag{6.81}$$

This expression makes the Lorentz invariance of the inner product explicit.

Finally, the inverse metric tensor is identical to the metric itself:

$$\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu{}_\nu, \tag{6.82}$$

where $\delta^\mu{}_\nu$ is the Kronecker delta. This property simplifies many calculations, as raising and lowering indices does not change the form of the metric in Minkowski spacetime.

6.8 Electromagnetism revisited

The four-vector formalism applies naturally to electromagnetism. Historically, electromagnetism is precisely the theory that led to the breakdown of Galilean invariance and the emergence of special relativity. In this framework, electric and magnetic phenomena are unified in a manifestly Lorentz-invariant way.

Multiplying the four-velocity by the proper charge density defines the four-current density:

$$\mathbf{J} = \rho_0 \mathbf{U} = \rho_0 \gamma_u (c, \mathbf{u}) = (c\rho, \mathbf{j}). \tag{6.83}$$

Here, ρ_0 is the charge density in the rest frame of the fluid, ρ is the charge density measured in the laboratory frame, and $\mathbf{j} = \rho\mathbf{u}$ is the usual three-dimensional current density. Thus, charge density and current density are unified into a single four-vector.

Charge conservation is expressed compactly as

$$\partial_\alpha J^\alpha = \frac{\partial J^\alpha}{\partial x^\alpha} = 0, \tag{6.84}$$

which is the continuity equation written in covariant form. This equation is Lorentz invariant.

Using the transformation properties of four-vectors, one directly obtains the Lorentz transformation of charge and current densities. For a boost along the x-direction:

$$\begin{aligned} \rho' &= \gamma\left(\rho - \frac{v j_x}{c^2}\right), \\ j'_x &= \gamma\left(j_x - v\rho\right), \end{aligned} \tag{6.85}$$

while the transverse components j_y and j_z remain unchanged. This shows explicitly how charge density and current mix under Lorentz transformations.

6.8.1 Electromagnetic tensor and equation of motion

The relativistic equation of motion for a charged particle is

$$\frac{dP^\mu}{d\tau} = \frac{q}{c} F^\mu{}_\nu U^\nu, \tag{6.86}$$

where U^ν is the four-velocity and $F^\mu{}_\nu$ is the electromagnetic field tensor. This equation is the covariant generalization of the Lorentz force.

The electromagnetic tensor can be written as a 4×4 matrix:

$$F^\mu{}_\nu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}. \tag{6.87}$$

This tensor combines the electric and magnetic fields into a single antisymmetric object.

The fully contravariant form is obtained by raising an index:

$$F^{\mu\nu} = \eta^{\nu\alpha} F^{\mu}{}_{\alpha} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{pmatrix}, \tag{6.88}$$

and the covariant form is obtained by lowering both indices:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -cB_3 & cB_2 \\ -E_2 & cB_3 & 0 & -cB_1 \\ -E_3 & -cB_2 & cB_1 & 0 \end{pmatrix}. \tag{6.89}$$

An equivalent and very useful definition of the electromagnetic tensor is in terms of the four-potential $A_\mu = (\phi, -\mathbf{A})$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.90}$$

This formulation automatically ensures several important properties.

6.8.2 Properties of the electromagnetic tensor

From its definition, the following properties can be deduced:

- The tensor is antisymmetric: $F_{\mu\nu} = -F_{\nu\mu}$. This reduces the number of independent components to six, corresponding to the three components of $\mathbf{E}$ and $\mathbf{B}$.

- The Lorentz-invariant scalar constructed from the tensor is

  $$F_{\mu\nu} F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right). \tag{6.91}$$

  This quantity has the same value in all inertial frames.

- The trace of the tensor vanishes:

  $$F^{\mu}{}_{\mu} = 0.$$

6.8.3 Maxwell equations in covariant form

Maxwell's equations take a particularly compact form when expressed in terms of the electromagnetic tensor. Gauss's law and the Maxwell–Ampère law combine into

$$\partial_\alpha F^{\beta\alpha} = -\mu_0 J^\beta, \tag{6.92}$$

where $J^\beta = (c\rho, \boldsymbol{j})$.

The remaining two equations (Faraday's law and the absence of magnetic monopoles) are encoded in the identity

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0, \tag{6.93}$$

which is a Bianchi identity. These equations follow directly from the definition of $F_{\mu\nu}$ in terms of the four-potential.

6.8.4 Lorentz transformation of the fields

Under a Lorentz transformation, the electromagnetic tensor transforms as

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}, \tag{6.94}$$

where $\Lambda^{\mu}{}_{\nu}$ is the Lorentz transformation matrix.

Since the tensor has only six independent components, this transformation determines how the electric and magnetic fields mix. For a boost along the x-direction:

$$\begin{aligned} E'_x &= E_x, \\ E'_y &= \gamma(E_y - vB_z), \\ E'_z &= \gamma(E_z + vB_y), \\ B'_x &= B_x, \\ B'_y &= \gamma\left(B_y - \frac{v}{c^2}E_z\right), \\ B'_z &= \gamma\left(B_z + \frac{v}{c^2}E_y\right). \end{aligned} \tag{6.95}$$

In a more compact vector form:

$$\begin{aligned} \mathbf{E}' &= \gamma\left(\mathbf{E} + \mathbf{V}\times\mathbf{B}\right) + (1-\gamma)\frac{\mathbf{V}\cdot\mathbf{E}}{V^2}\mathbf{V}, \\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \frac{\mathbf{V}}{c^2}\times\mathbf{E}\right) + (1-\gamma)\frac{\mathbf{V}\cdot\mathbf{B}}{V^2}\mathbf{V}. \end{aligned} \tag{6.96}$$

6.9 Relativistic Effects in Radiation: Bremsstrahlung, Cyclotron, and Synchrotron

We now examine three important examples in which relativistic effects become dominant compared to the non-relativistic (Galilean) description.

For a single charged particle (typically an electron), the Liénard–Wiechert potentials provide the exact electric and magnetic fields emitted in space and time, without approximation. These fields describe how electromagnetic signals propagate from an arbitrarily moving charge, taking into account the finite speed of light.

The magnetic field is related to the electric field through

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{c}\,[\hat{\mathbf{R}}\times\mathbf{E}(\mathbf{x})]_{\mathrm{ret}},$$

where

$$[\hat{\mathbf{R}}]_{\mathrm{ret}} = \frac{\mathbf{x}-\mathbf{r}(t_{\mathrm{ret}})}{|\mathbf{x}-\mathbf{r}(t_{\mathrm{ret}})|}$$

is the unit vector pointing from the retarded position of the particle to the observation point.

The Poynting vector, which represents the energy flux of the electromagnetic field, is then

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}[|\mathbf{E}|^2\hat{\mathbf{R}}]_{\mathrm{ret}}. \tag{6.97}$$

Using the Liénard–Wiechert expressions for the fields, one finds that the radiated power per unit solid angle is

$$\frac{dP}{d\Omega} = \kappa R^2\,\mathbf{S}\cdot\hat{\mathbf{R}} = \frac{q^2}{16\pi^2\epsilon_0 c^3\kappa^5}\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}} - \frac{\mathbf{v}}{c}\right)\times\mathbf{a}\right]\right|^2, \tag{6.98}$$

where

$$\kappa = 1 - \frac{\hat{\mathbf{R}} \cdot \mathbf{v}}{c},$$

$\mathbf{v}$ is the particle velocity, and $\mathbf{a}$ its acceleration.

The factor κ encodes relativistic beaming effects. When the particle velocity approaches the speed of light, κ becomes very small in the forward direction, leading to a strong enhancement of radiation in that direction.

By integrating over all solid angles, one obtains the total radiated power:

$$P = \frac{q^2}{6\pi\epsilon_0 c^3} \left(\gamma^6 a_\parallel^2 + \gamma^4 a_\perp^2\right), \tag{6.99}$$

which can also be written as

$$P = \frac{q^2\gamma^4}{6\pi\epsilon_0 c^3} \left(a^2 + \frac{\gamma^2}{c^2}(\mathbf{a} \cdot \mathbf{v})^2\right). \tag{6.100}$$

This is the relativistic generalization of the Larmor formula. At low velocities ($\gamma \approx 1$), it reduces to the classical result

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}.$$

At relativistic speeds, however, the additional term proportional to $(\mathbf{a} \cdot \mathbf{v})^2$ can dominate, especially when acceleration is parallel to the velocity.

We now consider three important physical situations.

6.9.1 Bremsstrahlung Radiation

In Bremsstrahlung ("braking radiation"), the charged particle follows a straight-line trajectory, so that velocity and acceleration are always parallel:

$$\mathbf{a} \parallel \mathbf{v}.$$

Starting from Eq. (6.98), and using

$$\hat{\mathbf{R}} \cdot \mathbf{v} = v \cos\theta,$$

we obtain

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c^3} \frac{\sin^2\theta}{\left(1 - \frac{v}{c}\cos\theta\right)^5}. \tag{6.101}$$

- At low velocities ($v \ll c$), the denominator is approximately constant, and the angular distribution is dominated by $\sin^2\theta$. Radiation is therefore maximal at $\theta \approx \pi/2$, i.e., perpendicular to the motion.
- At relativistic velocities ($v \to c$), the denominator becomes very small in the forward direction ($\theta \approx 0$). As a result, radiation is strongly beamed along the direction of motion.

Integrating over all angles yields the total power:

$$P = \frac{q^2\gamma^6 a^2}{6\pi\epsilon_0 c^3}. \tag{6.102}$$

Key result: The γ^6 scaling shows that radiation becomes extremely intense at relativistic speeds when acceleration is parallel to velocity.

6.9.2 Cyclotron and Synchrotron Radiation

We now consider circular motion, for which

$$\mathbf{v} \cdot \mathbf{a} = 0,$$

i.e., the acceleration is perpendicular to the velocity.

We define the observation direction as

$$\hat{\mathbf{R}} = \sin\theta\cos\phi\,\hat{\mathbf{e}}_x + \sin\theta\sin\phi\,\hat{\mathbf{e}}_y + \cos\theta\,\hat{\mathbf{e}}_z. \tag{6.103}$$

The angular distribution of radiation becomes

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{1}{(1-\frac{v}{c}\cos\theta)^3}\left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-(v/c)^2)\cos^2\theta}\right). \tag{6.104}$$

- At low velocities (cyclotron regime), the radiation pattern is nearly symmetric and governed by angular factors such as

  $$1 - \sin^2\theta\cos^2\phi,$$

  leading to a broad emission pattern.

- At high velocities (synchrotron regime), relativistic beaming dominates: radiation is concentrated in a narrow cone in the forward direction.

The total radiated power is

$$P = \frac{q^2\gamma^4 a^2}{6\pi\epsilon_0 c^3}. \tag{6.105}$$

Key result: Compared to Bremsstrahlung, the scaling is γ^4 instead of γ^6, reflecting the fact that acceleration is perpendicular to velocity.

In summary,

- Parallel acceleration (Bremsstrahlung): $P \propto \gamma^6$

- Perpendicular acceleration (Synchrotron): $P \propto \gamma^4$

This difference is fundamental in many applications, from particle accelerators to astrophysical sources such as pulsars and relativistic jets.

6.10 Collisions between relativistic particles

6.10.1 Pair annihilation

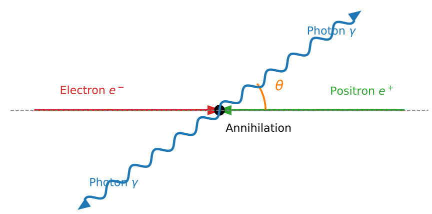

Figure 6.4: Pair annihilation

The conservation equations above allow us to study several examples of relativistic collisions. We first consider the head-on collision of an electron and a positron, which may annihilate into two (or more) photons (see Fig. 6.4). The case of a single photon is immediately ruled out by momentum conservation.

The incoming particles have the following four-momenta:

- $\mathbf{P}_1 = \gamma m_0(c, \mathbf{v})$ for the electron, where m_0 is the rest mass and $\mathbf{v}$ is the electron velocity in the laboratory frame;
- $\mathbf{P}_2 = \gamma m_0(c, -\mathbf{v})$ for the positron.

After annihilation, the two photons have the four-momenta

- $\mathbf{P}_3 = \frac{\hbar\omega}{c}(1, \mathbf{1})$ for the first photon, where ω is its angular frequency in the laboratory frame;
- $\mathbf{P}_4 = \frac{\hbar\omega}{c}(1, -\mathbf{1})$ for the second photon.

The temporal component of total four-momentum conservation gives

$$2\gamma m_0 c = 2\frac{\hbar\omega}{c}, \tag{6.106}$$

hence

$$\gamma m_0 c^2 = \hbar\omega. \tag{6.107}$$

This relation expresses the conversion of the total energy of the electron–positron pair into the energy of the photons.

6.10.2 Compton effect

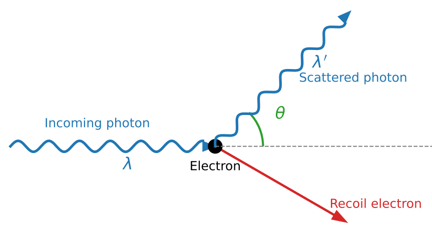

Figure 6.5: Compton diffusion

This corresponds to the inelastic diffusion of a photon with an electron at rest (see Fig.6.5). The conservation of the 4 momentum vector gives

$$\mathbf{P}_\gamma + \mathbf{P_e} = \mathbf{P'}_\gamma + \mathbf{P'_e} \tag{6.108}$$

which gives

$$\mathbf{P'}^2_e = \left(\mathbf{P}_\gamma + \mathbf{P}_e - \mathbf{P'}_\gamma\right)^2 \tag{6.109}$$

By defining the photon energies before and after the diffusion as $\mathcal{E}_\gamma$ and $\mathcal{E}'_\gamma$ and expanding the inner product, one obtains

$$\mathbf{P'}^2_e = \mathbf{P}^2_\gamma + \mathbf{P}^2_e + \mathbf{P'}^2_\gamma - 2\mathbf{P}_\gamma.\mathbf{P'}_\gamma - 2\mathbf{P}_e.(\mathbf{P}_\gamma - \mathbf{P'}_\gamma) \tag{6.110}$$

All terms are Lorentz invariants, which gives

$$\mathbf{P'}^2_e = \mathbf{P}^2_e = m_e^2 c^2 \tag{6.111}$$

The mass at rest of the photons is zero, which gives

$$\mathbf{P'}^2_\gamma = \mathbf{P}^2_\gamma = 0 \tag{6.112}$$

Eq. (6.110) becomes

$$\mathbf{P}_\gamma.\mathbf{P'}_\gamma = \mathbf{P}_e.(\mathbf{P'}_\gamma - \mathbf{P}_\gamma) \tag{6.113}$$

Before collision the electron is at rest, and the scalar product is given by

$$\mathbf{P}_e.(\mathbf{P'}_\gamma - \mathbf{P}_\gamma) = m_e(\mathcal{E}' - \mathcal{E}) \tag{6.114}$$

$$\mathcal{E}'_\gamma = \frac{m_e c^2 \mathcal{E}_\gamma}{m_e c^2 + \mathcal{E}_\gamma(1 - \cos(\theta))} \tag{6.115}$$

In the lab frame the scalar product is given by

$$\mathbf{P}_\gamma.\mathbf{P'}_\gamma = \frac{1}{c^2}\mathcal{E}'_\gamma \mathcal{E}_\gamma (1 - \cos(\theta)) \tag{6.116}$$

which gives

$$\mathcal{E}'_\gamma = \frac{m_e c^2 \mathcal{E}_\gamma}{m_e c^2 + \mathcal{E}_\gamma(1-\cos(\theta))} \tag{6.117}$$

Expressing the photon energy in term of the wavelength $\mathcal{E}_\gamma = \frac{hc}{\lambda}$, one obtains

$$\lambda = \lambda + \lambda_c(1-\cos(\theta)) = \lambda + \lambda_c \sin^2(\theta/2) \tag{6.118}$$

where λ_c is the Compton wavelength of the electron

$$\lambda_c = \frac{h}{m_e c} \tag{6.119}$$

In the original experiment done by Compton, the photons have a wavelength $\lambda = 0.022$Å, to be compared to the Compton wavelength of the electron $\lambda_c = 0.0243$Å which gives significant changes of the wavelength after diffusion.

6.10.3 Threshold creation of an antiparticle

A last example of the particle physics is the creation of an antiproton and a proton by starting from a collision of two protons. One proton is at rest.

$$p + p \to p + p + p + \overline{p} \tag{6.120}$$

The indices of the particles are the following: before collisions 1 and 2, after collision 3 and 4 for the same particles, 5 for the new proton and 6 for the anti proton. Taking the square norm of the conservation of the momentum give

$$(\mathbf{P}_1 + \mathbf{P}_2)^2 = (\mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5 + \mathbf{P}_6)^2 \tag{6.121}$$

The right-hand side of this equation gives

$$(\mathbf{P}_1 + \mathbf{P}_2)^2 = 2(m_p c) + 2\mathbf{P}_1.\mathbf{P}_2 = 2(m_p c)^2 + 2\mathcal{E}_2 m_p \tag{6.122}$$

For the left-hand side of the conservation of the momentum, the calculation is trivial in the center of masse and one obtains $(4m_p c)^2$. This gives a threshold energy

$$\mathcal{E}_2 = (16-2)/2mc^2 = 7mc^2 \tag{6.123}$$

Chapter 7

Transmission Lines and Waveguides

Applications of electromagnetism are numerous and are present in everyday life. In this chapter, we illustrate this broad field through two important examples: transmission lines and waveguides.

At high frequencies, when the wavelength of the signal propagating along a wire becomes comparable to the length of the wire, the wave nature of the signal can no longer be ignored. The phase then varies along the wire, so that different points of the wire are no longer at the same phase. In other words, the finite propagation speed of the signal must be taken into account when the voltage and current vary on a timescale comparable to, or shorter than, the time required for the signal to travel along the wire. In this regime, stray impedances along the wires can no longer be neglected, since they determine the signal velocity, the phase shift between voltage and current, and the reflections at the ends of the wires. Therefore, idealized AC circuits are not suitable for transporting high-frequency signals over long distances. Instead, one uses transmission lines or waveguides.

7.1 Transmission line

7.1.1 Introduction

A transmission line consists of two parallel conductors with a fixed cross section along the entire length of the line, separated by a dielectric. It is a two-port network connecting an input circuit, which generates the signal, to an output load. Although electromagnetic fields cannot penetrate far into good conductors, they can be guided over long distances by them.

The basic ingredients needed to describe a transmission line are the following:

- the time dependence of the current produces a time-dependent magnetic field, which induces an emf in the circuit and is characterized by a self-inductance;
- the potential difference between the two conductors leads to a charge distribution, characterized by a capacitance between them;
- the conductivity of the conductors is finite, which implies a finite resistance;
- the dielectric placed between the conductors has a very large but finite resistance. Only an ideal dielectric would have an infinite resistance.

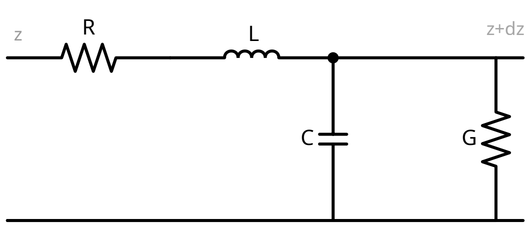

Figure 7.1: Sketch of an infinitesimal segment of a transmission line.

7.1.2 The telegrapher's equation

The theory of transmission lines was established by Oliver Heaviside, who is also known for introducing the step function. In a first simplified version, we consider a line where the resistance R can be neglected and the dielectric conductance is taken to be zero.

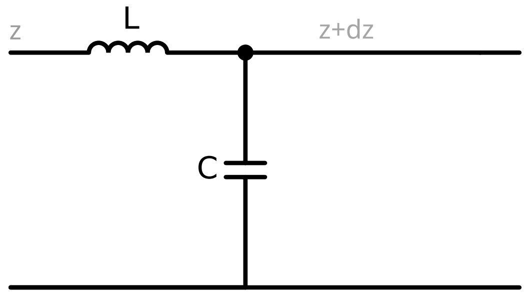

Figure 7.2: Sketch of an infinitesimal segment of an ideal transmission line.

If δz is much smaller than the wavelength λ, one may apply Kirchhoff's laws locally. Let $V(z,t)$ and $I(z,t)$ denote the voltage and current at position z and time t.

The first equation comes from the voltage drop across the inductance of length δz:

$$V(z+\delta z,t) = V(z,t) - L\delta z \frac{\partial I(z,t)}{\partial t} \tag{7.1}$$

where L is the inductance per unit length.

The second equation expresses current conservation in the presence of the capacitance:

$$I(z+\delta z,t) = I(z,t) - C\delta z \frac{\partial V(z+\delta z,t)}{\partial t} \tag{7.2}$$

where C is the capacitance per unit length.

In the limit $\delta z \to 0$, one obtains

$$\begin{aligned} \frac{\partial V}{\partial z} &= -L\frac{\partial I}{\partial t} \\ \frac{\partial I}{\partial z} &= -C\frac{\partial V}{\partial t} \end{aligned} \tag{7.3}$$

These are the telegrapher's equations for a lossless transmission line.

Taking the derivative of the first equation in Eq. (7.3) with respect to z, one gets

$$\frac{\partial^2 V}{\partial z^2} = -L\frac{\partial^2 I}{\partial z \partial t} \tag{7.4}$$

Using the second equation of Eq. (7.3), one obtains the wave equation for the voltage:

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} \tag{7.5}$$

Similarly, one finds for the current

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2} \tag{7.6}$$

Thus, both quantities satisfy the d'Alembert equation, with wave velocity

$$v = \frac{1}{\sqrt{LC}}. \tag{7.7}$$

7.1.3 Semi-infinite transmission line

We now consider a semi-infinite transmission line. On the left, a generator provides a sinusoidal voltage (see Fig. 7.3).

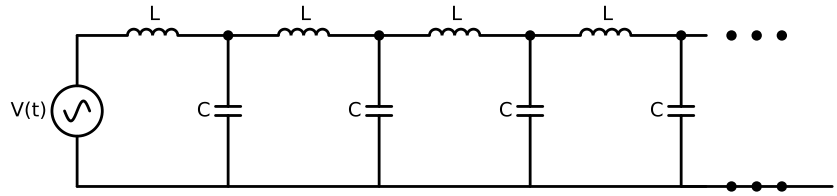

Figure 7.3: Sketch of a semi-infinite transmission line.

On the line, the solution is a progressive wave:

$$\begin{aligned} V(z,t) &= V_0 \cos(kz - \omega t) \\ I(z,t) &= I_0 \cos(kz - \omega t) \end{aligned} \tag{7.8}$$

with

$$k = \omega\sqrt{LC} = \frac{\omega}{v}. \tag{7.9}$$

Substituting these expressions into Eq. (7.3), one obtains

$$Z = \frac{V_0}{I_0} = \frac{L\omega}{k} = \sqrt{\frac{L}{C}} \tag{7.10}$$

where Z is the characteristic impedance of the line. For a lossless transmission line, this ratio is real and independent of frequency. If a generator has impedance $Z_g \equiv R$, it behaves like a resistor.

7.1.4 Two parallel wires

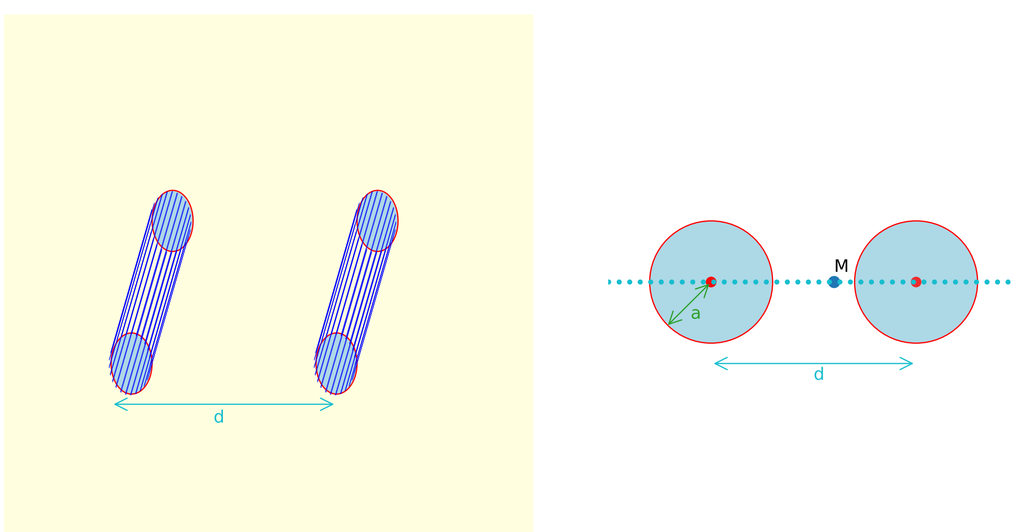

Figure 7.4: Top view (left) and side view of two cylindrical conductors embedded in a dielectric.

To determine the physical quantities L and C for a real device, we consider two parallel cylinders of radius a separated by a distance d. The cylinders are embedded in a dielectric of permittivity ϵ and permeability μ. For $d \gg a$, we assume that the surface charge density on each cylinder is uniform (see Fig. 7.4).

The left and right cylinders carry surface charges σ and $-\sigma$, respectively. Using Eq.(1.43), the The potential at a point of coordinate x is given as the sum of the potential of the two lines.

$$V(x) = \frac{a\sigma}{\epsilon}\left[\ln\left|x - \frac{d}{2}\right| - \ln\left|x + \frac{d}{2}\right|\right] + V_0 \tag{7.11}$$

The potential difference between the two cylinders is

$$\begin{aligned}\Delta V &= V(x = -d/2 + a) - V(x = d/2 - a) \\ &= \frac{2a\sigma}{\epsilon}\ln\left(\frac{d-a}{a}\right)\end{aligned} \tag{7.12}$$

The charge per unit length on each cylinder is $\pm\lambda = \pm 2\pi a\sigma$. Hence the capacitance per unit length is

$$\begin{aligned}C &= \frac{\lambda}{\Delta V} \\ &= \frac{\pi\epsilon}{\ln\left(\frac{d}{a}\right)}\end{aligned} \tag{7.13}$$

where we keep only the leading order in d/a.

To determine the inductance per unit length, we now consider currents I and $-I$ in the two cylinders (see Fig. 7.5).

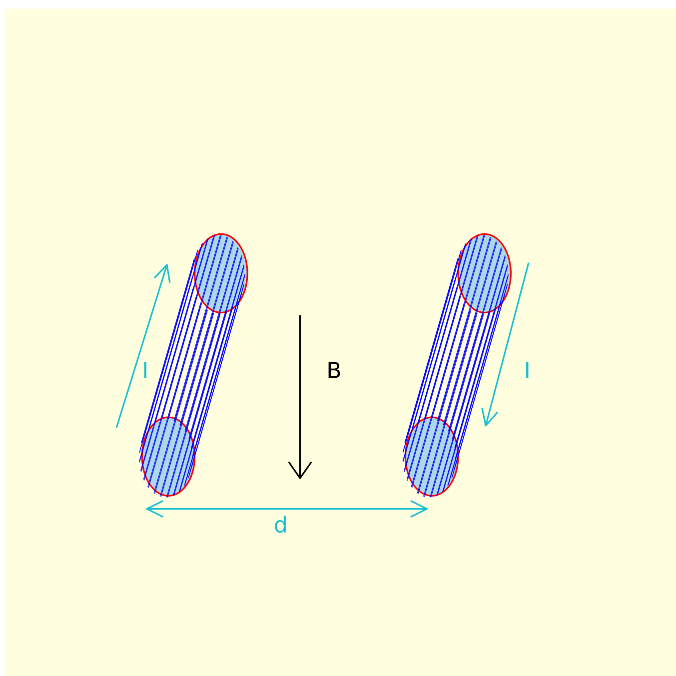

Figure 7.5: Magnetic field **B** generated by the currents in the two cylinders embedded in a dielectric.

The magnetic field is obtained by adding the contributions of both cylinders. Taking the origin at the midpoint between them, one finds

$$B = \frac{\mu I}{2\pi}\left(\frac{1}{\frac{d}{2} - x} + \frac{1}{\frac{d}{2} + x}\right) \tag{7.14}$$

The magnetic flux per unit length through the surface bounded by the two cylinders is

$$\phi = \int_{-d/2+a}^{d/2-a} B\,dx = \frac{\mu I}{\pi} \ln\left(\frac{d-a}{a}\right) \tag{7.15}$$

The inductance per unit length is defined by $L = \phi/I$, which gives

$$L = \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right) \tag{7.16}$$

again keeping only the leading term in a/d.

The wave velocity is therefore

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}} \tag{7.17}$$

which is precisely the wave velocity in the dielectric medium. It is noteworthy that the propagation velocity in a lossless transmission line is independent of the geometry of the device and depends only on the dielectric properties.

However, this simplified description neglects radiation losses. To minimize them, one often uses more sophisticated geometries, such as shielded twisted-pair cables.

7.1.5 Load impact

We now consider the effect of a terminal impedance $\tilde{Z}_o$ for a lossless transmission line. The general solution of the d'Alembert equation is the sum of a progressive and a regressive wave:

$$\begin{aligned} V(z,t) &= V_+ e^{i(kz-\omega t)} + V_- e^{i(-kz-\omega t)} \\ I(z,t) &= I_+ e^{i(kz-\omega t)} + I_- e^{i(-kz-\omega t)} \end{aligned} \tag{7.18}$$

in complex notation.

Let $V_+ \equiv V_0$ and define the reflection coefficient

$$r = \frac{V_-}{V_0}, \tag{7.19}$$

which is in general complex due to a phase shift between the progressive and regressive waves. Then

$$\begin{aligned} V(z,t) &= V_0 \left(e^{i(kz-\omega t)} + r e^{i(-kz-\omega t)}\right) \\ I(z,t) &= \frac{V_0}{Z} \left(e^{i(kz-\omega t)} - r e^{i(-kz-\omega t)}\right) \end{aligned} \tag{7.20}$$

where we used $I_+ = V_0/Z$ and $I_- = -V_0/Z$.

Choosing the end of the line at $z = 0$, the ratio of the complex voltage to the complex current defines the load impedance $\tilde{Z}_o = V/I$. This gives

$$\tilde{Z}_o = Z\frac{1+r}{1-r} \tag{7.21}$$

and therefore

$$r = \frac{\tilde{Z}_o - Z}{\tilde{Z}_o + Z}. \tag{7.22}$$

Let $\tilde{Z}_i$ denote the input impedance at the beginning of the line, $z = -L$:

$$\tilde{Z}_i = \left.\frac{V}{I}\right|_{z=-L}. \tag{7.23}$$

One obtains

$$\begin{aligned}\tilde{Z}_i &= Z\frac{e^{ikL}+re^{-ikL}}{e^{ikL}-re^{-ikL}} \\ &= Z\frac{\tilde{Z}_o\cos(kL)+iZ\sin(kL)}{Z\cos(kL)+i\tilde{Z}_o\sin(kL)}\end{aligned} \tag{7.24}$$

In general, $\tilde{Z}_i$ is complex. A special case occurs when $kL = n\pi$, where n is an integer, that is, when $L = n\lambda/2$. Then

$$\tilde{Z}_i = \tilde{Z}_o \tag{7.25}$$

so that the transmission line behaves like an ordinary conducting wire.

7.1.6 Impedance matching and quarter-wave transformer

When $\tilde{Z}_o = Z$, the reflection coefficient vanishes, meaning that no regressive wave is present. The transmission line is then perfectly matched: all the power delivered by the generator is absorbed by the load. This is called *impedance matching*:

$$\tilde{Z}_i = Z = \tilde{Z}_o. \tag{7.26}$$

Conversely, when $|r| = 1$, one has

$$|\tilde{Z}_o - Z| = |\tilde{Z}_o + Z|. \tag{7.27}$$

This requires $\tilde{Z}_o$ to be purely imaginary, corresponding to a purely reactive load.

In general, when the load is resistive, perfect matching is not automatically achieved. A workaround is to extend the transmission line by a segment of length $\lambda/4$ and characteristic impedance Z_1 (see Fig. 7.6).

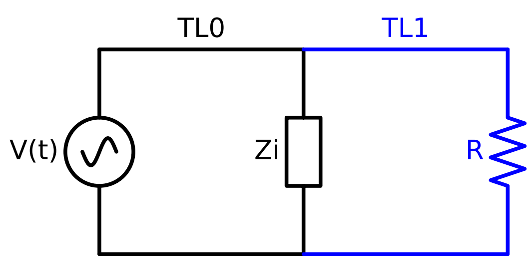

Figure 7.6: Two transmission lines of lengths L and $\lambda/4$.

The last segment can be viewed as a separate transmission line with input impedance Z_i and terminal impedance R. Using Eq. (7.24), one obtains

$$Z_i = \frac{Z_1^2}{R}. \tag{7.28}$$

For perfect matching, we require $Z_i = Z$, which gives

$$Z_1 = \sqrt{ZR}. \tag{7.29}$$

Note that this matching condition holds for a specific frequency. If the signal contains many frequencies, a more sophisticated device is needed.

7.2 Waveguides

A waveguide is a hollow metallic cylinder without end surfaces. If the ends are closed, the device is called a cavity. We consider a perfectly conducting cylinder filled with a nondissipative medium. In this idealized case, the energy is conserved during propagation along the axis of the cylinder. The medium is assumed to have constant permittivity ϵ and permeability μ.

We assume that the propagation direction is along the axis of the cylinder, chosen to be the z direction. Then, in complex notation,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathbf{E}(x,y)e^{i(kz-\omega t)} \\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}(x,y)e^{i(kz-\omega t)} \end{aligned} \tag{7.30}$$

We split the fields into longitudinal and transverse parts:

$$\begin{aligned} \mathbf{E}(x,y) &= E_z(x,y)\mathbf{e}_z + \mathbf{E}_t(x,y) \\ \mathbf{B}(x,y) &= B_z(x,y)\mathbf{e}_z + \mathbf{B}_t(x,y) \end{aligned} \tag{7.31}$$

where $\mathbf{E}_t$ and $\mathbf{B}_t$ are perpendicular to the propagation direction.

The nabla operator becomes

$$\boldsymbol{\nabla} = ik\mathbf{e}_z + \boldsymbol{\nabla}_t \tag{7.32}$$

where $\boldsymbol{\nabla}_t$ is the transverse gradient operator. Also,

$$\frac{\partial}{\partial t} = -i\omega, \qquad \boldsymbol{\nabla}^2 = -k^2 + \boldsymbol{\nabla}_t^2. \tag{7.33}$$

Using Maxwell's equations, one finds

$$-ik\mathbf{E}_t(x,y) + \boldsymbol{\nabla}_t E_z(x,y) = i\omega\mathbf{e}_z \times \mathbf{B}_t(x,y) \tag{7.34}$$

and similarly

$$-ik\mathbf{B}_t(x,y) + \boldsymbol{\nabla}_t B_z(x,y) = -i\omega\mu\epsilon\mathbf{e}_z \times \mathbf{E}_t(x,y) \tag{7.35}$$

Eliminating the transverse fields, one obtains

$$\begin{aligned} \mathbf{E}_t(x,y) &= \frac{k\boldsymbol{\nabla}_t E_z(x,y) - \omega\mathbf{e}_z \times \boldsymbol{\nabla}_t B_z(x,y)}{\mu\epsilon\omega^2 - k^2} \\ \mathbf{B}_t(x,y) &= \frac{k\boldsymbol{\nabla}_t B_z(x,y) + \mu\epsilon\omega\mathbf{e}_z \times \boldsymbol{\nabla}_t E_z(x,y)}{\mu\epsilon\omega^2 - k^2} \end{aligned} \tag{7.36}$$

Thus, once the longitudinal components E_z and B_z are known, the full fields can be reconstructed.

The longitudinal components satisfy

$$\begin{aligned} \left(\boldsymbol{\nabla}_t^2 - k^2 + \omega^2\mu\epsilon\right) E_z(x,y) &= 0 \\ \left(\boldsymbol{\nabla}_t^2 - k^2 + \omega^2\mu\epsilon\right) B_z(x,y) &= 0 \end{aligned} \tag{7.37}$$

Since the boundary is perfectly conducting, the boundary conditions are

$$\begin{aligned} \mathbf{E}^{\parallel} &= \mathbf{0} \\ \mathbf{B}^{\perp} &= \mathbf{0}. \end{aligned} \tag{7.38}$$

The different propagation modes are classified according to the presence or absence of longitudinal field components:

- Transverse Electric (TE): $E_z = 0$, $B_z \neq 0$
- Transverse Magnetic (TM): $B_z = 0$, $E_z \neq 0$
- Transverse Electromagnetic (TEM): $E_z = 0$, $B_z = 0$

7.2.1 Rectangular waveguide

A rectangular waveguide of dimensions $a \times b$ admits exact solutions. We first seek the solution for the magnetic field B_z for a TE mode.

Assuming separation of variables,

$$B_z(x, y) = F(x)G(y) \tag{7.39}$$

Substituting Eq. (7.39) into Eq. (7.37) and dividing by B_z gives

$$\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} + \left(\mu\epsilon\omega^2 - k^2\right) = 0. \tag{7.40}$$

Since the first term depends only on x and the second only on y, one must have

$$\begin{aligned} \frac{F''(x)}{F(x)} &= -k_x^2 \\ \frac{G''(y)}{G(y)} &= -k_y^2 \end{aligned} \tag{7.41}$$

with

$$k_x^2 + k_y^2 = k^2 - \mu\epsilon\omega^2. \tag{7.42}$$

Thus,

$$F(x) = A_1 e^{ik_x x} + A_2 e^{-ik_x x}. \tag{7.43}$$

For TE modes, the boundary conditions imply $\partial B_z/\partial x = 0$ at $x = 0$ and $x = a$, which leads to

$$A_1 = A_2, \tag{7.44}$$

and

$$\sin(k_x a) = 0. \tag{7.45}$$

Hence

$$k_x = \frac{n\pi}{a}, \qquad n \in \mathbb{N}, \tag{7.46}$$

and

$$F(x) = 2A_1 \cos(k_x x). \tag{7.47}$$

The same reasoning for $G(y)$ yields

$$G(y) = 2B_1 \cos(k_y y) \tag{7.48}$$

with

$$k_y = \frac{m\pi}{b}, \qquad m \in \mathbb{N}. \tag{7.49}$$

Combining the two solutions, the full field is

$$B_z(x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{TE}_{mn}(x, y) e^{i(kz-\omega t)} \tag{7.50}$$

with

$$\mathrm{TE}_{mn}(x, y) = C_{mn} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right). \tag{7.51}$$

Substituting into Eq. (7.42), one finds

$$k^2 = \mu\epsilon\omega^2 - \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right). \tag{7.52}$$

Wave propagation requires k to be real, which leads to a cutoff frequency:

$$\omega > \omega_{mn} \tag{7.53}$$

with

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}. \tag{7.54}$$

The dispersion relation is therefore

$$k = \sqrt{\mu\epsilon}\sqrt{\omega^2 - \omega_{mn}^2}. \tag{7.55}$$

The phase and group velocities are

$$\begin{aligned} v_\phi &= \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}} \\ v_g &= \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}. \end{aligned} \tag{7.56}$$

Hence

$$v_\phi v_g = \frac{1}{\mu\epsilon} = c^2 \tag{7.57}$$

where $c = 1/\sqrt{\mu\epsilon}$ is the propagation velocity in the medium. Therefore, the phase velocity is larger than c, whereas the group velocity is smaller than c.

If a broadband wave enters the waveguide, only a discrete set of modes with frequencies above their cutoff values can propagate.

7.2.2 TE_{10} mode

Without loss of generality, let us assume that $a > b$. Then the lowest cutoff frequency corresponds to $n = 1$, $m = 0$:

$$\omega_{10} = \frac{\pi}{a\sqrt{\mu\epsilon}}. \tag{7.58}$$

This is indeed the lowest mode, since the lowest TM mode is TM_{11}.

To understand how propagation occurs, we determine the fields for the TE_{10} mode. Keeping only the coefficient TE_{10} in Eq. (7.51), one obtains

$$B_z(x, y, z) = C\cos\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}. \tag{7.59}$$

Using Eq. (7.36) and the fact that $E_z = 0$ for TE modes, the electric field is

$$\mathbf{E}(x, y, z) = \frac{ia\omega C}{\pi}\sin\left(\frac{\pi x}{a}\right)e^{i(kz-\omega t)}\mathbf{e}_y \tag{7.60}$$

where we used

$$k^2 = \mu\epsilon\omega^2 - \frac{\pi^2}{a^2}. \tag{7.61}$$

Using the identity

$$i\sin\left(\frac{\pi x}{a}\right) = \frac{1}{2}\left(e^{i\pi x/a} - e^{-i\pi x/a}\right), \tag{7.62}$$

one finds

$$\mathbf{E}(x, y, z) = \frac{a\omega C}{2\pi}\left(e^{i(kz+\pi x/a-\omega t)} - e^{i(kz-\pi x/a-\omega t)}\right)\mathbf{e}_y. \tag{7.63}$$

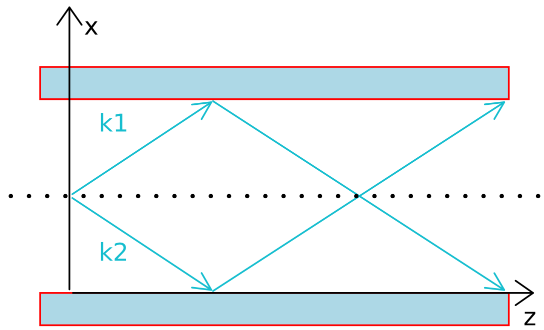

Figure 7.7: Illustration of wave propagation in a rectangular waveguide for the TE_{10} mode.

This shows that the solution is the superposition of two plane waves with wave vectors

$$\mathbf{k}_{\pm} = \pm\frac{\pi}{a}\mathbf{e}_x + k\mathbf{e}_z. \tag{7.64}$$

Figure 7.7 illustrates the propagation of the wave inside the guide. The two plane waves are repeatedly reflected by the walls, so that the energy remains confined within the waveguide.

Chapter A

Math recap

In this chapter, we review several mathematical definitions and properties that are useful throughout electromagnetism and relativity. In particular, Fourier transforms provide a very efficient tool for solving linear differential equations, studying wave propagation, and handling Green functions.

A.1 Fourier transforms

Let $f(\mathbf{r})$ be a function defined on $\mathcal{R}^d$, where d is the dimension of space. The Fourier transform of f is defined by

$$\hat{f}(\mathbf{k}) = \int_{\mathcal{R}^d} d^d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}), \tag{A.1}$$

where $\mathbf{r}$ and $\mathbf{k}$ are d-dimensional vectors:

$$\mathbf{r} = (r_1, r_2, \ldots, r_d), \qquad \mathbf{k} = (k_1, k_2, \ldots, k_d). \tag{A.2}$$

The inverse Fourier transform is

$$f(\mathbf{r}) = \frac{1}{(2\pi)^d} \int_{\mathcal{R}^d} d^d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{f}(\mathbf{k}). \tag{A.3}$$

These formulas require that the integrals be well defined, at least in the sense of distributions. In many applications of physics, this is sufficient, since objects such as Dirac distributions or plane waves are naturally treated within that framework.

Note that this convention is not unique. One may also use the alternative sign convention

$$\begin{aligned} \hat{f}(\mathbf{k}) &= \int_{\mathcal{R}^d} d^d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}), \\ f(\mathbf{r}) &= \frac{1}{(2\pi)^d} \int_{\mathcal{R}^d} d^d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} \hat{f}(\mathbf{k}), \end{aligned} \tag{A.4}$$

or the symmetric convention

$$\begin{aligned} \hat{f}(\mathbf{k}) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathcal{R}^d} d^d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}), \\ f(\mathbf{r}) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathcal{R}^d} d^d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{f}(\mathbf{k}). \end{aligned} \tag{A.5}$$

All these conventions are equivalent, provided the direct and inverse transforms are defined consistently.

A.2 Basic properties

We now collect the main properties of the Fourier transform in $\mathcal{R}^d$.

A.2.1 Linearity

For two functions f and g, and two constants a and b, one has

$$\mathcal{F}[af + bg] = a\hat{f} + b\hat{g}. \tag{A.6}$$

A.2.2 Translation

If

$$g(\mathbf{r}) = f(\mathbf{r} - \mathbf{r}_0), \tag{A.7}$$

then

$$\hat{g}(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{r}_0}\hat{f}(\mathbf{k}). \tag{A.8}$$

A translation in real space therefore becomes a phase factor in Fourier space.

A.2.3 Modulation

If

$$g(\mathbf{r}) = e^{-i\mathbf{k}_0\cdot\mathbf{r}}f(\mathbf{r}), \tag{A.9}$$

then

$$\hat{g}(\mathbf{k}) = \hat{f}(\mathbf{k} - \mathbf{k}_0). \tag{A.10}$$

Thus multiplication by a plane wave in real space corresponds to a translation in Fourier space.

A.2.4 Scaling

Let $a \neq 0$ be a real number and define

$$g(\mathbf{r}) = f(a\mathbf{r}). \tag{A.11}$$

Then

$$\hat{g}(\mathbf{k}) = \frac{1}{|a|^d}\hat{f}\left(\frac{\mathbf{k}}{a}\right). \tag{A.12}$$

This property expresses the reciprocal character of real space and Fourier space: a narrow function in real space corresponds to a broad function in $\mathbf{k}$-space, and conversely.

A.2.5 Complex conjugation

If $g(\mathbf{r}) = f(\mathbf{r})^*$, then

$$\hat{g}(\mathbf{k}) = \hat{f}(-\mathbf{k})^*. \tag{A.13}$$

In particular, if f is real-valued, then

$$\hat{f}(-\mathbf{k}) = \hat{f}(\mathbf{k})^*. \tag{A.14}$$

This is the Hermitian symmetry of the Fourier transform of a real function.

A.2.6 Even and odd functions

If f is even, that is

$$f(-\mathbf{r}) = f(\mathbf{r}), \tag{A.15}$$

then $\hat{f}$ is also even. If f is odd, then $\hat{f}$ is odd up to a factor of i, depending on the chosen convention.

A.2.7 Derivatives

Fourier transforms convert derivatives into multiplications by components of $\mathbf{k}$. For the first derivative,

$$\mathcal{F}\left[\frac{\partial f}{\partial r_j}\right] = -ik_j \hat{f}(\mathbf{k}). \tag{A.16}$$

More generally,

$$\mathcal{F}\left[\frac{\partial^n f}{\partial r_j^n}\right] = (-ik_j)^n \hat{f}(\mathbf{k}). \tag{A.17}$$

For the gradient,

$$\mathcal{F}[\boldsymbol{\nabla} f] = -i\mathbf{k}\, \hat{f}(\mathbf{k}), \tag{A.18}$$

for the divergence,

$$\mathcal{F}[\boldsymbol{\nabla} \cdot \mathbf{A}] = -i\mathbf{k} \cdot \hat{\mathbf{A}}(\mathbf{k}), \tag{A.19}$$

for the curl in three dimensions,

$$\mathcal{F}[\boldsymbol{\nabla} \times \mathbf{A}] = -i\mathbf{k} \times \hat{\mathbf{A}}(\mathbf{k}), \tag{A.20}$$

and for the Laplacian,

$$\mathcal{F}[\boldsymbol{\nabla}^2 f] = -k^2 \hat{f}(\mathbf{k}), \tag{A.21}$$

where $k = |\mathbf{k}|$.

This is one of the main reasons why Fourier methods are so useful: differential equations become algebraic equations in Fourier space.

A.2.8 Multiplication by coordinates

Multiplication by coordinates in real space becomes differentiation in Fourier space:

$$\mathcal{F}[r_j f(\mathbf{r})] = -i\frac{\partial \hat{f}}{\partial k_j}. \tag{A.22}$$

More generally,

$$\mathcal{F}[\mathbf{r} f(\mathbf{r})] = -i\boldsymbol{\nabla}_{\mathbf{k}} \hat{f}(\mathbf{k}). \tag{A.23}$$

A.2.9 Convolution theorem

For two functions f and g, define their convolution by

$$(f * g)(\mathbf{r}) = \int_{\mathcal{R}^d} d^d\mathbf{r}'\, f(\mathbf{r} - \mathbf{r}')g(\mathbf{r}'). \tag{A.24}$$

Then

$$\mathcal{F}[f * g] = \hat{f}\, \hat{g}. \tag{A.25}$$

Conversely,

$$\mathcal{F}[fg] = \frac{1}{(2\pi)^d} \hat{f} * \hat{g}. \tag{A.26}$$

This property is fundamental in electromagnetism, since many solutions are expressed as convolutions with Green functions.

A.2.10 Parseval–Plancherel relation

The Fourier transform preserves the scalar product up to a normalization factor:

$$\int_{\mathcal{R}^d} d^d\mathbf{r}\, f(\mathbf{r})g(\mathbf{r})^* = \frac{1}{(2\pi)^d}\int_{\mathcal{R}^d} d^d\mathbf{k}\, \hat{f}(\mathbf{k})\hat{g}(\mathbf{k})^*. \tag{A.27}$$

In particular,

$$\int_{\mathcal{R}^d} d^d\mathbf{r}\, |f(\mathbf{r})|^2 = \frac{1}{(2\pi)^d}\int_{\mathcal{R}^d} d^d\mathbf{k}\, |\hat{f}(\mathbf{k})|^2. \tag{A.28}$$

A.3 Dirac distribution

The Dirac distribution plays a central role in Fourier analysis. In d dimensions, one has

$$\delta^{(d)}(\mathbf{r}) = \frac{1}{(2\pi)^d}\int_{\mathcal{R}^d} d^d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{A.29}$$

Equivalently,

$$\int_{\mathcal{R}^d} d^d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} = (2\pi)^d\delta^{(d)}(\mathbf{k}). \tag{A.30}$$

If $\mathbf{r}_0$ is a fixed point,

$$\mathcal{F}\left[\delta^{(d)}(\mathbf{r}-\mathbf{r}_0)\right] = e^{i\mathbf{k}\cdot\mathbf{r}_0}. \tag{A.31}$$

In particular,

$$\mathcal{F}\left[\delta^{(d)}(\mathbf{r})\right] = 1. \tag{A.32}$$

A.4 Fourier transforms in one dimension

In one dimension, the Fourier transform reads

$$\hat{f}(k) = \int_{-\infty}^{+\infty} dx\, e^{ikx} f(x), \tag{A.33}$$

with inverse

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-ikx}\hat{f}(k). \tag{A.34}$$

Several useful examples are listed below.

A.4.1 Elementary examples in one dimension

$$\mathcal{F}[\delta(x-x_0)] = e^{ikx_0}. \tag{A.35}$$

$$\mathcal{F}[1] = 2\pi\delta(k). \tag{A.36}$$

For $a > 0$,

$$\mathcal{F}[e^{-a|x|}] = \frac{2a}{a^2+k^2}. \tag{A.37}$$

For the Gaussian,

$$\mathcal{F}\left[e^{-x^2/(2\sigma^2)}\right] = \sqrt{2\pi}\,\sigma\, e^{-\sigma^2k^2/2}. \tag{A.38}$$

The Gaussian is remarkable because its Fourier transform is again a Gaussian. This property explains why Gaussian wave packets play such an important role in many areas of physics.

A.4.2 Sine and cosine transforms

If $f(x)$ is even, then

$$\hat{f}(k) = 2\int_0^{+\infty} dx\ \cos(kx) f(x). \tag{A.39}$$

If $f(x)$ is odd, then

$$\hat{f}(k) = 2i\int_0^{+\infty} dx\ \sin(kx) f(x). \tag{A.40}$$

These formulas are often useful when the symmetry of the problem is known in advance.

A.5 Fourier transforms in two dimensions

In two dimensions,

$$\hat{f}(\mathbf{k}) = \int_{\mathcal{R}^2} d^2\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}), \tag{A.41}$$

with inverse

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{\mathcal{R}^2} d^2\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{f}(\mathbf{k}). \tag{A.42}$$

When the function is radially symmetric, that is

$$f(\mathbf{r}) = f(r), \qquad r = |\mathbf{r}|, \tag{A.43}$$

its Fourier transform depends only on $k = |\mathbf{k}|$ and can be written in polar coordinates as

$$\hat{f}(k) = 2\pi\int_0^{+\infty} r\, dr\, J_0(kr) f(r), \tag{A.44}$$

where J_0 is the Bessel function of the first kind of order zero.

Similarly, the inverse transform becomes

$$f(r) = \frac{1}{2\pi}\int_0^{+\infty} k\, dk\, J_0(kr)\hat{f}(k). \tag{A.45}$$

This is the two-dimensional Hankel transform of order zero. It is very useful in problems with cylindrical symmetry, for instance in waveguides, diffraction, and planar electrostatics.

A.5.1 Useful examples in two dimensions

One important result is

$$\mathcal{F}\left[-\frac{1}{2\pi}\ln r\right] = \frac{1}{k^2}, \tag{A.46}$$

in the sense of distributions.

This is the two-dimensional analogue of the Coulomb kernel. It appears naturally as the Green function of the Laplacian in two dimensions, since

$$\boldsymbol{\nabla}^2\left(-\frac{1}{2\pi}\ln r\right) = \delta^{(2)}(\mathbf{r}). \tag{A.47}$$

More generally, because

$$\mathcal{F}[\boldsymbol{\nabla}^2 f] = -k^2\hat{f}, \tag{A.48}$$

solving Poisson's equation in Fourier space in two dimensions reduces to a simple division by k^2.

A.6 Fourier transforms in three dimensions

In three dimensions,

$$\hat{f}(\mathbf{k}) = \int_{\mathcal{R}^3} d^3\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}), \tag{A.49}$$

with inverse

$$f(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{\mathcal{R}^3} d^3\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{f}(\mathbf{k}). \tag{A.50}$$

If the function is spherically symmetric,

$$f(\mathbf{r}) = f(r), \tag{A.51}$$

then $\hat{f}(\mathbf{k}) = \hat{f}(k)$ also depends only on $k = |\mathbf{k}|$, and the angular integration gives

$$\hat{f}(k) = 4\pi \int_0^{+\infty} r^2 dr\, \frac{\sin(kr)}{kr} f(r). \tag{A.52}$$

The inverse relation is

$$f(r) = \frac{1}{2\pi^2} \int_0^{+\infty} k^2 dk\, \frac{\sin(kr)}{kr} \hat{f}(k). \tag{A.53}$$

These formulas are especially useful in electrostatics and scattering theory, where spherical symmetry is common.

A.6.1 Useful examples in three dimensions

The most important example for electromagnetism is

$$\mathcal{F}\left[\frac{1}{r}\right] = \frac{4\pi}{k^2}, \tag{A.54}$$

again in the sense of distributions.

This result is directly related to the fact that

$$\boldsymbol{\nabla}^2 \left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\mathbf{r}). \tag{A.55}$$

Indeed, taking the Fourier transform of both sides gives

$$-k^2\mathcal{F}\left[\frac{1}{r}\right] = -4\pi, \tag{A.56}$$

hence the result above.

Using the derivative property, one immediately obtains

$$\mathcal{F}\left[\frac{\mathbf{r}}{r^3}\right] = -i\mathbf{k}\,\mathcal{F}\left[\frac{1}{r}\right] = -\frac{4\pi i\,\mathbf{k}}{k^2}, \tag{A.57}$$

since

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}. \tag{A.58}$$

Another useful example is the Yukawa kernel:

$$\mathcal{F}\left[\frac{e^{-\alpha r}}{r}\right] = \frac{4\pi}{k^2+\alpha^2}, \qquad \alpha > 0. \tag{A.59}$$

This expression appears in screened Coulomb interactions and in several wave and field equations.

For a three-dimensional Gaussian,

$$f(\mathbf{r}) = e^{-r^2/(2\sigma^2)}, \tag{A.60}$$

one finds

$$\hat{f}(\mathbf{k}) = (2\pi)^{3/2}\sigma^3 e^{-\sigma^2 k^2/2}. \tag{A.61}$$

A.7 Applications to differential equations

One major application of Fourier transforms is the solution of linear differential equations.

A.7.1 Poisson equation

Consider Poisson's equation in d dimensions:

$$\boldsymbol{\nabla}^2\phi(\mathbf{r}) = -\rho(\mathbf{r}). \tag{A.62}$$

Taking the Fourier transform gives

$$-k^2\hat{\phi}(\mathbf{k}) = -\hat{\rho}(\mathbf{k}), \tag{A.63}$$

so that

$$\hat{\phi}(\mathbf{k}) = \frac{\hat{\rho}(\mathbf{k})}{k^2}. \tag{A.64}$$

Thus the differential equation becomes an algebraic equation.

In three dimensions, the inverse transform gives the well-known Coulomb solution

$$\phi(\mathbf{r}) = \frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{A.65}$$

A.7.2 Wave equation

For the wave equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\boldsymbol{\nabla}^2\phi = 0, \tag{A.66}$$

the spatial Fourier transform gives

$$\frac{\partial^2\hat{\phi}}{\partial t^2} + c^2k^2\hat{\phi} = 0. \tag{A.67}$$

For each wave vector $\mathbf{k}$, one obtains an ordinary differential equation in time:

$$\hat{\phi}(\mathbf{k},t) = A(\mathbf{k})e^{-ickt} + B(\mathbf{k})e^{ickt}. \tag{A.68}$$

This decomposition shows that a general solution can be viewed as a superposition of plane waves.

A.8 Summary table

Some Fourier transforms frequently used in electromagnetism are collected below.

$f(\mathbf{r})$	$\hat{f}(\mathbf{k})$
In d dimensions	
$\delta^{(d)}(\mathbf{r})$	1
$\delta^{(d)}(\mathbf{r}-\mathbf{r}_0)$	$e^{i\mathbf{k}\cdot\mathbf{r}_0}$
1	$(2\pi)^d\delta^{(d)}(\mathbf{k})$
In one dimension	
$e^{-a\|x\|}$ $(a>0)$	$\dfrac{2a}{a^2+k^2}$
$e^{-x^2/(2\sigma^2)}$	$\sqrt{2\pi}\,\sigma\, e^{-\sigma^2k^2/2}$
In two dimensions	
$-\dfrac{1}{2\pi}\ln r$	$\dfrac{1}{k^2}$
In three dimensions	
$\dfrac{1}{r}$	$\dfrac{4\pi}{k^2}$
$\dfrac{\mathbf{r}}{r^3}$	$-\dfrac{4\pi i\,\mathbf{k}}{k^2}$
$\dfrac{e^{-\alpha r}}{r}$	$\dfrac{4\pi}{k^2+\alpha^2}$
$e^{-r^2/(2\sigma^2)}$	$(2\pi)^{3/2}\sigma^3e^{-\sigma^2k^2/2}$

A.9 Final remarks

Fourier transforms are central in theoretical physics because they connect local differential operators in real space with algebraic operations in Fourier space. In electromagnetism, they are used to solve Poisson and wave equations, to construct Green functions, and to analyze the spectral content of fields. In relativity and field theory, they also play a central role in the study of wave propagation and dispersion relations.

For this reason, mastering their main properties in one, two, and three dimensions is extremely useful before moving on to more advanced applications.