Exercises : Electromagnetism

1 Electrostatics

I Electric fields

1. A circular disk of radius a has uniform surface charge densityσ. Compute the potential at a point on the axis of symmetry at distance z from the center. Compute the electric field at this point. Find the discontinuity in the normal electric field at the center of the disk. Show that, far along the axis of symmetry, the electric field looks approximately like that of a charged point particle.

2. A hemispherical surface of radius R is uniformly charged with surface charge density σ. Evaluate the electric field and potential at the center of curvature (hint: start from the electric field of a uniformly charged ring along its axis)

   .

3. One considers a regular one-dimensional lattice where particles of charge q occur each lattice site. The lattice space is a. Determine the electric field at a distance r perpendicular to the lattice above one site. For $r >> a$, one can use

$$a \sum_{i=-\infty}^{\infty} f(ai) = \int_{\infty}^{\infty} dx f(x) \tag{12}$$

   and

$$\int_{-\infty}^{\infty} dx \frac{1}{(x^2 + r^2)^3/2} = \frac{2}{r^2} \tag{13}$$

II Charged Sphere with Internal Spherical Cavity

A sphere of radius a has uniform charge density over all its volume, excluding a spherical cavity of radius $b < a$, where $\rho = 0$. The center of the cavity, O_b is located at a distance d, with $|d| < (a - b)$, from the center of the sphere, O_a. The mass distribution of the sphere is proportional to its charge distribution.

4. Determine the electric field inside the cavity.

   Now we apply an external, uniform electric field $\mathbf{E}_0$.

5. Calculate the force on the sphere,

6. Calculate the torque with respect to the center of the sphere.

III Energy of a Charged Sphere

A total charge Q is distributed uniformly over the volume of a sphere of radius R.

Evaluate the electrostatic energy of this charge configuration in the following three alternative ways:

7. Evaluate the work needed to assemble the charged sphere by moving successive infinitesimals shells of charge from infinity to their final location.

8. Evaluate the volume integral of $u_E = \epsilon_0 \mathbf{E}^2/2$ where $\mathbf{E}$ is the electric field

9. Evaluate the volume integral of $\rho\phi/2$ where ρ is the charge density and ϕ is the electrostatic potential. Discuss the differences with the calculation previously

IV Collision of two Charged Spheres

Two rigid spheres have the same radius R and the same mass M, and opposite charges $\pm Q$. Both charges are uniformly and rigidly distributed over the volumes of the two spheres. The two spheres are initially at rest, at a distance $x_0 >> R$ between their centers, such that their interaction energy is negligible compared to the sum of their 'internal' (construction) energies.

10. Evaluate the initial energy of the system.

    The two spheres, having opposite charges, attract each other, and start moving at $t = 0$.

11. Evaluate the velocity of the spheres when they touch each other (i.e. when the distance between their centers is $x = 2R$).

12. Assume that, after touching, the two spheres penetrate each other without friction. Evaluate the velocity of the spheres when the two centers overlap ($x = 0$).

V Electrostatic Potential with Image Charges

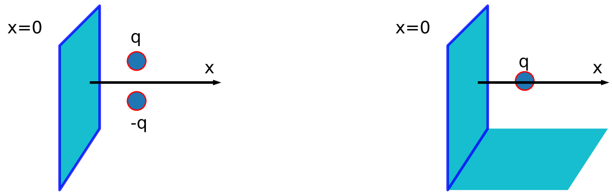

Figure 1: (left)A plane conductor plane with a charge q (middle) with two opposite charges (right) two plane conductors with a charge q.

Consider the configurations of point charges in the presence of conducting planes. (see Fig.1) For each case, find the solution for the electrostatic potential over the whole space and evaluate the electrostatic potential of the system. Use the method of image charges.

13. A charge q is located at a distance a from an infinite conducting plane.

14. Two opposite charges $+q$ and $-q$ are at a distance d from each other, both at the same distance a from an infinite conducting plane.

15. A charge q is at distances a and b, respectively, from two infinite conducting half planes forming a right dihedral angle.

VI Dipoles and sphere

An electric dipole $\mathbf{p}$ is located at a distance d from the center of a conducting sphere of radius $a < d$. In all three cases consider the two possibilities of i) a grounded sphere, and ii) an electrically unchar- ged isolated sphere. Evaluate the electrostatic potential ϕ over the whole space assuming that

16. $\mathbf{p}$ is perpendicular to the direction from $\mathbf{p}$ to the center of the sphere,

17. $\mathbf{p}$ is directed towards the center of the sphere.

18. $\mathbf{p}$ forms an arbitrary angle θ with respect to the straight line passing through the center of the sphere and the dipole location.

VII Electrically Connected Spheres

Two conducting spheres of radii a and $b < a$, respectively, are connected by a thin metal wire of negligible capacitance. The centers of the two spheres are at a distance $\gg a \gg b$ from each other. A total net charge Q is located on the system. Evaluate to zeroth order approximation, neglecting the induction effects on the surfaces of the two spheres,

19. How the charge Q is partitioned between the two spheres,

20. the value V of the electrostatic potential of the system (assuming zero potential at infinity) and the capacitance $C = Q/V$,

21. the electric field at the surface of each sphere, comparing the intensities and discussing the limit $b \to 0$.

22. Now take the electrostatic induction effects into account and improve the preceding results to the first order in a/d and b/d.

VIII Magnetic fields

1. A constant magnetic field points along the z-axis: $\mathbf{B} = B\hat{\mathbf{z}}$. Verify that each of the following vector potentials satisfies $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$:

   $\mathbf{A} = xB\hat{\mathbf{y}}$.

2. $\mathbf{A} = \frac{1}{2}\left(xB\hat{\mathbf{y}} - yB\hat{\mathbf{x}}\right)$
3. In cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}rB\hat{\phi}$ with $r^2 = x^2 + y^2$
4. In spherical polar coordinates, $\mathbf{A} = \frac{1}{2}r\sin(\theta)B\hat{\phi}$ with $r^2 = x^2 + y^2 + z^2$
5. Show that all potential vectors satisfy the Coulomb gauge

IX Hollow cylinder

A steady current I flows in the z-direction uniformly in the the region between the cylinders $x^2 + y^2 = a^2$ and $(x + d)^2 + y^2 = b^2$, where $0 < d < (b - a)$ and $b > a$.

6. Show that the associated magnetic field B throughout the region $x^2 + y^2 < a^2$ is given by

$$\mathbf{B} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)}\hat{\mathbf{y}} \tag{74}$$

X Pinch Effect in a Cylindrical Wire

A uniform current density J flows in an infinite cylindrical conductor of radius a. The current carriers are electrons (charge $-e$) of number volume density n_e and drift velocity v, parallel to the axis of the cylinder. Ions can be considered as fixed in space, with uniform number density ni and charge Ze. The system is globally neutral.

7. Evaluate the magnetic field generated by the current, and the resulting magnetic force on the electrons.

   The magnetic force modifies the volume distribution of the electrons and this, in turn, gives origin to a static electric field. At equilibrium the magnetic force on the electrons is compensated by the electrostatic force.

8. Evaluate the electric field that compensates the magnetic force on the electrons, and the corresponding charge distribution.

9. Evaluate the effect in 'standard' conditions for a good Ohmic conductor. The typical values (e.g. copper-like metal) are for the electron density $n_e \sim 10^{29}, \mathrm{m}^{-3}$ and the $c \approx 3 \times 10^8, \mathrm{m/s}$ for light velocity , the drift velocity (for ordinary currents): $v \sim 10^{-4}, \mathrm{m/s}$ and speed of light: $c = 3 \times 10^8 m/s$

XI Magnetic Levitation

In a given region of space we have a static magnetic field, which, in a cylindrical reference frame (r, ϕ, z), is symmetric around the z axis, i.e., is independent of ϕ, and can be written $\mathbf{B} = \mathbf{B}(r, z)$. The field component along z is $B_z(z) = B_0 0z/L$, where B_0 and L are constant parameters.

10. Find the radial component B_r close to the z axis.

    A particle of magnetic polarizability α (such that it acquires an induced magnetic dipole moment $\mathbf{m} = \alpha\mathbf{B}$ in a magnetic field $\mathbf{B}$), is located close to the z axis.

11. Find the potential energy of the particle in the magnetic field.

12. Discuss the existence of equilibrium positions for the particle, and find the frequency of oscillations for small displacements from equilibrium either along z or r (let M be the mass of the particle).

XII Conducting Cylinder in a Magnetic Field

A conducting cylinder of radius a and height $h \gg a$ rotates around its axis at constant angular velocity ω in a uniform magnetic field $\mathbf{B}_0$, parallel to the cylinder axis.

13. Evaluate the magnetic force acting on the conduction electrons, assuming $\omega = 2\pi \times 10^2 s^{-1}$ and $B_0 = 5 \times 10^{-5} T$ (the Earth's magnetic field), and the ratio of the magnetic force to the centrifugal force. $m_e = 9.1 \times 10^{-31}, \mathrm{kg}$

14. Assume that the cylinder is rotating in stationary conditions. Evaluate the electric field inside the cylinder, and the volume and surface charge densities;

15. The magnetic field B_1 generated by the rotation currents inside the cylinder, and the order of magnitude of B_1/B_0 (assume $a \simeq 0.1m$).

XIII A Coil Moving in an Inhomogeneous Magnetic Field

A magnetic field has rotational symmetry around a straight line, that we choose as the longitudinal axis, z, of a cylindrical reference frame (r, ϕ, z). The z- component of the field , $B_z(0, z)$, is known and equals $B_z(0, z) = B_0 z/L$, where L is a constant.

A circular coil has radius a, resistance R, and axis coinciding with the z axis of our reference frame. The coil performs a translational motion at constant velocity $\mathbf{v} = v\mathbf{e}_z$, and its radius a is assumed to be small enough that the magnetic field is always approximately uniform over the surface limited by the coil.

16. Find the current I flowing in the coil.

17. Find the power P dissipated by the coil due to Joule heating, and the corresponding frictional force $\mathbf{f}$ on the coil.

18. Calculate $\mathbf{f}$ as the resultant magnetic force on the loop carrying the current I.

XIV Charge and dipole moment

1. For a charge density $\rho(\mathbf{r}, t)$ and current $\mathbf{J}(\mathbf{r}, t)$, both localised within a finite volume V, show that

$$\int_V d\mathbf{r}^3 \mathbf{J}(\mathbf{r}, t) = \frac{d\mathbf{p}}{dt} \tag{134}$$

where $\mathbf{p}$ is the dipole moment of the volume.

XV Eddy Currents in a Solenoid

A long solenoid consists of a helical coil of n turns per unit length wound around a soft ferromagnetic cylinder of radius R and length $l \gg R$. The ferromagnetic material has a relative magnetic permittivity μ_r , and an electrical conductivity σ. An AC current $I = I_0 \cos \omega t$ flows in the coil.

2. Find the electric field induced in the solenoid.

3. Explain why the cylinder warms up and evaluate the dissipated power.

4. Evaluate how the induced currents affect the magnetic field in the solenoid. (Boundary effects and the displacement current are assumed to be negligible).

XVI A Magnetized Cylinder as DC Generator

A long hard-iron cylinder has height h, radius $a \ll h$, and permanent, uniform magnetization M throughout its volume. The magnetization is parallel to the cylinder axis, which we choose as the z axis of a cylindrical coordinate system (r, ϕ, z).

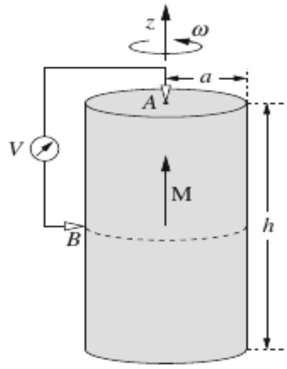

Figure 2: Magnetized cylinder

5. Show that the magnetic field inside the cylinder, far from the two bases, is $\mathbf{B}_0 \simeq \mu_0 M$ in SI units. Show that the magnitude of the z component of the field at the two bases is $B_z \simeq B_0/2$.

6. Two brush contacts (the white arrows in Fig. 2 connect the center of the upper base of the cylinder, A, and a point on the equator of the cylinder, B, to a voltmeter. The cylinder is kept in rotation around the z axis with constant angular velocity electromotive force measured by the voltmeter. ω. Evaluate the electromotive force measured by the voltmeter.

XVII Mutual Induction between a Solenoid and a Loop

A conducting loop of radius a and resistance R is located with its center at the center of solenoid of radius $b > a$ and n turns per unit length. The loop rotates at constant angular velocity *omega* around a diameter perpendicular to the solenoid axis, while a steady current I flows in the solenoid.

7. Evaluate the flux of the magnetic field through the rotating coil as a function of time.

8. Evaluate the torque exerted by the external forces on the loop in order to keep it rotating at constant angular velocity.

   Now assume that the solenoid is disconnected from the current source, and that the rotating loop is replaced by a magnetic dipole m, still rotating at constant angular velocity ω, as in Fig

9. Evaluate the electromotive force induced in the solenoid.

XVIII Wave propagation between two conducting planes

Perfectly conducting planes are positioned at $y = 0$ and $y = a$.

10. Show that a monochromatic wave may propagate between the plates in the direction z if the field components are

$$\begin{aligned} E_x &= A\omega \sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \\ B_y &= Ak\sin\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \\ B_z &= \frac{n\pi A}{a}\cos\left(\frac{n\pi y}{a}\right)\sin(kz-\omega t) \end{aligned} \tag{168}$$

with A a constant and $n \in \mathcal{Z}$. Show that the wavelength λ is given by

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_\infty^2} + \frac{n^2}{4a^2} \tag{169}$$

where λ_∞ is the wavelength of waves of the same frequency in the absence of conducting plans

XIX Plane with a surface current

In the plane $z = 0$, it exists surface current given by

$$\mathbf{j}_s = j_s^0 e^{i(\omega t - \alpha x)}\mathbf{e}_y \tag{181}$$

with $\alpha < \frac{\omega}{c}$. These currents generate a electromagnetic field in space. Except the plane ($z = 0$), there is no charge.

11. Find the surface charge density in the plane $z = 0$

12. Explain why a solution can be found by using the ansatz :

$$\mathbf{E} = f(z)e^{i(\omega t - \alpha x)}\mathbf{e}_y \tag{187}$$

13. Find the equation satisfied by $f(z)$. One denotes $\beta = \sqrt{\frac{\omega^2}{c^2} - \alpha^2}$.

14. What is the electric field in each half-space $z > 0$ and $z < 0$

15. Find the relationship between the module of the wave vector and the pulsation ω

16. Calculate the Poynting vector and the electromagnetic energy density. Check the energy conservation

XX A Bent Dipole Antenna

1. A dipole antenna consists of two identical conductive elements, usually two metal rods, each of length a and resistance R. The driving current is applied between the two halves of the antenna , so that the current flows as shown in Fig.3.

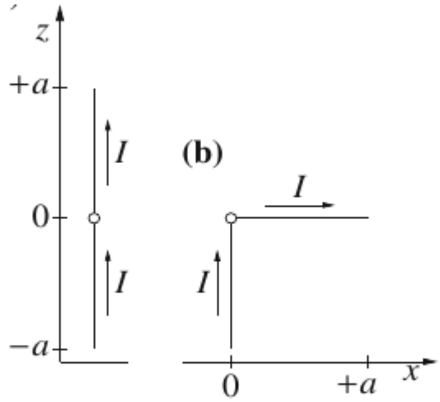

Figure 3: Dipole antenna: (left) : straight antenna (right) bent antenna

For a “short” antenna ($a \ll \lambda = \frac{2\pi c}{\omega}$ the current can be approximately specified as

$$I(z,t) = 2I_0 Re\left[\left(1 - \frac{|z|}{a}\right) e^{-i\omega t}\right] \tag{230}$$

The dependence of the current oscillation amplitude on z is shown in Fig. 10.3. Calculate the dissipated power.

2. Calculate the linear charge density λ on the rods of the antenna, and the antenna electric dipole moment $\mathbf{p}$.

3. Calculate the cycle-averaged radiated power P_{rad} and the ratio P_{rad}/P_{diss}.

XXI Cyclotron Radiation

An electron moves in the xy plane in the presence of a constant and uniform magnetic field $\mathbf{B} = B_0\mathbf{e}_z$. The initial velocity is $v0 \ll c$, so that the motion is non relativistic and the electron moves on a circular orbit of radius $r_L = v_0/\omega_L$ and frequency $\omega_L = eB_0/m_e c$ (Larmor frequency).

4. Describe the radiation emitted by the electron in the dipole approximation specifying its frequency, its polarization for radiation observed along the z axis, and along a direction lying in the xy plane, and the total irradiated power P_{rad} .

5. Discuss the validity of the dipole approximation.

6. The electron gradually loses energy because of the emitted radiation. Use the equation $P_{rad} = -dU/dt$, where U is the total energy of the electron, to show that the electron actually spirals toward the "center" of its orbit.

7. Evaluate the time constant τ of the energy loss, assuming $\tau \gg \omega_L^{-1}$, and provide a numerical estimate.

8. The spiral motion cannot occur if we consider the Lorentz force $\mathbf{f}_L = -(e/c)\mathbf{v} \times \mathbf{B}$ as the only force acting on the electron. Show that a spiral motion can be obtained by adding a friction force $\mathbf{f}_f$ proportional to the electron velocity $\mathbf{f}_f$.

XXII Radiation Emitted by Orbiting Charges

Two identical point charges q rotate with constant angular velocity ω on the circular orbit $x^2 + y^2 = R^2$ on the $z = 0$ plane of a Cartesian reference frame.

9. Write the most general trajectory for the charges both in polar coordinates $r_i = r_i(t)$, $\phi = \phi(t)$ and in Cartesian coordinates $x_i = x_i(t), y_i = y_i(t)$ (where $i = 1, 2$) and calculate the electric dipole moment of the system.

10. Characterize the dipole radiation emitted by the two-charge system, discussing how the power depends on the initial conditions, and finding the polarization of the radiation emitted along the three directions of the cartesian frame.

11. Answer above questions and in the case where the charges are orbiting with opposite angular velocity.

12. Now consider a system of three identical charges on the circular orbit with the same angular velocity. Find the initial conditions for which the radiation power is either zero or has its maximum.

13. Determine whether the magnetic dipole moment gives some contribution to the radiation, for each of the above specified cases.

XXIII A Receiving Circular Antenna

A receiving circular antenna is a circular coil of radius a and resistance R. The amplitude of the received signal is proportional to the current induced in the antenna by an incoming EM wave (Fig.4).

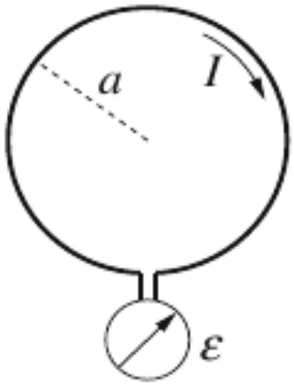

Figure 4: Cicular antenna

14. Assume that the incoming signal is a monochromatic, linearly polarized wave of wavelength $\lambda \gg a$, and electric field amplitude $\mathbf{E}_0$. Find how the antenna must be oriented with respect to the wave vector $\mathbf{k}$ and to the polarization in order to detect the maximum signal, and evaluate the signal amplitude.

15. In a receiving linear antenna the signal is approximately proportional to $E_{\|}l$, where $E_{\|}$ is the component of the electric field of the wave parallel to the antenna, and l is the length of the antenna. Old portable TV sets were provided with both a linear and a circular antenna, typical dimensions were $l \simeq 50cm$ and $a \simeq l/2$. Which antenna is best suited to detect EM waves with λ in the $10^2 - 10^3 cm$ range?

16. Calculate the power P_{rad} scattered by the antenna, and the ratio P_{rad}/P_{dis} , where P_{dis} is the power dissipated in the antenna by Joule heating.

XXIV Polarization Effects on Thomson Scattering

An electron is in the field of an elliptically polarized plane wave of frequency ω propagating along the z axis of a Cartesian reference frame. The electric field of the wave can be written as

$$\mathbf{E} = E_0 \cos(\theta) \cos(kz - \omega t)\mathbf{e}_x + \sin(\theta) \sin(kz - \omega t)\mathbf{e}_y \tag{304}$$

where θ is a constant real number with $0 \leq \theta \leq \pi/2$ such that we have linear polarization along the x axis for $\theta = 0$, linear polarization along the y axis for $\theta = \pi/2$, and circular polarization for $\theta = \pi/4$. First, neglecting the effects of the magnetic force $-e\mathbf{v} \times \mathbf{B}/c$.

17. Characterize the radiation scattered by the electron by determining the frequency and the polarization observed along each axis (x, y, z), and find a direction along which the radiation is circularly polarized;

18. Calculate the total (cycle-averaged) scattered power and comment the result.

    Now consider the effect of the magnetic force on the scattering process.

19. Evaluate the $-e\mathbf{v} \times \mathbf{B}/c$ term by calculating the $\mathbf{B}$ field and using the result of the first question for v. Discuss the direction and frequency of the magnetic force and its dependence on θ as well.

20. Discuss how the scattering of the incident wave is modified by the magnetic force by specifying which new frequencies are observed, in which direction and with which polarization, and the modification of the scattered power.

XXV Charge in Front of a Dielectric Half-Space

1. A plane divides the whole space into two halves, one of which is empty and the other filled by a dielectric medium of relative permittivity ϵ_r. A point charge q is located in vacuum at a distance d from the medium.

   Find the electric potential and electric field in the whole space, using the method of image charges.

2. Evaluate the surface polarization charge density on the interface plane, and the total polarization charge of the plane.

3. Find the field generated by the polarization charge in the whole space.

XXVI Refraction of the Electric Field at a Dielectric Boundary

A dielectric slab of thickness h, length $L \gg h$, and dielectric permittivity ϵ_r , is placed in an external uniform electric field $\mathbf{E}_0$. The angle between $\mathbf{E}_0$ and the normal to the slab surface is θ.

4. Find the electric field $\mathbf{E}'$ inside the slab and the angle θ' between $\mathbf{E}'$ and the normal to the slab surface.

5. Find the polarization charge densities in the dielectric medium.

6. Evaluate the torque exerted by the external field on the slab, if any. Neglect all boundary effects.

XXVII A Dielectric Slab in Contact with a Charged Conductor

A dielectric slab of relative permeability ϵ_r, thickness h and surface $S \gg h$ is in contact with a plane conducting surface, carrying a uniform surface charge density σ, as in Fig. 5 Boundary effects are negligible.

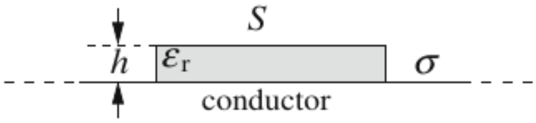

Figure 5: (Dielectric slab in contact with a charged copnductor

7. Evaluate the electric field in the whole space.

8. Evaluate the polarization surface-charge densities on the dielectric surfaces.

9. c) How do the answers to points a) and b) change if the slab is moved at a distance $s < h$ from the conducting plane? How does the electrostatic energy of the system depend on s? Is there an interaction force between slab and conductor?

XXVIII Energy Densities in a Free Electron Gas

A plane, monochromatic, transverse electromagnetic wave propagates in a medium containing n_e free electrons per unit volume. The electrons move with negligible friction.

10. Calculate the dispersion relation of the wave, the phase (v_ϕ) and group (v_g) velocities, and the relation between the amplitudes of the electric (E_0) and magnetic (B_0) fields.

11. Calculate the EM energy density u_{EM} (averaged over an oscillation period) as a function of E_0 .

12. Calculate the kinetic energy density u_K (averaged over an oscillation period), defined as $u_K = n_e m_e \langle v^2 \rangle/2$, where v is the electron oscillation velocity, and the total energy density $u = u_{EM} + u_K$.

13. Assume that the medium fills the half-space $x > 0$, while we have vacuum in the half-space $x < 0$. An EM wave, propagating along the x axis, enters the medium. Assume that both v_g and v_ϕ are real quantities. Use the above results to verify the conservation of the energy flux, expressed by the relation

$$c(u_i - u_r) = v_g u_t, \tag{366}$$

where u_i , u_r and u_t are the total energy densities for the incident, reflected and transmitted waves, respectively.

XXIX Longitudinal Waves

Consider a longitudinal monochromatic plane wave, propagating in a medium along the x axis of a Cartesian reference frame. "Longitudinal" means that the electric field $\mathbf{E}$ of the wave is parallel to the wavevector $\mathbf{k}$. Assume that the electric and magnetic fields of the wave are

$$\begin{aligned} \mathbf{E}(x,t) &= E_0 e^{i(kx-\omega t)}\hat{\mathbf{x}}, \\ \mathbf{B} &= 0, \end{aligned} \tag{375}$$

respectively, and that the optical properties of the medium are described by a given frequency-dependent dielectric permittivity $\epsilon_r(\omega)$.

14. Show that the possible frequencies for the wave correspond to zeros of the dielectric permittivity, $\epsilon_r(\omega) = 0$.

15. Find the charge and current densities in the medium associated to the presence of the wave fields.

16. Assuming that the optical properties of the medium are determined by ne classical electrons per unit volume, bound to atoms by an elastic force $-m_e\omega_0^2\mathbf{r}$, determine $\epsilon_r(\omega)$ and the dispersion relation for the longitudinal wave.

6 Relativity

XXX Hafele & Keating experiment

1. In 1971 Hafele and Keating performed an experiment illustrating the "twins paradox". They synchronized several atomic clocks and flew them on airplanes. Then gathering the clocks, they compared the elapsed time measured by each of them in flight, i.e., the proper duration. Numerical values: typical plane velocity with respect to ground $V = 900km/h$, flight duration $T_f = 2\pi R/V = 45h$, Earth radius $R = 6380km$. Using the time dilation relationship, derive the time lag between a clock on the ground and a clock on a plane. How much lifetime does a pilot spare if he flies 1000 h per year during 30 years ? Same question for a student who performs 200 Paris-Orsay return trips per year.

2. In this experiment, it is observed that clocks traveling to the east are lagged back and those traveling to the west are ahead of time with respect to a clock on the ground. Given that γ is independent of the velocity direction, how can you explain this result with special relativity?

XXXI Inelastic collisions of two protons

One studies a two protons collision ($m_pc^22 = 938.25MeV$) resulting in a deuteron d+ ($m_dc^2 = 1875.56MeV$) and a π^+ meson ($m_\pi c^2 = 139.6MeV$):

$$p^+ + p^+ \to d^+ + \pi^+$$

3. Determine the threshold energy of this reaction in the center of mass reference frame $\mathcal{R}^*$.

4. Compute, at threshold, the energy of the incident proton in the lab frame ${}_calR$ in which the target proton is at rest. Give the analytical formula and the numerical value.

XXXII Accelerated motion

A particle of rest mass m and charge q moves in a constant uniform electric field $\mathbf{E} = (E, 0, 0)$. It starts from the origin with initial momentum $\mathbf{p} = (0, p0, 0)$.

5. Show that the particle traces out a path in the (x, y) plane given by

$$x = \frac{\mathcal{E}_0}{qE} \cosh\left(\frac{qEy}{p_0c} - 1\right) \tag{407}$$

where $\mathcal{E}_0 = \sqrt{p_0c^2 + m2c^4}$

XXXIII The Four-Potential of a Plane Wave

Consider a monochromatic plane wave, propagating in vacuum along the x axis of the Cartesian laboratory frame S , linearly polarized along $\mathbf{e}_y$, and of frequency ω.

6. Show that the electric field $\mathbf{E} = \mathbf{E}(x, t)$ and the magnetic field $\mathbf{B} = \mathbf{B}(x, t)$ of the wave can be obtained from a suitable four-potential $A_\mu = (\phi, \mathbf{A}) = (0, 0, A_y, 0)$.

   Now consider the same wave observed in a frame S' , moving with velocity $\mathbf{v} = v\mathbf{e}_y$ with respect to S.

7. Evaluate the frequency ω' and the wave vector $\mathbf{k}'$ of the wave in S'.
8. Calculate the electric field $\mathbf{E}' = \mathbf{E}'(\mathbf{r}', t')$ and the magnetic field $\mathbf{B}' = \mathbf{B}'(\mathbf{r}', t')$ as functions of $\mathbf{E}$ in the S frame.
9. Verify that the wave is linearly polarized in S' and show that $\mathbf{E}'$ and $\mathbf{B}'$ can be obtained from a four-potential $A'_\mu = (0, \mathbf{A}')$, where $\mathbf{A}' = \mathbf{A}'(\mathbf{r}', t')$.
10. Find the four-potential A'_μ obtained from A_μ through a Lorentz transformation.
11. Verify that $\mathbf{E}'$ and $\mathbf{B}'$ can be obtained also from A'_μ .
12. Show that A_μ and A'_μ are related by a gauge transformation.

XXXIV Non-Dispersive Transmission Line

1. The 'elementary cell' scheme of a transmission line is sketched in the figure. In addition to the inductance L and capacitance C typical of the ideal "LC" transmission line, there is a resistance R in series with L, which accounts for the finite resistivity of the two conductors which form the line. In addition, we assume a finite leakage of current between the two conductors (i.e., in the direction "transverse" to the propagation) which is modeled by a second resistance R_L in parallel to C. The corresponding conductance is $G = 1/R_L$. In the limit of a continuous system with homogeneous, distributed properties, we define all quantities per unit length by replacing R with $R_l dx$, L with $L_l dx$, C with $C_l dx$ and G with $G_l dx$ (it is proper to use G as a quantity defined per unit length instead of R_L because the latter is proportional to the inverse of the length of the line).

   Show that the current intensity $I(x,t)$ satisfies the equation

$$(\partial_x^2 - L_l\partial_t^2)I = (R_lC_l + L_lG_l)\partial_t I + R_lG_lI \tag{448}$$

2. Study the propagation of a monochromatic current signal of frequency ω, i.e., search for solutions

$$I = I_0 e^{i(kx-\omega t)} \tag{458}$$

   for $x > 0$ with the boundary condition $I(0,t) = I_0 e^{i-\omega t}$, and determine the dispersion relation $k = k(\omega)$.

3. Find the condition on the line parameters for which a wave packet traveling along the lines undergoes attenuation of the amplitude but no dispersion. This condition corresponds to solutions having the general form

$$I(x,t) = e^{\kappa x} f(x - vt) \tag{462}$$

   where $f(x)$ is an arbitrary differentiable function.

4. Find the expression for v and κ.

XXXV A Wave Packet along a Weakly Dispersive Line

A transmission line extends from $x = 0$ to $x = \infty$. A generator at $x = 0$ inputs a signal

$$f(t) = Ae^{-i\omega_0 t}e^{-(t/\tau)^2} \tag{471}$$

where A and τ are constant and $\omega_0\tau \gg 1$, i.e., the signal is 'quasi-monochromatic', The dispersion relation of the transmission line can be written

$$\omega = \omega(k) = kv(1 + bk), \tag{472}$$

where v and b are known constants, and we assume $k > 0$.

5. Find the expression $f(x,t)$ for the propagating signal, i.e., for the wave packet traveling along the line, assuming $b = 0$.

   From now on, assume dispersive effects to be small but not negligible, i.e., assume $bk_0 \ll 1$, where $k_0 = k(\omega_0)$ according to Eq.(472).

6. Within the above approximation, write the phase and group velocities as functions of ω_0 to the lowest order at which dispersive effects are present.

7. Give an estimate of the instant t_x when the "peak" of the signal reaches the position x, and of the corresponding length of the wave packet.

8. Now find the expression of the wave-packet shape as a function of (x,t), by calculating the integral

$$f(x,t) = \int d\omega e^{ik(\omega)x - i\omega t}\tilde{f}(\omega) \tag{478}$$

   where $\tilde{f}(\omega)$ is the Fourier transform of the wave packet. As a reasonable approximation, keep only factors up to the second order in $(k - k_0)^2$.

XXXVI Propagation in an Optical Fiber

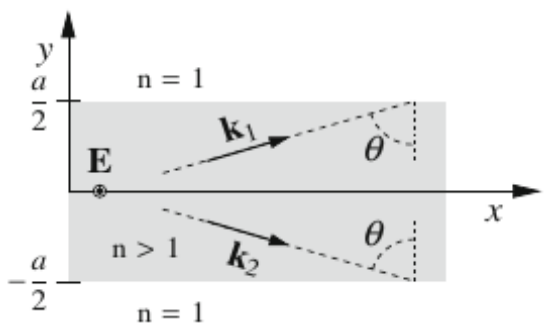

Figure 6: Scheme of an optical fiber.

Figure 6 represents a simple model for an optical fiber. In a Cartesian reference frame (x, y, z) the space between the planes $y = \pm a/2$ is filled by a material of a real and positive refractive index $n > 1$ (in the frequency range of interest), while we have vacuum $(n = 1)$ in the regions $y > a/2$ and $y < -a/2$. A mono- chromatic electromagnetic wave of frequency ω propagates parallel to bfe_x inside the fiber. We assume that the only nonzero component of the electric field $\mathbf{E}$ of the wave is parallel to z (i.e. perpendicular to the plane of the figure). Further, we assume that the wave is the superposition of two plane waves with wavevectors $\mathbf{l}_1 \equiv ({}_k x, k_y, 0) \equiv k(\sin(\theta), \cos(\theta), 0)$, and $k_2 \equiv (k_x, -k_y, 0) \equiv k(\sin(\theta), -\cos(\theta), 0)$, where θ is the angle of incidence shown in the figure. We have, in complex notation,

$$\begin{aligned} \mathbf{E} &= \mathbf{e}_z \left(E_1 e^{i(\mathbf{k}_1 . \mathbf{r} - \omega t)} + E_2 e^{i(\mathbf{k}_2 . \mathbf{r} - \omega t)} \right) \\ &= \mathbf{e}_z \left(E_1 e^{i(kx + ky - \omega t)} + E_2 e^{i(kx - ky - \omega t)} \right) \end{aligned} \tag{485}$$

9. Find the relation between k and ω, and the range of θ for which the wave propagates without energy loss through the boundary surfaces at $y = \pm a/2$.

10. The amplitude reflection coefficient $r = E_r/E_i$ is the ratio of the complex amplitude of the reflected wave to the amplitude of the incident wave, at the surface separating two media. In the case of total reflection we have $r = e^{i\delta}$, with δ a real number. Show that, in our case, we have

$$k_y a + \delta = m\pi \tag{494}$$

with $m \in \mathcal{N}$, and write the equation for the cut-off frequencies of the fiber. Find the values of k_y explicitly at the $n \sin(\theta) \gg 1$, $\theta \to \pi/2$ limit.

11. How do the results change if $\mathbf{E}$ lies in the xy plane?

XXXVII Square and Triangular Waveguides

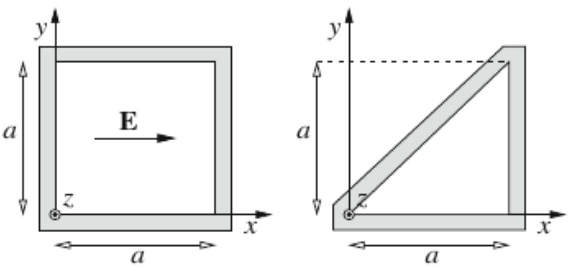

Figure 7: Square and triangular waveguides.

A waveguide has perfectly conducting walls and a square section of side a, as shown in Fig.7. We choose a Cartesian coordinate system (x, y, z) where the interior of the waveguide is delimited by the four planes $x = 0$, $x = a$, $y = 0$ and $y = a$. Consider the propagation along ẑ of a wave of frequency ω, whose electric field $\mathbf{E}(x, y, z, t)$ is perpendicular to $\mathbf{e}_z$ (a TE mode). Assume that the electric field can be written as

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(x, y) e^{i(kz - \omega t)} \tag{508}$$

where $\mathbf{E}(x, y)$ on x and y only.

12. Assuming that $\mathbf{E}$ is parallel to $\mathbf{e}_x$, i.e. $\mathbf{E} = E_x \mathbf{e}_x$ determine the lowest value of ω for which the TE mode can propagate in the waveguide, and the corresponding expressions for the electric and magnetic fields.

13. Determine the lowest frequency and the EM fields for a waveguide delimited by the conducting planes $x = 0$, $y = 0$, and $y = x$, whose cross section is the right isosceles triangle shown in Fig.7