1 A bound particle coupled to two thermostats

One considers a particle of mass $m$ in one dimension subjected to a harmonic force and coupled to two heat reservoirs. The equations of motion are given by

\[
\frac{dx}{dt} = v \tag{1}
\]

\[
m \frac{dv}{dt} = -(\gamma_1 + \gamma_2)v - kx + \xi_1(t) + \xi_2(t) \tag{2}
\]

where $\gamma_i$ are the viscosity coefficients and the two Gaussian white noises are given by

\[
\langle \xi_i(t) \rangle = 0 \tag{3}
\]

\[
\langle \xi_i(t) \xi_j(t') \rangle = 2\gamma_i T_i \delta_{ij} \delta(t - t') \tag{4}
\]

with $\delta_{ij}$ is the Kronecker symbol. ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$) and $\delta(t)$ is the Dirac distribution. At $t = 0$, one has $v(0) = 0$ and $x(0) = 0$. Let us define the Laplace transform

\[
\tilde{f}(u) = \int_0^\infty dt e^{-ut} f(t) \] (see Glossary for additional properties of Laplace transforms).

1) Show that the Laplace transform of the velocity $\tilde{v}(u)$ can be expressed as

\[
\tilde{v}(u) = \tilde{G}(u)[\tilde{\xi}_1(u) + \tilde{\xi}_2(u)] \tag{5}
\]

where $\tilde{G}(u)$ is a function to be determined.

2) Show that $\tilde{G}(u)$ can be written as

\[
\tilde{G}(u) = \frac{1}{m(u_1 - u_2)} \left( \frac{u_1}{u - u_1} - \frac{u_2}{u - u_2} \right) \tag{6}
\]

where $u_1$ and $u_2$ are functions of $\gamma_1, \gamma_2$ and $k$.

3) Determine the values of $k$ for which the system is overdamped and for which one has a damped oscillatory behavior.

4) By taking the inverse Laplace transform, give the velocity $v(t)$.

5) Express the rates of heat $\frac{dQ_1}{dt}$ and $\frac{dQ_2}{dt}$ received by the reservoir 1 and 2, respectively.

6) Calculate $\langle v(t)^2 \rangle$.

7) Calculate $\langle \xi_i(t)v(t) \rangle$.

8) Show that the two above quantities converge to stationary values when $t$ is larger than a typical time to be determined.

9) Show that the mean stationary value of $\frac{d(Q_i)}{dt}$ is given by

\[
\frac{d\langle Q_i \rangle}{dt} = \frac{\gamma_1 \gamma_2}{m(\gamma_1 + \gamma_2)} (T_i - T_j) \tag{7}
\]

where $j = 2$ for $i = 1$ and $j = 1$ for $i = 2$.

10) Why is the heat transport independent of the spring constant?
Glossary

— The inverse Laplace transform of \( \frac{1}{u-u_1} \) is \( e^{u_1 t} \)

— The inverse Laplace transform of a product of two Laplace transforms \( \tilde{f}(u)\tilde{g}(u) \) is given by

\[
\int_0^\infty d\tau f(t-\tau)g(\tau)
\]  

(8)

— \( \int_0^t dt' f(t')\delta(t'-t) = f(t)/2 \)

Références