Stochastic thermodynamics

Références

1 Optimal finite time process in thermodynamics

We consider below a system which is driven through the application of an external time-dependent control parameter \( \lambda(\tau) \). The following Langevin equation describes the driven overdamped motion of a single degree of freedom \( x \) in a 1D potential \( V(x, \lambda) \),

\[
\dot{x}(\tau) = -\mu \frac{\partial V(x, \lambda(\tau))}{\partial x} + \xi(\tau),
\]

where \( \xi(\tau) \) is a Gaussian white noise such that \( \langle \xi(\tau) \rangle = 0 \) and \( \langle \xi(\tau)\xi(\tau') \rangle = 2\mu k_B T \delta(\tau - \tau') \). For the sake of simplicity, one set \( \mu = k_B T = 1 \).

Initially, the system is in equilibrium in the potential \( V(x, \lambda_0) \), and we choose \( \lambda_0 = 0 \). During the interval \( 0 \leq \tau \leq t \), the control parameter is varied from \( \lambda_0 \) at the initial time to \( \lambda_f \neq 0 \) at the final time \( \tau = t \). Below, we are particularly interested in the optimal protocol \( \lambda^*(\tau) \) which minimizes the mean work for an imposed finite duration \( t \) of the process.

\( \triangleright \) 1 Write the expression of the work for an arbitrary potential which corresponds to a realization of this process. Explain why the work is in fact a stochastic variable, and give the expression of the mean work.

In the following, we assume that the potential is harmonic of the form

\[
V(x, \tau) = \frac{1}{2} (x - \lambda(\tau))^2.
\]

Let us first consider two simple cases where : (i) the control parameter is varied infinitely slowly, corresponding to the quasi-static limit when \( t \to \infty \) and (ii) the control parameter is varied very fast, corresponding to the limit \( t \to 0 \).

\( \triangleright \) 2 For both cases, obtain the expression of the mean work \( \langle W \rangle \) and the variation of free energy \( \Delta F \). Why can one say that the results are compatible with the second law of thermodynamics.

\( \triangleright \) 3 Let us now return to the general problem of determining the optimal the control parameter \( \lambda(\tau) \) which optimizes the mean work in a fixed given time \( t \). To progress, introduce the variable \( u(t) = \langle x(t) \rangle \) and obtain an evolution equation for this variable using the Langevin equation.

\( \triangleright \) 4 With the equation obtained above, write the mean work in terms of only \( u(t) \) and its derivatives. This mean work should have the form of a sum of a boundary term plus an integral of the form \( \int_0^t d\tau \dot{u}^2 \), where the dot means the time derivative.

\( \triangleright \) 5 In order to optimize the mean work, one should solve the Euler-Lagrange equations associated with the mean work obtained above. In the present case, the Euler-Lagrange equation is very simple \( \ddot{u} = 0 \). What is the solution of this equation compatible with the boundary conditions ?

\( \triangleright \) 6 Minimize the mean work for fixed \( \lambda_f \) and \( t \), and deduce from this that the optimal protocol has the form

\[
\lambda^*(\tau) = \lambda_f \frac{1 + \tau}{2 + t}.
\]
Deduce from the above, the expression of the optimal mean work. The result should have a compact simple form. Verify that the results obtained in question 2 for cases (i) and (ii) are recovered when \( t \to \infty \) and \( t \to 0 \) respectively.

The optimal protocol differs from the simple linear protocol \( \lambda(\tau) = \lambda_f \tau/t \). Verify that the mean work associated with that protocol is not optimal in the sense that

\[
W^{lin} = \left(\frac{\lambda_f}{t}\right)^2 (t + e^{-t} - 1) > W^*,
\]

where \( W^* \) is the optimal mean work found previously.