# 1 Kuramoto model

We propose to study by numerical simulation the paradigmatic model of synchronization, the Kuramoto model. Introduced by Kuramoto in 1975, it illustrates the propensity of coupled oscillators to evolve towards a collective oscillation when the coupling exceeds a critical value. After 35 years, a lot of situations have been considered, and we restrict to this homework to some properties.

You can choose any language for writing your codes. The simulations are not very demanding and you can use Python by using numpy and by avoiding loops in many situations.

The typical execution time in Python should be close to 2 mins on a Laptop and less than 10 mins in a last part of the homework.

One considers $N$ one-dimensional rotors connecting in a ring. The equations of motion are given by

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \frac{K}{N} \sin(\theta_j - \theta_i) \tag{1}$$

where $\theta_i$ and $\omega_i$ are the phase and the frequency of the rotor $i$. $K$ is the coupling strength and the $1/N$ factor ensures that the model is well-behaved in the thermodynamic limit. The frequencies $\{\omega_i\}$ are random numbers chosen from a centered random distribution.

To visualize the collective dynamics of the system, one introduces the complex order parameter

$$r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i(t)} \tag{2}$$

1. Show that the equations of motion can be rewritten as

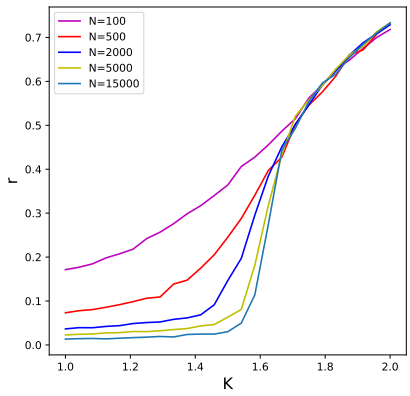
$$\dot{\theta}_i = \omega_i + Kr \sin(\Psi - \theta_i) \tag{3}$$

2. Because the relaxation time of the system is relatively short for values of $K$, write a code solving the differential equations by using a adaptive Runge-Kutta method [1]. The frequencies are slected from a centered Gaussian distribution. The initial phases of the system are chosen from an uniform distribution from $-\pi$ to $\pi$. One takes $N = 10$, Solve and plot $\theta(t)/t$ for all rotors for a time interval $[0, 100]$. Four figures should be created corresponding to different values of $K = 0.0, 1.0, 5, 1, 2$. Comment the time evolution of the rotoros for the different values of couplings. For $K = 1$ and $K = 2$, the time evolution depends strongly of the random frequencies. By considering several set of trajectories, give an interpretation of these different final states.
3. One searches to obtain results for a large number of rotors $N = 100$. Modify the previous code by computing the absolute value of the order parameter $r$ of the system as a function of time. To suppress some spurious oscillations, once selected the random frequencies, subtract the mean value to each frequency in order to keep a mean value

1. **Scipy** provides **solve_ivp** for solving ode and uses parallelization automatically)

| N | Nrep |
|---|---|
| 100 | 300 |
| 500 | 100 |
| 2000 | 50 |
| 5000 | 30 |
| 15000 | 10 |

TABLE 1 – Number of particles and of independent simulations




of frequencies equal to 0. Plot **r** as a function of time for different values of $K$ between 1 and 2. (Take 25 values of $K$ between 1 and 2)

4. In order to suppress fluctuations due to the disorder, it is necessary to take an average of the order parameter over 100 independent simulations. After a first relaxation time which is less than 50, the order parameter **r** oscillates for a long time. In order to save computer time, one estimates the stationary value of the order paramter by averaging $r$ on the interval $[50, 100]$ for each simulation. By performing $Nrep$ independent simulations, obtain mean values of the stationary order parameter $\bar{r}$ for each value of $K$. Plot $\bar{r}$ as a function of $K$
5. Run the previous code for different system sizes $N$ and several simulations $Nrep$. See table1 for the 5 simulations to be performed.
6. Plot the mean stationary values of $\bar{r}$ as a function ok $K$. You should obtain a figure similar to Fig.1
7. One can show that the thermodynamic limit give a second order phase transition with a value of the coupling strength $K_c$

$$K_c = \frac{2}{\pi g(0)} \tag{4}$$

where $g(0)$ is the value of the probability distribution at the center.

Add to the previous figure the location of the thermodynamic limit. Even if more extensive simulation should be necessary to obtain the critical exponents, show that the simulation results are compatible with the location of the phase transition.