Hyperuniformity

We propose to study by numerical simulation the hyperuniform states which appears in various situations (biology, astrophysics, granular media). The hyperuniformity has a very recent reseach activity, because unusual physical properties seem to be related to the disordered structure of these systems. The goal of this homework is to study by numerical simulation some properties characterizing the state of the matter.

The hyperuniformity corresponds to the situation when the variance of the number of particles in a finite volume of the system scales less than the number of particles with the linear size L' of the volume.

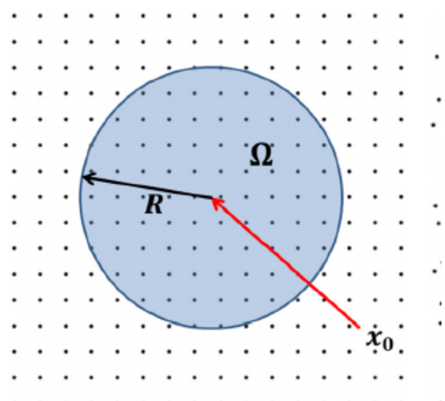

FIGURE 1 – Observation window of radius R

For this homework, you can choose the preferred language for your codes. The simulations are not very demanding for this first homework and you can use Python by using massively numpy and by suppressing loops in many situations. The typical execution time in Python should be close to 2mns on a Laptop.

1 Lattice

One first considers a square lattice of linear size L, where point particles are located at the lattice nodes. One set the lattice step as a unit length, which gives a total number of particles equal to L^2. To avoid boundary effects, one must implement periodic boundary conditions

1. Write a simple code displaying point particles on a square lattice of linear size $L = 10$.
2. To measure the hyperuniformity of the system, one introduces a circular window of observation of radius. x_0 denotes the center of the window. (see Fig.1) The mean variance of the number of particles for a window of radius R is given by

$$\sigma^2(R) = \langle N^2(R)\rangle - \langle N(R)\rangle^2$$

where the brackets denote the average of over uniform random positions x_0 and $N(R)$ the number of particles of the observation window. In your code, you must include the periodic boundary conditions. Write a code for calculating this quantity as a function of R. Justify why the maximum distance for measuring the variance is $L/2$. For convenience, one chooses the minimum distance of observation equal to 0.01. The number of random points is equal to 10000 and the number of bins for calculating the rescaled variance is equal to 200 and the different distances R in the interval $[0.01, L/2]$ must be choosen with a geometric sequence.

Hint : Finally, you should obtain a figure similar to Fig.2.

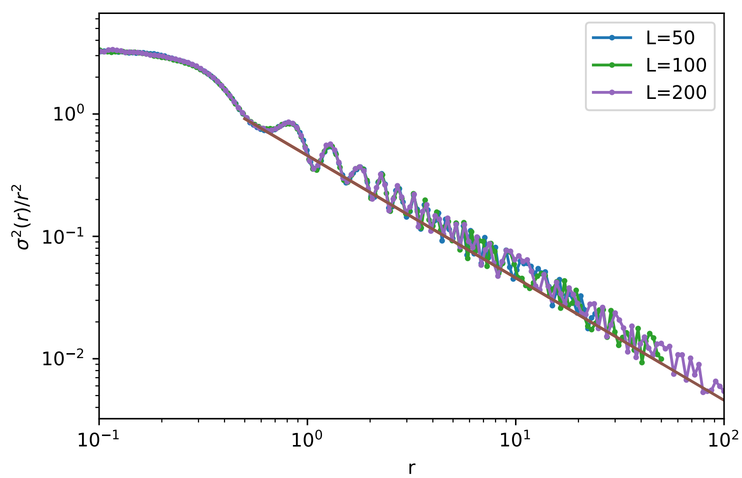

FIGURE 2 – Rescaled variance $\frac{\sigma^2(r)}{r^2}$ as a function of the linear size r of the window.

3. Fig.2 shows the rescaled variance versus r for 3 different system sizes. Comment the results : why do $\sigma^2(r)/r^2$ goes to a constant at short distance ? What are the reasons of the oscillations of the curve ? The straight line corresponds to the asymptotic behavior of variance at long distance.

$$\sigma^2(r)/r^2 = \frac{0.457648}{r}$$

.

Why do the amplitudes of the oscillations decrease at long distance ?

2 Shuffled lattice

We obtained that the ordered system displays hyperuniformity as expected . In this second part of the homework, one considers particles which undergoes a single stochastic displacement from the initial lattice configuration.

1. Starting from the lattice configuration move all particles by a displacement in all directions by selecting random numbers from a centered uniform distribution with the range $[-\Delta/2, \Delta/2]$

   Modify the previous code for simulating this new system and plot rescaled variance as a function of the distance for different systems when $\Delta = 1, 2, 3, 4, 5$. ($L = 100$). Save the program and generate the pdf files for the different figures. What do you oberve ? Give a physical interpretation of the observation. .

2. Modify the previous code in order to plot for $\Delta = 2$ the results for $L = 50, 100, 200$. Comment your results.