Feuille 7

$\boxed{\mathrm{I}}$ 1) $[F] = MLT^{-2}$

$[\hbar] = ML^2T^{-1}$

$[c] = LT^{-1}$

$[a] = L$

$$F = K \frac{\hbar c}{a^2}$$

2) $\langle 0 | H | 0 \rangle = \sum_k \frac{\omega_k}{2}$

$k_n = \frac{\pi n}{a} \Rightarrow \omega_k = \frac{\pi n}{a}$

$$E(a) = \sum_{n=1}^{\infty} \frac{n\pi}{2a}$$

3) Si L est l'extension spatiale ~~tas~~ de notre système :

$$E = \sum_{n=1}^{\infty} \frac{n\pi}{2L}$$

Energie dans le segment de taille a :

$E = \frac{a}{L} \sum_{n=1}^{\infty} \frac{n\pi}{2L}$ dans la limite où $L \to \infty$

$\frac{a}{L} \Rightarrow dx$

$\frac{an}{L} \Rightarrow x$

$$E(a) = \int_0^{\infty} \frac{x\pi}{2a} dx$$

$$\hookrightarrow \Delta E(a) = \frac{\pi}{2a} \left\{ \sum_{n=0}^{\infty} n - \int_0^{\infty} n\,dn \right\}$$

4) $f(\delta) = \sum_{n=0}^{\infty} n e^{-\delta n}$

$$g(\delta) = \sum_{n=0}^{\infty} e^{-\delta n} = \frac{1}{1-e^{-\delta}}.$$

$$-g'(\delta) = \sum_{n=0}^{\infty} n e^{-\delta n} = f(\delta) = \frac{e^{-\delta}}{(1-e^{-\delta})^2} = \frac{1}{(e^{\frac{\delta}{2}} - e^{-\frac{\delta}{2}})^2}.$$

$$\hookrightarrow f(\delta) = \frac{1}{4(\operatorname{sh}\frac{\delta}{2})^2}.$$

$$h(\delta) = \int_0^\infty n e^{-\delta n} \, dn \qquad X = \delta n. \quad n = x/\delta.$$

$$= \int_0^\infty \frac{dx}{\delta} \frac{x}{\delta} e^{-x} = \frac{1}{\delta^2} \int_0^\infty dx \, x e^{-x}.$$

$$= \frac{1}{\delta^2} \Gamma(2) = \frac{1}{\delta^2}.$$

$$\frac{1}{(\delta + \frac{\delta^3}{24})^2} = \frac{1}{\delta^2}\left(1 - \frac{\delta^3}{12}\right)$$

$$f(\delta) - h(\delta) = \frac{1}{4(\operatorname{sh}\frac{\delta}{2})^2} - \frac{1}{\delta^2} \simeq \frac{1}{4\left(\frac{\delta}{2} + \left(\frac{\delta}{2}\right)^3 \frac{1}{6}\right)^2} - \frac{1}{\delta^2}$$

$$\simeq \frac{1}{\delta^2}\left(1 - \frac{\delta^2}{12}\right) - \frac{1}{\delta^2} \simeq -\frac{1}{12} \qquad \text{Fini !!!}$$

$$\hookrightarrow \Delta E(a) = -\frac{\pi}{24a}.$$

5) $$\sum_{n=0}^{\infty} n f_\delta(n) - \int_0^\infty n f_\delta(n) \, dn = \frac{1}{2} \times 0 f_\delta(0)$$

$$+ i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left(it f_\delta(it) + it f_\delta(-it) \right)$$

$$= - \int_0^\infty \frac{t \, dt}{e^{2\pi t} - 1} \left(f_\delta(it) + f_\delta(-it) \right)$$

L'exponentielle fait converger l'intégrale, même si $\delta = 0$ ↝

$$\sum_{n=0}^{\infty} n f_\delta(n) - \int_0^\infty n f_\delta(n)\,dn \xrightarrow[\delta\to 0]{} -2\int \frac{t\,dt}{e^{2\pi t}-1}$$

$$= -\frac{1}{12}$$ indépendant de la fonction f_δ...

$\hookrightarrow \Delta E(a) = -\frac{\pi}{24a}$.

6] $F = -\frac{dE_p}{da} = -\frac{\pi}{24a^2}$

$\rightarrow$ indéterminé.

7) $F = (k) S \frac{\hbar c}{a^4}$

2 polarisations possibles de $\vec{E}$ ($\vec{B}$)

$\hookrightarrow$ sommes sur $n>0$ et $n<0$.

sauf pour $n=0 \Rightarrow$ 1 polarisation

8] $$E(a) = \frac{1}{2} S \int \frac{dk_x\,dk_y}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \omega_k$$

$$= \frac{1}{2} S \int \frac{dk_x\,dk_y}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \sqrt{k_x^2+k_y^2+\frac{n^2\pi^2}{a^2}}$$

$$= \frac{1}{2}\frac{S}{2\pi} \int_0^\infty y\,dy \sum_{n=-\infty}^{+\infty} \sqrt{y^2+\frac{n^2\pi^2}{a^2}}$$

Si l'existence spatiale dans la direction $\perp$ aux plaques,

l'énergie comprise dans un volume $S\times a$:

$$E_0(a) = \frac{1}{2}\frac{S}{2\pi} \int_0^\infty y\,dy \sum_{n=-\infty}^{+\infty} \frac{a}{L} \sqrt{y^2+\frac{n^2\pi^2}{L^2}}$$

$$= ~~\frac{Sa}{2\pi} \int_0^\infty y\,dy \sum_{n=-\infty}^{+\infty} \frac{1}{yL} \sqrt{\cdots (n\pi)^2 \cdots}~~$$

(4)

$$E_0(a) = \frac{1}{2}\frac{S}{2\pi}\int_0^{\infty} y\,dy \sum_{n=-\infty}^{+\infty} \frac{a}{L}\sqrt{y^2+\frac{\pi^2}{a^2}\left(\frac{na}{L}\right)^2}$$

$$\left.\begin{array}{l} dx = \frac{a}{L} \\ x = \frac{na}{L}\end{array}\right\} \Rightarrow E_0(a) = \frac{1}{2}\frac{S}{2\pi}\int_0^{\infty} y\,dy \int_{-\infty}^{+\infty} dx \sqrt{y^2+\frac{\pi^2 x^2}{a^2}}$$

$$\Delta E(a) = \frac{1}{2}\frac{S}{2\pi}\int_0^{\infty} y\,dy \underbrace{\left\{\sum_{n=-\infty}^{+\infty}\sqrt{y^2+\frac{n^2\pi^2}{a^2}} - \int_{-\infty}^{\infty}\sqrt{y^2+\frac{n^2\pi^2}{a^2}}\,dn\right.}_{A(y)}$$

Expression pour $A(y)$

a) régulariser

$$A_\delta(y) = 2\left\{\sum_{n=0}^{\infty} f_\delta(n)\sqrt{y^2+\frac{n^2\pi^2}{a^2}} - \int_0^{\infty} f_\delta(n)\sqrt{y^2+\frac{n^2\pi^2}{a^2}}\,dn\right\} - f_\delta(0)y$$

$$= 2\left\{\frac{1}{2}\cancel{f_\delta(0)y} + i\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}\left\{f_\delta(it)\sqrt{y^2-\frac{t^2\pi^2}{a^2}} - f_\delta(-it)\sqrt{y^2-\frac{t^2\pi^2}{a^2}}\right\}\right.$$

$$\cancel{-f_\delta(0)y}$$

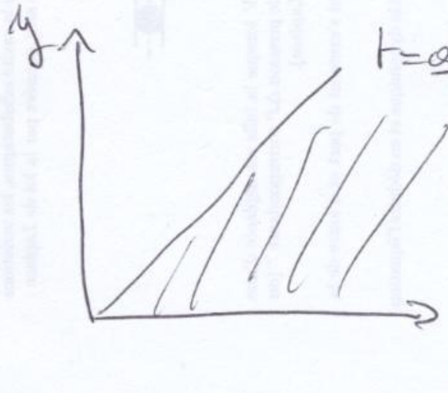
grâce à $e^{2\pi t}$, l'intégral est finie, même pour $\delta = 0 \Rightarrow f_\delta \to 1$.

Re: si $y > \frac{t\pi}{a}$, $\sqrt{y^2-\frac{t^2\pi^2}{a^2}} - \sqrt{y^2-\frac{t^2\pi^2}{a}} = 0$

Si $y < \frac{t\pi}{a}$ $\rightsquigarrow 2i\sqrt{\frac{t^2\pi^2}{a^2}-y^2}$.

$$A_s(y) = -4\int_{\frac{ay}{\pi}}^{\infty} \frac{dt}{e^{2\pi t}-1}\sqrt{\frac{t^2\pi^2}{a^2}-y^2}$$

$$\hookrightarrow \Delta E(a) = -\frac{\cancel{2}S}{2\pi}\int_0^{\infty} y\,dy \int_{\frac{ay}{\pi}}^{\infty}\frac{dt}{e^{2\pi t}-1}\sqrt{\frac{t^2\pi^2}{a^2}-y^2}$$



$t=\frac{ay}{\pi} \iff y=\frac{\pi t}{a}$

zone d'intégration

en permutant l'ordre des intégrales:

$$\Delta E(a) = -\frac{S}{\pi}\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}\underbrace{\int_0^{\frac{\pi t}{a}} dy\, y\sqrt{\frac{t^2\pi^2}{a^2}-y^2}}_{\left[-\frac{1}{3}\left(\frac{t^2\pi^2}{a^2}-y^2\right)^{3/2}\right]_0^{\frac{\pi t}{a}}}$$

$$= -\frac{S}{3\pi}\frac{\pi^3}{a^3}\int_0^{\infty}\frac{t^3\,dt}{e^{2\pi t}-1}$$

$$= -\frac{S}{720\pi^2 a^3}$$

$$F = -\frac{S}{240\pi^2 a^4}$$

2) $D_\mu \phi \to \partial_\mu (\phi e^{-ie\Lambda}) + ie(A_\mu + \partial_\mu \Lambda)\phi e^{-ie\Lambda}$

$= e^{-ie\Lambda}\left(\partial_\mu \phi - i\partial\Lambda\phi + ieA_\mu\phi + ie\partial_\mu\Lambda\phi\right).$

$= e^{-ie\Lambda}\left(D_\mu \phi\right)$

$\to (D_\mu\phi)^* \to e^{ie\Lambda}(D_\mu\phi)^*$

et $(D_\mu\phi)^* D^\mu\phi$ inchangé

$\phi^*\phi$ inchangé

$F_{\mu\nu} \to \partial_\mu A_\nu + \partial_\mu\partial_\nu\Lambda - \partial_\nu A_\mu - \partial_\nu\partial_\mu\Lambda$

$= F_{\mu\nu}$ inchangé

$F_{\mu\nu}F^{\mu\nu}$ inchangé

$\to \mathcal{L}$ inchangé sous transformation de jauge

3) Λ infinitésimal infime $\Rightarrow A$ inchangé

$\delta\phi \to -ie\Lambda\phi$

$\delta\phi^* \to ie\Lambda\phi^*$.

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$$J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(-ie\Lambda\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(ie\Lambda\phi^*)$$

$$= -ie\Lambda\phi\,(D^\mu\phi)^* + ie\Lambda\phi^* D^\mu\phi$$

$$= -ie\Lambda\,\phi\,(\partial^\mu\phi^* - ieA^\mu\phi^*) + ie\Lambda\,\phi^*(\partial^\mu\phi + ieA^\mu$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= -m^2\phi^* + ieA^\mu(D_\mu\phi)^* \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} &= (D^\mu\phi)^* \end{aligned}\right\} \begin{aligned} &+m^2\phi^* + \partial_\mu(D^\mu\phi)^* = 0 \\ &-ieA^\mu(D_\mu\phi)^* \end{aligned}$$

$$\hookrightarrow \quad m^2\phi^* + (D_\mu D^\mu\phi)^* = 0$$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi^*} &= -m^2\phi - ieA_\mu D^\mu\phi \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)} &= D^\mu\phi \end{aligned}\right\} \begin{aligned} &m^2\phi + ieA_\mu D^\mu\phi + \partial_\mu D^\mu\phi = 0 \\ &\hookrightarrow \quad m^2\phi + D_\mu D^\mu\phi = 0 \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = ie\phi D_\mu \phi^* - ie\phi^* D_\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)} = -F^\nu{}_\mu$$

$$\hookrightarrow \quad ie\left[\phi D_\mu \phi^* - \phi^* D_\mu \phi\right] + \partial_\nu F^\nu{}_\mu = 0$$

$$\hookrightarrow \quad \partial_\mu F^{\mu\nu} = -ie\left[\phi D^\nu \phi^* - \phi^* D^\nu \phi\right]$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$ Equations de Maxwell inhomogènes.