Feuille 6.

1] Champ scalaire sous Poincaré

$$\phi^*\phi \to e^{-i\theta} e^{i\theta} \phi^*\phi \text{ invariant}$$

↳ termes invariants : autant de ϕ^* que de ϕ

+ des contractions $\partial_\mu \partial^\mu$

↳ $(\phi^*\phi)^n$

$(\partial_\mu\phi)(\partial^\mu\phi)\,\phi^{*2}\,(\phi^*\phi)^n$ + c.c.

$(\partial_\mu\phi)(\partial^\mu\phi^*)\,(\phi^*\phi)^n$.

2] $$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right] = 0$$

$$-m^2\phi - \partial_\mu\partial^\mu\phi = 0$$

$$(\Box + m^2)\phi = 0$$

$P_\mu = i\partial_\mu \quad P_\mu P^\mu = -\Box \Rightarrow P_\mu P^\mu - m^2 = 0$

$P^2 = m^2$

pour ϕ^* : $(\Box + m^2)\phi^* = 0$.

3] $\phi' = \phi + i\theta\phi$.

$\phi^{*\prime} = \phi^* - i\theta\phi^*$.

②

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^I)}\delta\phi^I + \delta x^\nu\left[\mathcal{L}\delta^\mu_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^I)}\partial_\nu\phi^I\right.$$

Dans notre cas:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^*$$

$$= (\partial^\mu\phi^*)\theta\phi + (\partial^\mu\phi)(-\theta\phi^*)$$

$$= i\theta\left[\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right]$$

on pluck (e [illegible] θ)

$$J^\mu = i(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi)$$

$$\partial_\mu J^\mu = i(\partial_\mu\phi\,\partial^\mu\phi^* - \partial_\mu\phi^*\partial^\mu\phi + \phi\Box\phi^* - \phi^*\Box\phi)$$

$$= i(-m^2\phi\phi^* + m^2\phi\phi^*) = 0$$

δJ

$$J^\mu = a^\nu\left[\mathcal{L}\delta^\mu_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\partial_\nu\phi^*\right]$$

$$T^{\mu\nu} = -\mathcal{L}g^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\partial^\nu\phi^*$$

$$= -g^{\mu\nu}\mathcal{L} + \partial^\mu\phi^*\partial^\nu\phi + \partial^\mu\phi\,\partial^\nu\phi^*$$

$$\partial_\mu T^{\mu\nu} = -\partial^\nu\left[(\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi\right] + \Box\phi^*\partial^\nu\phi + \partial^\mu\phi^*(\partial_\mu\partial^\nu\phi) + c.c.$$

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transverse component of $\Gamma_k^{(2)}$ evaluated for uniform fields and zero momentum:

[illegible] (4)

where $C_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$, u_{yy}, v_y and $v_{yy'}$ stand, respectively, for $\partial_y^2 u$, $\partial_y v$ and $\partial_y\partial_{y'}v$ evaluated for fields that minimize u; the subscript k has been dropped for simplicity, and [illegible] are dimensionless threshold functions described in [13]. In eq.(4) we have set for clarity $T_k = T = 0$, but the same FPs are reached when $T > 0$ provided (as indeed found) that $\theta = \theta_{k\to 0} > 0$. As discussed previously [12], breakdown of DR is associated with the appearance of a strong enough nonanalycity in the renormalized 2-replica potential as the two replica fields become equal. This nonanalycity is related to the presence of metastable states in the renormalized random potential generated along the RG flow.

We have numerically integrated the flow equations for [illegible] for a variety of initial conditions and determined in this way the FPs and the associated stability behavior (and critical exponents). To cover the whole (N,d) diagram for continuous values of N and d with a tractable computational effort, we have used an additional approximation that consists in expanding [illegible] around the field configuration [illegible]. One should point out that the resulting flow equations reproduce all 1-loop perturbative results in the appropriate region of the (N,d) plane, including the FRG equation at first order in $\epsilon = d - 4$ [15].

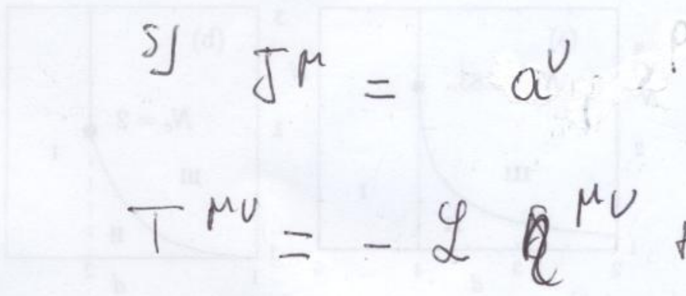

FIG. 1: Predicted phase diagram for (a) the RFO(N)M (nonperturbative FRG) and (b) pure O(N) model (sketch). Regions I and II correspond to transitions to a ferromagnetic and a QLRO phase, respectively.

The central results of our study are summarized in the phase diagram displayed in Fig. 1a. In region III there are no phase transitions, the RFO(N)M always being disordered. In region I there is a transition to a ferromagnetic phase at a critical point governed by a zero-temperature nonanalytic FP. The nonanalycity is strong enough (in itself) for breaking the DR predictions; however, when approaching a critical line [illegible]

k and its flow obeys an exact equation:

$$\partial_t\Gamma_k[\{\Phi_a\}] = \frac{1}{2}\mathrm{Tr}\{\partial_t R_k(q)(\Gamma_k^{(2)}[\{\Phi_a\},q] + \hat{R}_k(q))^{-1}\} \quad (2)$$

where ∂_t is a derivative with respect to $t = \ln(k/\Lambda)$, $\Gamma_k^{(2)}$ is the tensor formed by the second functional derivatives of Γ_k with respect to the fields $\phi_a(q)$, $\hat{R}$ is the regulator tensor, and the trace involves an integration over momenta as well as a sum over replica indices and N vector components. $R_k(q)$ is the infrared regulator that suppresses the low-momentum modes. In practice, solving eq.(2) numerically requires the introduction of approximation schemes which amount to truncate the functional form of Γ_k. Guided by the physics of the problem at hand and by the [illegible] work on the method [13], one can then formulate a nonperturbative RG description. For the RFO(N)M we have argued that the two main ingredients allowing to study the long-distance physics are (i) the derivative expansion, which approximates the momentum dependence of the 1PI vertex functions [13], and (ii) the expansion in increasing number of free replica sums, which is equivalent to include increasing-order cumulants of the renormalized distribution of the quenched disorder [12].

The simplest nonperturbative FRG description of the RFO(N)M is then the following approximation:

$$\Gamma_k[\{\Phi_a\}] = \int_x\left\{\sum_{a=1}^n\left[\frac{1}{2}Z_{m,k}\partial\phi_a\cdot\partial\phi_a + U_k(\phi_a)\right] - \frac{1}{2}\sum_{a,b=1}^n V_k(\phi_a,\phi_b)\right\} \quad (3)$$

with one single wavefunction renormalization for all fields $Z_{m,k}$ evaluated at the field configuration that minimizes the 1-replica potential U_k (pseudo first-order derivative expansion [13]); the 1-replica U_k and 2-replica V_k parts of the effective potential are obtained from Γ_k taken for uniform fields. Inserting eq.(3) into eq.(2), leads to coupled partial differential equations for the functions $U_k(\rho)$ and $V_k(\rho,\rho',z)$ where $\rho = \frac{1}{2}\phi^2$ and $z = \phi\cdot\phi'/\sqrt{(4\rho\rho')}$, and a running anomalous dimension defined as $\eta_k = -\partial_t \log Z_{m,k}$. The sought FP being at zero temperature, one also introduces a running temperature T_k and the associated exponent $\theta_k = \partial_t \log T_k$. This is most conveniently done by defining a temperature-like order strength $\Delta_{m,k} = (2\rho_{m,k})^{-1}\partial_z V_k(\rho_{m,k},\rho_{m,k},z=1)$, where $\rho_{m,k}$ corresponds to the minimum of $U_k(\rho)$. Defining then T_k as [illegible], one can see that it reduces to the bare temperature T at the microscopic scale Λ.

The flow equations can be expressed in a scaled form by introducing renormalized dimensionless quantities, [illegible]. The expression for η_k is derived by considering the flow of the

(2)

$$\partial_\mu T^{\mu\nu} = -\cancel{(\partial_\mu \partial^\nu \phi^*)(\partial^\mu \phi)} + cc + m^2 (\partial^\nu \phi^*)\phi + cc$$

$$+ \Box \phi^* \partial^\nu \phi + \cancel{\partial^\mu \phi^* (\partial_\mu \partial^\nu \phi)} + cc$$

$$= \partial^\nu \phi^* \left[(\Box + m^2)\phi \right] + cc = 0.$$

$$T^{00} = -(\partial_\mu \phi^*)(\partial^\mu \phi) + m^2 \phi \phi^* + 2\partial^0 \phi \, \partial^0 \phi^* .$$

$$= -(\partial_t \phi^*)(\partial_t \phi) + (\vec\nabla \phi^*)(\vec\nabla \phi) + m^2 \phi \phi^* + 2(\partial_t \phi)(\partial_t \phi^*)$$

$$= +(\partial_t \phi^*)(\partial_t \phi) + (\vec\nabla \phi^*)(\vec\nabla \phi) + m^2 \phi \phi^* .$$

$$T^{0i} = \partial^0 \phi^* \partial^i \phi + \partial^0 \phi \, \partial^i \phi^* .$$

$$= -\left(\partial_t \phi^* \vec\nabla \phi + cc \right)$$

[2] $\mathcal{L} = \frac{i}{2} \bar\psi \gamma^\mu \partial_\mu \psi - \frac{i}{2} (\partial_\mu \bar\psi) \gamma^\mu \psi - m \bar\psi \psi .$

$$\times \; \frac{\partial \mathcal{L}}{\partial \bar\psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} = 0$$

$$-m\psi + \frac{i}{2} \gamma^\mu \partial_\mu \psi - \partial_\mu \left(-\frac{i}{2} \gamma^\mu \psi \right) = 0 .$$

$$i \gamma^\mu \partial_\mu \psi - m\psi = 0 \Rightarrow (i \not\partial - m)\psi = 0 .$$

$$\times \; \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0$$

$$-m\bar\psi - \frac{i}{2} (\partial_\mu \bar\psi) \gamma^\mu - \partial_\mu \left(\frac{i}{2} \bar\psi \gamma^\mu \right) = 0$$

$$\hookrightarrow \; \bar\psi \left(-m - i \overleftarrow{\not\partial} \right) = 0 \qquad \text{avec } \bar\psi \overleftarrow{\partial_\mu} \psi = (\partial_\mu \bar\psi) \psi .$$

$$\mathcal{L} = \bar\Psi \left[\underbrace{(i\not\partial - m)\Psi}_{=0} \right] + \left[\underbrace{\bar\Psi(-i\overleftarrow{\not\partial} - m)}_{=0} \right]\Psi .$$

$= 0$

2) $$J^\mu = -a^\nu T^\mu{}_\nu$$

$$T^\mu{}_\nu = -\mathcal{L}\eta^\mu{}_\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \partial_\nu \Psi + (\partial_\nu \bar\Psi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\Psi)}$$

$$= \frac{i}{2} \bar\Psi \gamma^\mu \partial_\nu \Psi - \frac{i}{2} \partial_\nu \bar\Psi \gamma^\mu \Psi$$

$$= \frac{i}{2} \bar\Psi \overleftrightarrow{\partial_\nu} \gamma^\mu \Psi$$

$$\partial_\mu T^{\mu\nu} = \frac{i}{2} \partial_\mu \bar\Psi \gamma^\mu \partial^\nu \Psi + \frac{i}{2} \bar\Psi \gamma^\mu \partial_\mu \partial^\nu \Psi$$

$$- \frac{i}{2} (\partial_\mu \partial^\nu \bar\Psi) \gamma^\mu \Psi - \frac{i}{2} \partial^\nu \bar\Psi \gamma^\mu \partial_\mu \Psi$$

$$= \frac{i}{2} m \left\{ - \bar\Psi \partial^\nu \Psi + \bar\Psi \partial^\nu \Psi + (\partial^\nu \bar\Psi)\Psi - (\partial^\nu \bar\Psi)\Psi \right\} = 0 .$$

3) $$T^{00} = \frac{i}{2} \bar\Psi \overleftrightarrow{\partial^0} \gamma^0 \Psi = \frac{i}{2} \left(-(\partial_t \bar\Psi) \gamma^0 \Psi + \bar\Psi \gamma^0 \partial_t \Psi \right)$$

$$= \frac{i}{2} \left(\Psi^+ \partial_t \Psi - (\partial_t \Psi^+) \Psi \right)$$

$$T^{0i} = -\frac{i}{2} \left[\bar\Psi \gamma^0 \vec\nabla \Psi - (\vec\nabla \bar\Psi) \gamma_0 \Psi \right];$$

$$= -\frac{i}{2} \left[\Psi^+ \vec\nabla \Psi - (\vec\nabla \Psi^+) \Psi \right];$$

3) 1) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - F^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

$$\hookrightarrow \quad \underline{\partial_\mu F^{\mu\nu} = 0}$$

2) $\delta J^\mu = -a^\nu T^\mu{}_\nu$

$$T^\mu{}_\nu = -\eta^\mu{}_\nu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial_\nu A_\rho$$

$$= -\eta^\mu{}_\nu \mathcal{L} - F^{\mu\rho} \partial_\nu A_\rho$$

3) $\partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \underbrace{\partial_\mu \partial_\rho X^{\rho\mu\nu}}_{=0}$

(symétrique $\partial_\mu \partial_\rho$, antisymétrique $X^{\rho\mu\nu}$)

$$= 0.$$

$$\int \tilde{T}^{0\nu} d^3x = \int T^{0\nu} d^3x + \int \partial_\rho X^{\rho 0 \nu} d^3x$$

$$= \int T^{0\nu} d^3x + \underbrace{\int \partial_t X^{00\nu} d^3x}_{=0} + \underbrace{\int \vec{\nabla} X^{i0\nu} d^3x}_{\text{rene de Gauss} \Rightarrow 0}$$

$$= \int T^{0\nu} d^3x$$

4] $X^{\mu\rho\nu} = -X^{\rho\mu\nu}$ OK.

$$\tilde{T}^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} - F^{\mu\rho}\partial^{\nu}_{\nu} A_\rho - \partial_\rho\left(F^{\rho\mu} A^\nu\right)$$

$$= -\eta^{\mu\nu}\mathcal{L} - F^{\mu\rho}\partial^{\nu} A_\rho - \underbrace{\left(\partial_\rho F^{\rho\mu}\right)}_{=0} A^\nu - F^{\rho\mu}\partial_\rho A^\nu$$

$$= \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta} - F^{\mu\rho}F^{\nu}{}_{\rho}$$

Symétrique et invariant de Jauge !

5]
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\alpha\beta}F^{\alpha\beta} = \left(F^{00}\right)^2 - 2\sum_i \left(F^{0i}\right)^2 + \sum_{ij}\left(F^{ij}\right)^2$$

$$= 0 - 2E^2 + 2B^2$$

$$= 2\left(B^2 - E^2\right)$$

$$\tilde{T}^{00} = \frac{B^2 - E^2}{2} - F^{0\rho}F^{0}{}_{\rho}$$

$$= \frac{B^2-E^2}{2} + \sum_i \left(F^{0i}\right)^2 = \frac{B^2+E^2}{2}$$ densité d'énergie.

$$\tilde{T}^{0i} = -F^{0\rho}F^{i}{}_{\rho}$$

$$\tilde{T}^{01} = -F^{02}F^{1}{}_{2} - F^{03}F^{1}{}_{3} = F^{02}F^{12} + F^{03}F^{13}$$

$$= +E_y B_z - E_z B_y = \left(\vec{E}\times\vec{B}\right)_1$$

$\tilde{T}^{0i} = (\vec{E} \times \vec{B})_i$ vecteur de Poynting
densité d'impulsion

4

1) $J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \left(-\frac{i}{2} \omega^{\mu\nu} (\Sigma_{\mu\nu})^{ab} \phi_b\right) - \omega^{\nu\rho} x_\rho T^\mu{}_\nu$

$= \frac{\omega^{\alpha\beta}}{2} \left[-i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} (\Sigma_{\alpha\beta})^{ab} \phi_b - x_\beta T^\mu{}_\alpha + x_\alpha T^\mu{}_\beta \right]$

$= \frac{\omega^{\alpha\beta}}{2} M^\mu{}_{\alpha\beta}$

avec $\Delta^{\mu\alpha\beta} = -i \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} (\Sigma^{\alpha\beta})^{ab} \phi_b$

2) $\partial_\mu J^\mu = 0 \Rightarrow \frac{\omega^{\alpha\beta}}{2} \partial_\mu M^\mu{}_{\alpha\beta} = 0 \quad \forall \omega$

$\hookrightarrow \partial_\mu M^\mu{}_{\alpha\beta} = 0$

$\hookrightarrow \partial_\mu M^{\mu\nu\rho} = T^{\nu\rho} + x^\nu \underbrace{(\partial_\mu T^{\mu\rho})}_{=0} - T^{\rho\nu} - x^\rho \underbrace{(\partial_\mu T^{\mu\nu})}_{=0}$
$+ \partial_\mu \Delta^{\mu\nu\rho}$

$\hookrightarrow T^{\nu\rho} - T^{\rho\nu} = -\partial_\mu \Delta^{\mu\nu\rho}$

$T^{\mu\nu} - T^{\nu\mu} = -\partial_\rho \Delta^{\rho\mu\nu}$

3] $$X^{\mu\rho\nu} = \frac{1}{2}\left(\Delta^{\mu\rho\nu} - \Delta^{\rho\mu\nu} - \Delta^{\nu\mu\rho}\right)$$

$$= \frac{1}{2}\left(-\Delta^{\rho\mu\nu} + \Delta^{\mu\rho\nu} + \Delta^{\nu\rho\mu}\right)$$

$$= -X^{\mu\nu\rho} \quad \text{OK.}$$

$$\tilde{T}^{\mu\nu} - \tilde{T}^{\nu\mu} = T^{\mu\nu} - T^{\nu\mu} + \frac{1}{2}\Big(\partial_\rho\Delta^{\rho\mu\nu} - \partial_\rho\Delta^{\rho\nu\mu}$$

$$- \partial_\rho\Delta^{\mu\rho\nu} + \partial_\rho\Delta^{\nu\rho\mu} - \partial_\rho\Delta^{\nu\rho\mu} + \partial_\rho\Delta^{\mu\rho\nu}\Big)$$

$$= T^{\mu\nu} - T^{\nu\mu} + \partial_\rho\Delta^{\rho\mu\nu}$$

$$= 0 \quad (\text{voir } 2])$$

4] $$A'^\mu = A^\mu + \omega^\mu{}_\nu A^\nu$$

$$= A^\mu - \frac{i}{2}\,\omega^{\alpha\beta}\left(\Sigma_{\alpha\beta}\right)^{\mu\sigma} A_\sigma$$

$$\omega^{\alpha\beta}\left[\eta^\mu_\alpha \eta_{\beta\nu} A^\nu + \frac{i}{2}\left(\Sigma_{\alpha\beta}\right)^{\mu\nu} A_\nu\right] = 0$$

$$\frac{\omega^{\alpha\beta} A_\nu}{2}\left[\eta^\mu_\alpha \eta^\nu_\beta - \eta^\nu_\alpha \eta^\mu_\beta + i\left(\Sigma_{\alpha\beta}\right)^{\mu\nu}\right] = 0$$

$$\hookrightarrow \left(\Sigma_{\alpha\beta}\right)^{\mu\nu} = i\left(\eta^\mu_\alpha \eta^\nu_\beta - \eta^\nu_\alpha \eta^\mu_\beta\right).$$

$$\Delta^{\mu\alpha\beta} = i\, F^\mu{}_\rho \left(\Sigma^{\alpha\beta}\right)^{\rho\sigma} A_\sigma$$

~~$= - F^\mu{}_\rho\left(\eta^\rho_\alpha \eta^\nu_\beta - \eta^\nu_\alpha \eta^\rho_\beta\right) A$~~

$$= - F^{\mu}{}_{\rho} \left(\eta^{\alpha\rho} \eta^{\beta\sigma} - \eta^{\alpha\sigma} \eta^{\beta\rho} \right) A_{\sigma}$$

$$= - F^{\mu\alpha} A^{\beta} + F^{\mu\beta} A^{\alpha}$$

$$X^{\rho\mu\nu} = \frac{1}{2}$$

$$\Delta^{\rho\mu\nu} = - F^{\rho\mu} A^{\nu} + F^{\rho\nu} A^{\mu}$$

$$X^{\rho\mu\nu} = \frac{1}{2} \Big(- F^{\rho\mu} A^{\nu} + F^{\rho\nu} A^{\mu} + F^{\mu\rho} A^{\nu} - F^{\mu\nu} A^{\rho} + F^{\nu\rho} A^{\mu} - F^{\nu\mu} A^{\rho} \Big)$$

$$= F^{\mu\rho} A^{\nu} = - F^{\rho\mu} A^{\nu}$$

qui est bien la recette qu'on avait donnée dans l'exercice 3

5] $\delta\psi = -\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \psi$ avec $\sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}]$

$$= -\frac{i}{2} \omega^{\mu\nu} (\Sigma_{\mu\nu}) \psi$$

↳ $\Sigma_{\mu\nu} = \frac{\sigma_{\mu\nu}}{2} = \frac{i}{4} [\gamma_{\mu}, \gamma_{\nu}]$

$$\delta\bar{\psi} = \frac{i}{2} \psi^{+} (\Sigma^{+}_{\mu\nu}) \omega^{\mu\nu} \gamma^{0}$$

$$= \frac{i}{2} \bar{\psi} \gamma^{0} (\Sigma^{+}_{\mu\nu}) \gamma^{0} \omega^{\mu\nu}$$

$$= \frac{i}{2} \bar{\Sigma}_{\mu\nu} \omega^{\mu\nu} \psi$$

$$\bar{\Sigma}_{\mu\nu} = \gamma^{0} (\Sigma^{+}_{\mu\nu}) \gamma^{0}$$