1 Flow equations for the $O(N)\times O(2)/O(N-2)\times O(2)$ model.

The recursion equations at quartic order for $\kappa, \lambda, \mu, \omega$ and $\eta = -d\log Z_k/dt$ are written in terms of dimensionless renormalized quantities:

$$\kappa_r = Z\ k^{2-d}\kappa\ ;\ \lambda_r = Z^{-2}\ k^{\frac{d-4}{2}}\lambda\ ;\ \mu_r = Z^{-\frac{3}{2}}\ k^{\frac{d-6}{2}}\mu\ ;\ \omega_r = Z^{-2}\ k^{\frac{d-2}{2}}\omega \tag{1}$$

$$\left\{\begin{aligned}
\frac{d\kappa_r}{dt} &= -(d-2+\eta)\kappa_r - 2v_d\Big[\frac{1}{2}L^d_{01}[0,0,\kappa_r\omega_r] + \frac{\omega_r}{\lambda_r}L^{2+d}_{0,1}[0,0,\kappa_r\omega_r] + (N-2)L^d_{10}[0,0,0] \\
&\quad + \frac{3}{2}L^d_{10}[\kappa_r\lambda_r,0,0] + \left(1+2\frac{\mu_r}{\lambda_r}\right)L^d_{10}[\kappa_r\mu_r,0,0]\Big] \\
\frac{d\lambda_r}{dt} &= (d-4+2\eta)\lambda_r - \frac{v_d}{2}\Big[2\lambda_r^2(N-2)L^d_{20}[0,0,0] + \lambda_r^2 L^d_{02}[0,0,\kappa_r\omega_r] + 9\lambda_r^2 L^d_{20}[\kappa_r\lambda_r,0,0] \\
&\quad + 2(\lambda_r+2\mu_r)^2 L^d_{20}[\kappa_r\mu_r,0,0] + 4\lambda_r\omega_r L^{2+d}_{02}[0,0,\kappa_r\omega_r] + 4\omega_r^2 L^{4+d}_{02}[0,0,\kappa_r\omega_r]\Big] \\
\frac{d\mu_r}{dt} &= (d-4+2\eta)\mu_r + v_d\mu_r\Big[-\frac{2}{\kappa_r}L^d_{01}[0,0,\kappa_r\omega_r] + \frac{3(2\lambda_r+\mu_r)}{\kappa_r(\mu_r-\lambda_r)}L^d_{10}[\kappa_r\lambda_r,0,0] \\
&\quad + \frac{8\lambda_r+\mu_r}{\kappa_r(\lambda_r-\mu_r)}L^d_{10}[\kappa_r\mu_r,0,0] + \mu_r L^d_{11}[\kappa_r\mu_r,0,\kappa_r\omega_r] + \mu_r(N-2)L^d_{20}[0,0,0]\Big] \\
\eta &= -\frac{v_d}{d\kappa_r}\Big[(4-d)\kappa_r\omega_r L^d_{01}[0,0,\kappa_r\omega_r] + 2\kappa_r^2\omega_r^2 L^{2+d}_{02}[0,0,\kappa_r\omega_r] + 2(-2+d)\kappa_r\omega_r L^d_{10}[0,0,0] \\
&\quad + 2M^d_{02}[0,0,\kappa_r\omega_r] - 4M^d_{11}[0,0,\kappa_r\omega_r] + 2M^d_{20}[0,0,\kappa_r\omega_r] + 2\kappa_r^2\lambda_r^2 M^d_{2,2}[\kappa_r\lambda_r,0,0] + \\
&\quad + 4\kappa_r^2\mu_r^2 M^d_{2,2}[\kappa_r\mu_r,0,0] + 4\kappa_r\omega_r N^d_{02}[0,0,\kappa_r\omega_r] - 8\kappa_r\omega_r N^d_{11}[0,0,\kappa_r\omega_r] + 4\kappa_r\omega_r N^d_{20}[0,0,\kappa_r\omega_r]\Big] \\
\frac{d\omega_r}{dt} &= (d-2+2\eta)\omega_r - \frac{2v_d}{d\kappa_r^2}\Big[\kappa_r\omega_r\Big\{\frac{(4-d)}{2}L^d_{01}[0,0,\kappa_r\omega_r] + \frac{(d-16)}{2}L^d_{01}[\kappa_r\lambda_r,0,\kappa_r\omega_r] \\
&\quad + \kappa_r\omega_r L^{2+d}_{02}[0,0,\kappa_r\omega_r] - 3\kappa_r\omega_r L^{2+d}_{02}[\kappa_r\lambda_r,0,\kappa_r\omega_r] + (d-2)L^d_{10}[0,0,0] - (d-8)L^d_{10}[\kappa_r\lambda_r,0,0] \\
&\quad + 8\kappa_r\lambda_r L^d_{11}[\kappa_r\lambda_r,0,\kappa_r\omega_r] + 2\kappa_r\omega_r(N-2)L^{2+d}_{20}[0,0,0] + 2\kappa_r\omega_r L^{2+d}_{20}[\kappa_r\mu_r,0,0]\Big\} + M^d_{02}[0,0,\kappa_r\omega_r] \\
&\quad - M^d_{02}[\kappa_r\lambda_r,0,\kappa_r\omega_r] - 2M^d_{11}[0,0,\kappa_r\omega_r] + 2M^d_{11}[\kappa_r\lambda_r,0,\kappa_r\omega_r] + M^d_{20}[0,0,\kappa_r\omega_r] - M^d_{20}[\kappa_r\lambda_r,0,\kappa_r\omega_r] \\
&\quad + \kappa_r^2\lambda_r^2 M^d_{22}[\kappa_r\lambda_r,0,0] + 2\kappa_r^2\mu_r^2 M^d_{22}[\kappa_r\mu_r,0,0] + 2\kappa_r\omega_r N^d_{02}[0,0,\kappa_r\omega_r] - 4\kappa_r\omega_r N^d_{02}[\kappa_r\lambda_r,0,\kappa_r\omega_r] \\
&\quad - 4\kappa_r\omega_r N^d_{11}[0,0,\kappa_r\omega_r] + 8\kappa_r\omega_r N^d_{11}[\kappa_r\lambda_r,0,\kappa_r\omega_r] + 2\kappa_r\omega_r N^d_{20}[0,0,\kappa_r\omega_r] - 4\kappa_r\omega_r N^d_{20}[\kappa_r\lambda_r,0,\kappa_r\omega_r]\Big]
\end{aligned}\right. \tag{2}$$

where $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$.

2 Threshold Functions

Definition of the threshold functions which enter in Eq.(2):

$$\bullet L^d_{n_1 n_2}[M_1, M_2, u] = \int_0^\infty dy\ 2\ y^{1+d/2}\ r'(y) \left\{ \frac{n_1}{[M_1 + y(1 + r(y))]^{n_1+1}[M_2 + y(1 + u + r(y))]^{n_2}} \right.$$
$$\left. + \frac{n_2}{[M_2 + y(1 + u + r(y))]^{n_2+1}[M_1 + y(1 + r(y)])]^{n_1}} \right\} \tag{3}$$

$$\bullet M^d_{n_1 n_2}[M_1, M_2, u] = \int_0^\infty dy\ y^{1+d/2}\ \frac{(1 + r(y) + y\, r'(y))^2}{[M_1 + y\ (1 + r(y)])^{n_1}\ [M_2 + y\ (1 + u + r(y))]^{n_2}} \times$$
$$\left\{ 2\, y\ r'(y) \left(\frac{n_1}{M_1 + y\ (1 + r(y))} + \frac{n_2}{M_2 + y\ (1 + u + r(y))} \right) - \frac{4\ (2\, r'(y) + y\, r''(y))}{1 + r(y) + y\, r'(y)} \right\} \tag{4}$$

$$\bullet N^d_{n_1 n_2}[M_1, M_2, u] = \int_0^\infty dy y^{1+\frac{d}{2}}\ \frac{(1 + r(y) + y\, r'(y))}{[M_1 + y\ (1 + r(y)])^{n_1}\ [M_2 + y\ (1 + u + r(y))]^{n_2}} \times$$
$$\left\{ 2\, y\ r'(y) \left(\frac{n_1}{M_1 + y\ (1 + r(y))} + \frac{n_2}{M_2 + y\ (1 + u + r(y))} \right) - \frac{2\ (2\, r'(y) + y\, r''(y))}{1 + r(y) + y\, r'(y)} \right\} \tag{5}$$

with $y = q^2/k^2$, $r(y) = 1/(e^y - 1)$

3 Expansion of the flow equations in powers of $\epsilon = d - 2$

For $d = 2 + \epsilon$ the values of the masses $m_{1r} = \kappa_r \lambda_r$ and $m_{2r} = \kappa_r \mu_r$ at the fixed point behave as $1/\epsilon$. We can thus expand our flow equations in powers ϵ. To do so, we need to know the behaviour of the threshold functions for large values of the first two arguments M_1 and M_2. At dominant order, we get:

$$v_2 = 1/(8\pi), \tag{6}$$

$$\begin{aligned}
&L^2_{10}[0,0,u] \simeq -2, && L^2_{01}[M_1,0,u] \simeq -2/(1+u),\\
&L^4_{11}[0,0,u] \simeq -2/(1+u), && L^2_{11}[M_1,0,u] \simeq -2/(M_1(1+u)),\\
&L^4_{20}[0,0,0] \simeq -2, && L^4_{02}[M_1,0,u] \simeq -2/(1+u)^2,\\
&L^2_{20}[0,0,0] \simeq -4\ln[2], && L^6_{12}[0,0,u] \simeq -2/(1+u)^2,\\
&L^2_{30}[0,0,0] \simeq -6\ln[27/16], && L^2_{40}[0,0,0] \simeq -176\ln[2] + 108\ln[3],\\
&M^2_{11}[0,0,u] \simeq -2/(1+u), && M^2_{02}[M_1,0,u] \simeq -2/(1+u)^2,\\
&M^2_{20}[0,0,u] \simeq -2, && M^2_{22}[M_1,0,0] \simeq -2/M_1^2,\\
&M^2_{22}[0,M_1,0] \simeq -2/M_1^2, && N^2_{02}[M_1,0,u] \simeq -2/(1+u)^2,\\
&N^2_{11}[0,0,u] \simeq -2/(1+u), && N^2_{20}[0,0,u] \simeq -2
\end{aligned} \tag{7}$$

The other threshold functions give subdominant contributions.

4 Expansion of the flow equations in powers of $\epsilon = 4 - d$

In dimension $d = 4 - \epsilon$ the values of the coupling constants λ_r and μ_r at the fixed point take values of order 1 in ϵ, whereas $\kappa_r \omega_r$ is of order 2. We can expand our flow equations around the fixed point. We need to know the behaviour of the threshold functions for M_1 and M_2 small, and the third argument being null. We get, to dominant order:

$$
\begin{aligned}
& v_4 = 1/(32\pi^2), \\
& L_{10}^4[M_1, 0, 0] \simeq -2(1 - M_1), \\
& L_{11}^4[M_1, 0, 0] \simeq -2, \\
& L_{01}^4[0, M_1, 0] \simeq -2(1 - M_1), \\
& L_{20}^4[M_1, 0, 0] \simeq -2, \\
& L_{02}^4[0, M_1, 0] \simeq -2, \\
& M_{22}^4[M_1, 0, 0] \simeq -1,
\end{aligned}
\tag{8}
$$

The other threshold functions give subdominant contributions.