# Precise large deviations for statistical field theories with weak noise

LPTMC seminar, 03/25/2025


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based on:
**J. Phys. A: Math. Theor. 54 235003 (2021)**
**J. Stat. Phys. 190, 50 (2023)**
**Stat. Comput. 33, 137 (2023)**
**arXiv:2502.20114 (2025)**

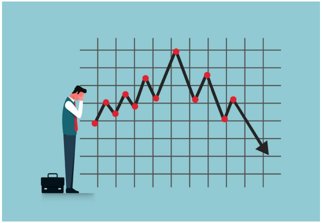
# Very rough outline

1. Motivation and introduction
2. Technical results and details
3. Applications of the theory

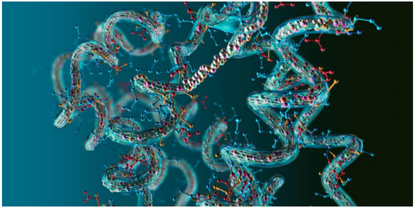
(about 15 minutes & 10 slides each)

# Motivation and introduction

# Extreme events: rare, but high impact. . .

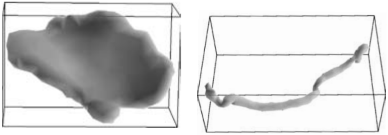


# ...or maybe not so rare on a different timescale

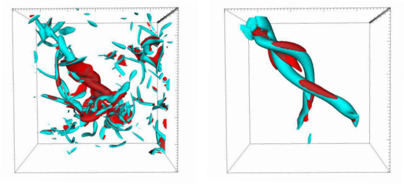


# Extreme events in statistical physics: Intermittency in fluids

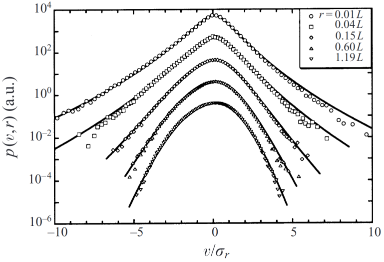
Incompressible 3D Navier–Stokes turbulence at high Re dominated by intense localized events of vortex stretching and energy dissipation



(Moisy, Jiménez 2004)



(Buaria, Pumir, Bodenschatz, Yeung 2019)

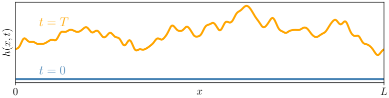



(Renner, Peinke, Friedrich 2000)

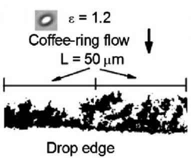
**Goal**: Use these structures to calculate tail statistics in turbulence

Began with: Gurarie, Migdal, Falkovich, Kolokolov, Lebedev, Chernykh, Stepanov, . . . in 1990s

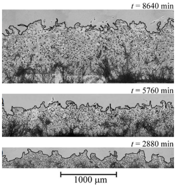
**Task**: Find detailed height statistics for Kardar–Parisi–Zhang (KPZ) eq.



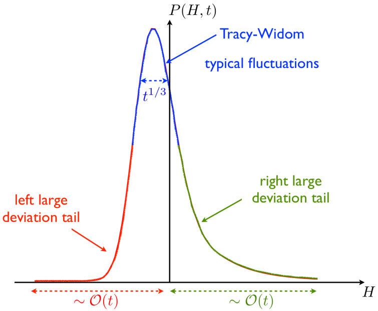

$$\partial_t h = \nu \partial_{xx} h + \tfrac{\lambda}{2}(\partial_x h)^2 + \sqrt{\mathcal{D}}\eta$$




(Yunker, Lohr, Still, Borodin, Durian, Yodh 2013)




(Huergo, Pasquale, Bolzán, Arvia 2010)

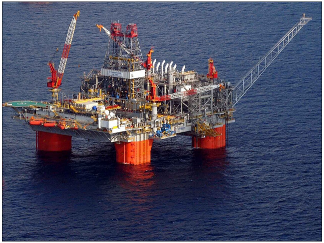



(Le Doussal, Majumdar, Schehr 2016)

**Question**: How do non-Gaussian statistics start to arise at small times $t$?

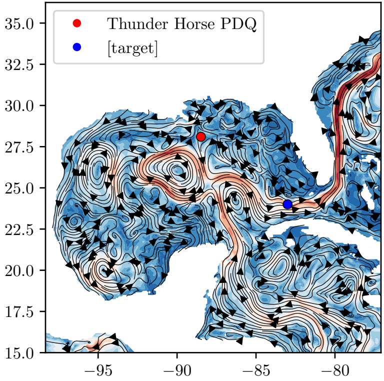
Studied by: Kolokolov, Korshunov, Meerson, Katzav, Vilenkin, Kamenev, Sasorov, Smith, Krajenbrink, Le Doussal, . . . since 2010s

# Extreme events in fluid dynamics: oil spill events



Thunder Horse PDQ offshore oil platform

After Hurricane Dennis 2005



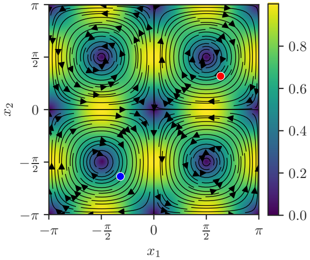

- pollutant diffuses & advected by currents
- limited measurements $\Rightarrow$ add randomness
- what is $\mathbb{P}[\text{pollutant concentration} \geq z]$?

# Extreme events in fluid dynamics: oil spill events

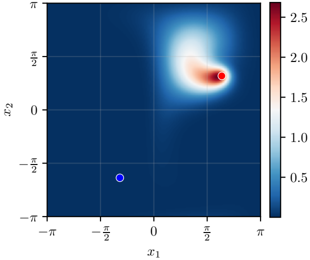
2d advection-diffusion equation for pollutant concentration $c(x,t)$

$$\partial_t c = \underbrace{-(v(x)\cdot\nabla)c}_{\text{advection}}\ \underbrace{-(w(x,t)\odot\nabla)c}_{\text{random advection}}\ \underbrace{+D_0\Delta c}_{\text{diffusion}}\ \underbrace{+s}_{\text{source}}$$

with $v$ given, $w$ Gaussian random vector field (white-in-time, smooth in space)




background field $v$




$c(x,T)$ for $w=0$

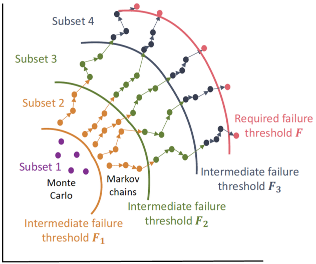
$\Rightarrow$ want to estimate tail probability $P(z) := \mathbb{P}[c(\text{target}, T) \geq z] \ll 1$

# Estimating rare event probabilities $\mathbb{P}[F(\eta) \geq z] \ll 1$

## 1. rare event sampling

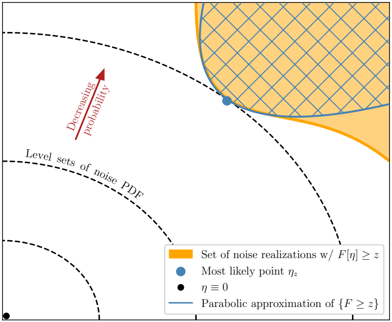
$$\cancel{\mathbb{P}[F(\eta) \geq z] \approx \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{F(\eta_i) \geq z\}}}$$

$\Rightarrow$ use importance sampling or subset simulation




(Chakroborty et al. 2023)

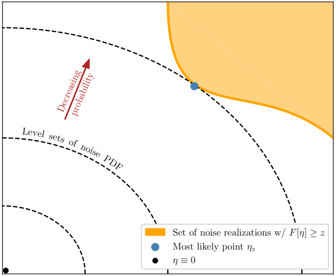
## 2. asymptotic approximations




large deviation theory (LDT), instanton calculus

$\Rightarrow$ systematic; provides insights into physics of rare events

# LDT: how it works in finite dimensions




Setup:

- $\eta \sim \mathcal{N}(0, \mathrm{Id})$ random vector in $\mathbb{R}^N$
- want to calculate $\mathbb{P}[F(\eta) \geq z] \ll 1$
- approximate event set by most likely point

Requires:

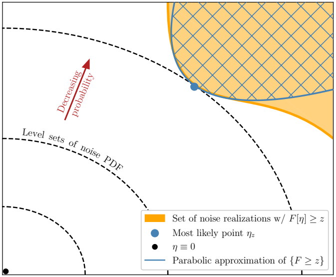
1. most likely point $\eta_z$

Optimize $\eta_z = \arg\min \frac{1}{2}\|\eta\|^2$ s.t. $F(\eta) = z$, then approximate

$$\boxed{P(z) \asymp \exp(-I(z))}$$

log-asymptotic estimate with **rate function** $I(z) = \frac{1}{2}\|\eta_z\|^2$

# Precise LDT: how it works in finite dimensions




Setup:

- $\eta \sim \mathcal{N}(0, \mathrm{Id})$ random vector in $\mathbb{R}^N$
- want to calculate $\mathbb{P}[F(\eta) \geq z] \ll 1$
- approximate event set by paraboloid

Requires:

1. most likely point $\eta_z$
2. curvatures at $\eta_z$ via $\nabla^2 F(\eta_z)$

**First** optimize $\eta_z = \arg\min \frac{1}{2}\|\eta\|^2$ s.t. $F(\eta) = z$, **then**:

$$\boxed{P(z) \sim (2\pi)^{-1/2} C(z) \exp(-I(z))}$$

$$I(z) = \frac{1}{2}\|\eta_z\|^2 \quad \& \quad C(z) = \left[2I(z) \det\left(\mathrm{Id} - \lambda_z \mathrm{pr}_{\eta_z^\perp} \nabla^2 F(\eta_z) \mathrm{pr}_{\eta_z^\perp}\right)\right]^{-1/2}$$

# Central question of this project

Can we calculate the same kind of **precise LDT estimates** in infinite dimensions, so for Langevin equations/field theories?

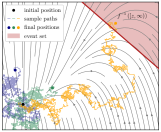
$\Rightarrow$ develop robust&general methods i.p. for prefactor computation

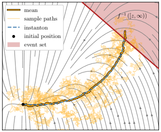
# Technical results and details

# Setup: Extreme events for Langevin equations via LDT

Consider the Langevin equation

$$\begin{cases} \dot{\phi} & = b(\phi) + \sqrt{\varepsilon}\sigma(\phi)\eta(t)\,, \\ \phi(0) & = \phi_0 \in \mathbb{R}^n\,, \end{cases}$$




with small $\varepsilon$ and final-time observable $f\,(\phi(T)) \in \mathbb{R}$.

**Our goal**:

- Precise estimate for tail probability

$$\boxed{P(z) \overset{\varepsilon \downarrow 0}{\sim} (\varepsilon/2\pi)^{1/2} C(z) \exp\left(-S[\phi_z]/\varepsilon\right)}$$

  of $f(\phi(T))$ with leading **pre-exponential** term $C(z)$
- Closed-form result for $C(z)$, can evaluate for field theories $n \to \infty$
- For non-gradient system $b \neq -\nabla V$, finite times $T < \infty$

# Setup: Extreme events for Langevin equations via LDT

Consider the Langevin equation

$$\begin{cases} \dot{\phi} & = \\ \phi(0) & = \end{cases}$$

with small $\varepsilon$ and final

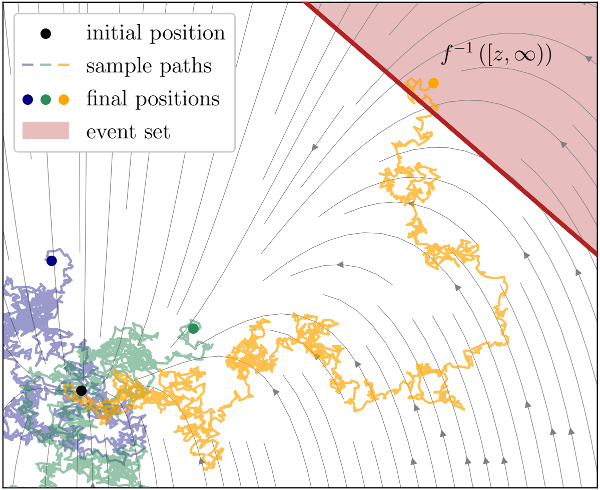
**Our goal**:

- Precise estimate

$$\boxed{P(z}$$

of $f(\phi(T))$ with

- Closed-form resu
- For non-gradient



Consider the Langevin equation

$$\begin{cases} \dot{\phi} & = \\ \phi(0) & = \end{cases}$$

with small $\varepsilon$ and final

**Our goal**:

- Precise estimate

$P(z$

of $f(\phi(T))$ with

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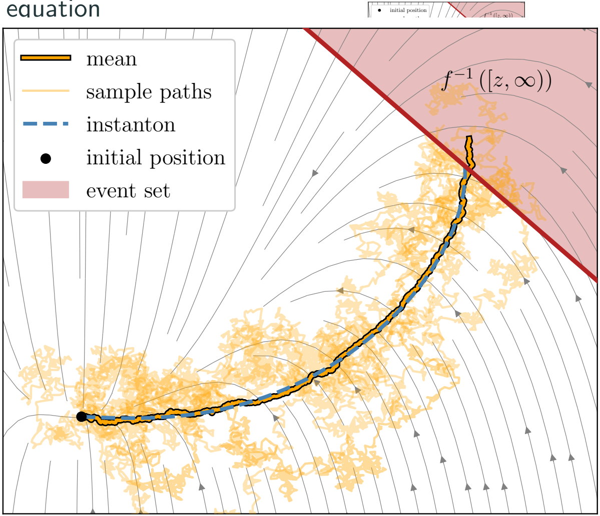
- For non-gradient

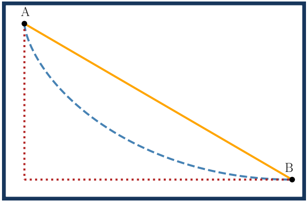


# The approximation: Instanton & Gaussian fluctuations

Semiclass. expansion of path integral for PDF $\rho(z) = \mathbb{E}[\delta(f(\phi(T)) - z)]$

$$\rho(z) = \int_{\phi(0)=\phi_0} \mathscr{D}\phi \; \delta(f(\phi(T)) - z) J[\phi] \exp\left(-S[\phi]/\varepsilon\right)$$

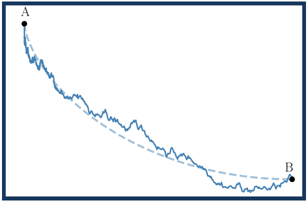
**Step 1: instanton**




Find $\phi_z = \arg\min_\phi S[\phi]$ under boundary conditions

$\Rightarrow \rho(z) \asymp \exp\left(-S[\phi_z]/\varepsilon\right)$

**Step 2: fluctuations**

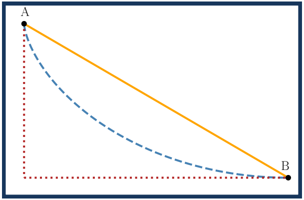



Expand path integral to second order around $\phi_z$

$\Rightarrow \rho(z) \sim \mathcal{N} J[\phi_z] e^{-\frac{S}{\varepsilon}} / \sqrt{\mathrm{Det}(\delta^2 S)}$

# Our contributions: optimization algorithms and determinants

**Step 1: instanton**

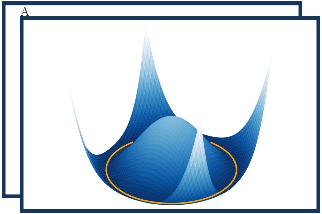



Formulated minimization as **optimal control problem**

$$\min_\eta \left\{ S[\eta] - \lambda(f(\phi[\eta](T)) - z) + \frac{\mu}{2}(f(\phi[\eta](T)) - z)^2 \right\}$$

to automate, apply to non-convex settings, and scale to large $n$

**Step 2: fluctuations**




Derived 2 strategies to **compute prefactor** $C(z)$ in practice:

(i) solve $n \times n$ matrix Riccati diff. eq. initial value problem

(ii) iteratively find largest eigenvalues for Fredholm det.

Extended results to zero modes

$\Rightarrow$ instanton calc. more robust, prefactor calc. now possible at all

# Details on the prefactor: (i) Riccati approach

**Idea:** Derive diff. eq. for covariance matrix of fluctuations around $\phi_z$

**Riccati equation** for $Q\colon [0, T] \to \mathbb{R}^{n \times n}$ with $Q(0) = 0$ and

$$\dot{Q} = \sigma\sigma^\top + Q\nabla b\left(\phi_z\right)^\top + \nabla b\left(\phi_z\right) Q + Q\left\langle \nabla^2 b(\phi_z), \theta_z \right\rangle Q\,,$$

(for additive noise $\sigma = \text{const.}$, generalization to multiplicative noise straightforward) s.t.

$$C(z) = |\lambda_z|^{-1} \exp\left\{\frac{1}{2}\int_0^T \operatorname{tr}\left[\left\langle \nabla^2 b(\phi_z), \theta_z \right\rangle Q\right] \mathrm{d}t\right\} \times$$
$$\times \left[\det\left(U_z\right)\left\langle \nabla f\left(\phi_z(T)\right), Q(T) U_z^{-1} \nabla f\left(\phi_z(T)\right)\right\rangle\right]^{-1/2}$$

**Many roads to Rome:** WKB analysis for KBE, Feynman–Kac, path integral discretization, Gel'fand–Yaglom method, Forman's theorem

See: Maier, Stein 1996; Lehmann, Reimann, Hänggi 2003; TS, Grafke, Grauer 2021; Bouchet, Reygner 2022; TS, Grafke, Grauer 2023; Grafke, Schäfer, Vanden-Eijnden 2024; Rosinberg, Tarjus, Munakata 2024

# Discussion of the prefactor: (i) Riccati approach

Properties:

- $n \times n$ symmetric matrix diff. eq. for Langevin eq. in $\mathbb{R}^n$
- initial-value problem, solve once
- nonlinear, but can linearize by Radon transformation $Q = \gamma \zeta^{-1}$

Advantages :)

- can (always formally) solve exactly via time-ordered exponential
- can solve exactly for simple enough instanton
- can use perturbation theory in additional small parameters

Disadvantages :(

- numerical time integration can be tricky (pseudo-singularities)
- very hard to use in high dimensions $n \gg 1$, i.e. beyond (1+1) dim. field theories $\Rightarrow$ $n \times n$ too large, and $\gamma, \zeta$ blow up

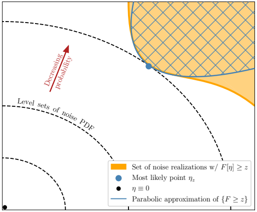
# Details on the prefactor: (ii) Fredholm determinant

Analogous to finite dim. for **additive noise**

$$\begin{cases} \dot{\phi} & = b(\phi) + \sigma \eta \,, \\ \phi(0) & = \phi_0 \in \mathbb{R}^n \,, \end{cases}$$

with e.g. final-time observable $f\left(\phi(T)\right) \in \mathbb{R}$.

Then with $F \colon \eta \mapsto f(\phi(T))$ for $\eta = \mathrm{d}W/\mathrm{d}t$:




- keep noise variable as main object of interest:

$$\boxed{P(z) \approx (2\pi)^{-1/2} C(z) \exp(-I(z))}$$

with $I(z) = \frac{1}{2}\|\eta_z\|^2_{L^2}$; where $\eta_z = \arg\min \frac{1}{2}\|\eta\|^2_{L^2}$ s.t. $F(\eta) = z$

- prefactor with **Fredholm determinant of trace-class operator**

$$C(z) = \left[ 2I(z) \det\left( \mathrm{Id} - \lambda_z \mathrm{pr}_{\eta_z^\perp} \left.\frac{\delta^2 F}{\delta \eta^2}\right|_{\eta_z} \mathrm{pr}_{\eta_z^\perp} \right) \right]^{-1/2}$$

# Details on the prefactor: (ii) Fredholm determinant

This prefactor is easy to evaluate numerically, even for very large $n$!!!

$$C(z) = \left[ 2I(z) \det \left( \mathrm{Id} - \lambda_z \mathrm{pr} \left. \frac{\delta^2 F}{\delta \eta^2} \right|_{\eta_z} \mathrm{pr} \right) \right]^{-1/2} \approx \left[ 2I(z) \prod_{i=1}^{M} (1 - \mu_i) \right]^{-1/2}$$

Need to find largest eigenvalues $\mu_i$ of (projected) second variation of noise-to-observable map $F$, can do that **iteratively**.

$$\boxed{\left. \frac{\delta^2 F}{\delta \eta^2} \right|_{\eta_z} \delta\eta = \sigma^\top \zeta}$$

with $\begin{cases} \dot{\gamma} = \nabla b(\phi_z)\gamma + \sigma \delta\eta \,, \\ \dot{\zeta} = -\left\langle \nabla^2 b(\phi_z), \theta_z \right\rangle \gamma - \nabla b^\top(\phi_z)\zeta \,, \end{cases}$ & $\begin{cases} \gamma(0) = 0 \,, \\ \zeta(T) = \lambda_z \nabla^2 f(\phi_z(T))\gamma(T) \,. \end{cases}$

see also: Bureković, Schäfer, Grauer 2024, for an application to stochastic NLS

# What about multiplicative noise? arXiv:2502.20114 (2025)

Now consider

$$\begin{cases} \dot{\phi} & = b(\phi) + \sigma(\phi)\eta\,, \\ \phi(0) & = \phi_0 \in \mathbb{R}^n\,, \end{cases}$$

- still have $P(z) \approx (2\pi)^{-1/2} C(z) \exp(-I(z))$ with same $I(z)$
- BUT: Hessian $A = \delta^2 F / \delta\eta^2$ no longer guaranteed to be trace-class!

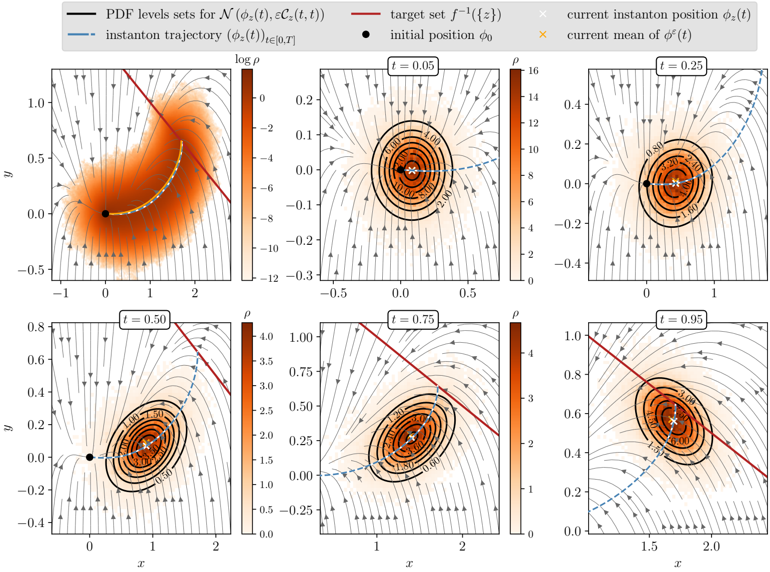
$A$ is still *Hilbert–Schmidt*; then natural replacement of Fredholm determinant is **Carleman–Fredholm determinant**

$$\boxed{\det{}_2(\mathrm{Id} - A) := \det\left((\mathrm{Id} - A)\exp(A)\right)}$$

[think $\det_2(\mathrm{Id} - A) \approx \det\left((\mathrm{Id} - A)(\mathrm{Id} + A)\right) = \det\left(\mathrm{Id} - A^2\right)$]

$\Rightarrow$ generalize $C(z)$ w/ $\det_2$ (cf. Ben Arous (1988)), idea: $\pm\,\mathrm{tr}\,A$ counter term

# Instantons and fluctuations can be observed in practice



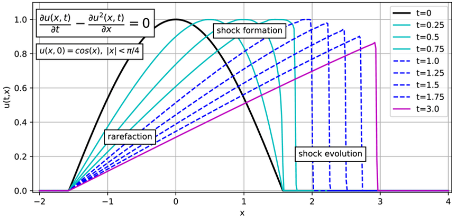
# Applications of the theory

# Application: Gradient statistics in viscous 1D Burgers eq.

Burgers eq. for $u\colon [0, L] \times [0, T] \to \mathbb{R}$ with stochastic large-scale forcing

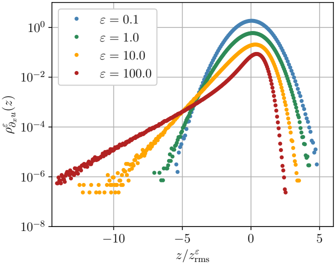
$$\begin{cases} \partial_t u + u\partial_x u - \partial_{xx} u = \sqrt{\varepsilon}\sigma * \eta\,, \\ u(\cdot, 0) = 0 \end{cases}$$

Shock formation by nonlinear advection




(Feo and Celico 2021)

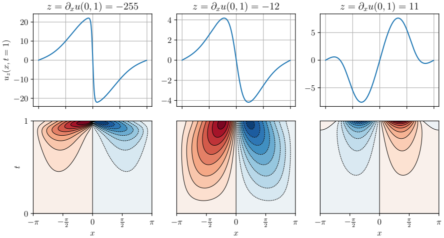
PDF of $\partial_x u(x = 0, t = T)$



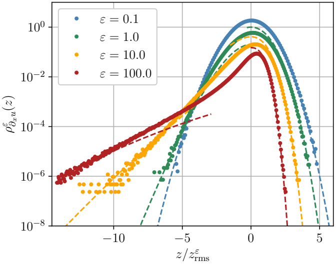
# Gradient statistics in 1D Burgers: instanton computation

$$\min_{\eta, u} \frac{1}{2} \int_0^T \mathrm{d}t \int_0^L \mathrm{d}x \, \eta(x,t)^2 \quad \text{s.t.} \quad \begin{cases} \partial_t u + u \partial_x u - \partial_{xx} u = \sigma * \eta \,, \\ u(\cdot, 0) = 0 \,, \\ \partial_x u(0, T) = z \end{cases}$$

Instanton fields

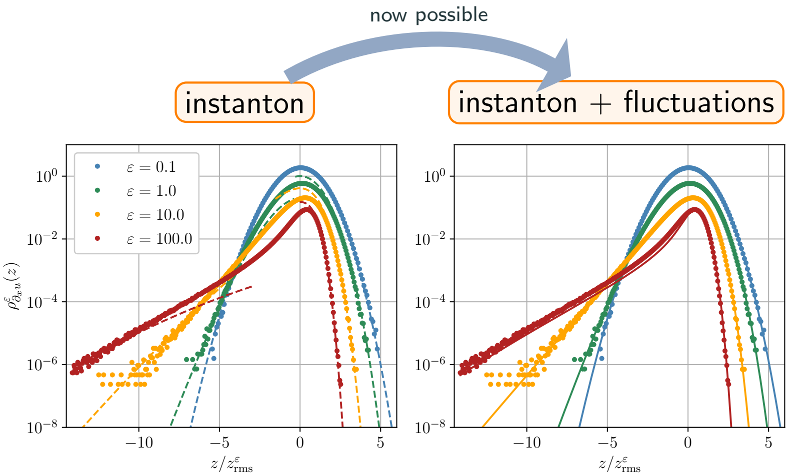



PDF of $\partial_x u(0, T)$




$\Rightarrow$ log-asymptotic PDF estimate $\propto \exp\{-S[u_z]/\varepsilon\}$, fit constant prefactor

# Gradient statistics in 1D Burgers: prefactor computation

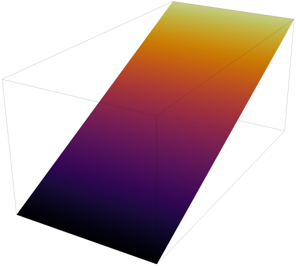



$$\partial_t Q(x,y,t) = \sigma^{*2}(x-y) - \left[\partial_x\left(u_z(x)\cdot\right) + \partial_y\left(u_z(y)\cdot\right)\right] Q(x,y,t) + \left[\partial_{xx} + \partial_{yy}\right] Q(x,y,t) + \int_0^L Q(x,x',t)\partial_{x'} p_z(x',t) Q(x',y,t)\mathrm{d}x'$$

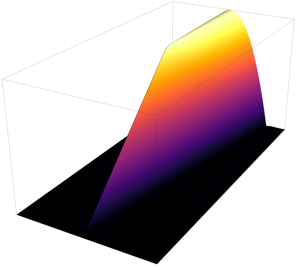
# Application: Extreme average growth events in 1D KPZ eq.

Consider KPZ eq. $\partial_t h = \nu \partial_{xx} h + \frac{\lambda}{2}(\partial_x h)^2 + \sqrt{\mathcal{D}}\eta$ on ring of length $L$ with $h(x,0) = 0$, and analyze tail statistics of $\frac{1}{L}\int_0^L h(x,T)\mathrm{d}x$ at small $T$.

**Instantons** (with multiple DPTs here, depending on $L$):



Spatially uniform growth

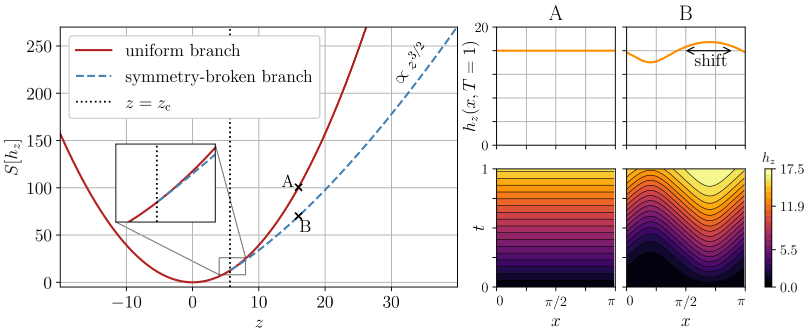


Soliton

(Joint work with P. Sasorov and B. Meerson; further analysis of DPT with O. Shpielberg)

# Application: Extreme average growth events in 1D KPZ eq.

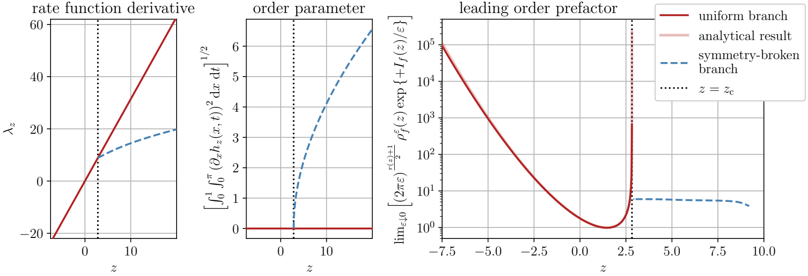
Rate function at small $L$ (2nd order dynamical phase transition to slight modulation):




Interesting $L$-$z$ phase diagram in system, but no time to talk about it

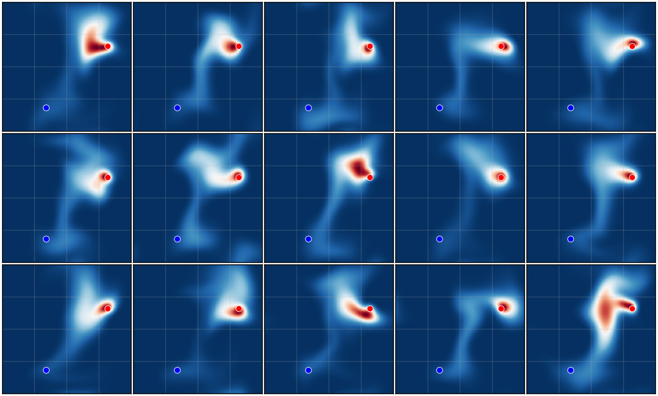
# Application: Extreme average growth events in 1D KPZ eq.

Gaussian fluctuations at small $L$ after Cole–Hopf transform
(analytical result below DPT, numerical result including zero mode w/ Riccati afterwards):

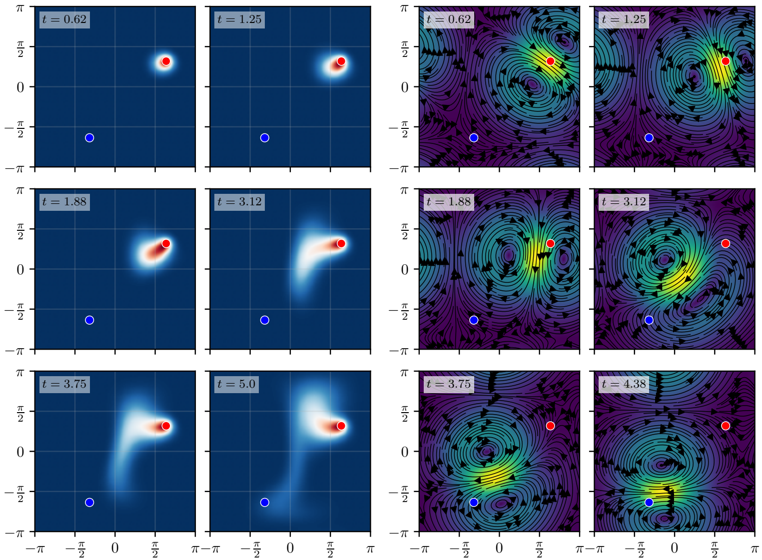


# 2d advection-diffusion example: reminder

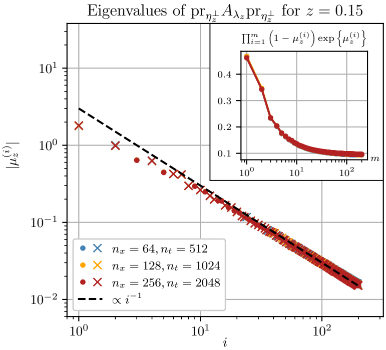
$$\partial_t c = \underbrace{-(v(x) \cdot \nabla) c}_{\text{advection}} \underbrace{-(w(x,t) \odot \nabla) c}_{\text{random advection}} \underbrace{+D_0 \Delta c}_{\text{diffusion}} \underbrace{+s}_{\text{source}}, \quad \mathbb{P}[c(\text{target}, T) \geq z] = ?$$



# First: optimization, finding the instanton



# Then: computing prefactor $C(z)$ as Carleman–Fredholm det



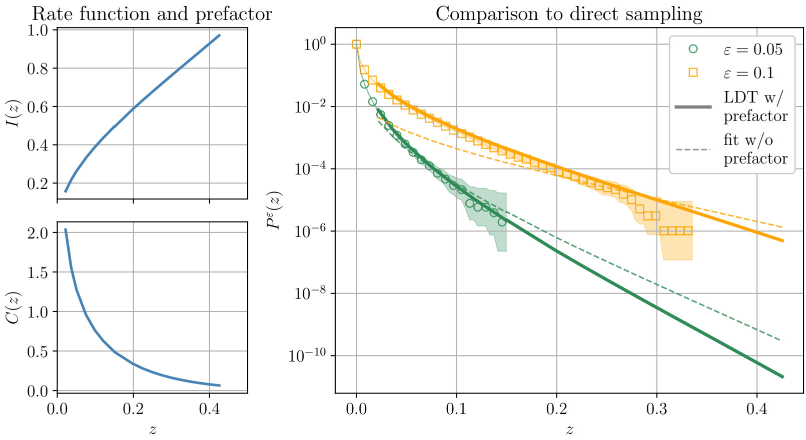

Use dominant eigenvalues $\mu_i$

$$\det_2(\mathrm{Id} - A) \approx \prod_{i=1}^{M} (1-\mu_i) e^{\mu_i}$$

found iteratively and matrix-free with 2nd order adjoints in JAX

scaling $\mu_i \propto 1/i$, indeed not trace-class but Hilbert–Schmidt

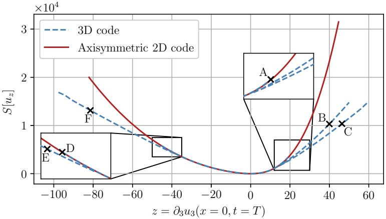
# Resulting estimates compared to direct sampling




at different noise strengths $\varepsilon$ of the random advecting field

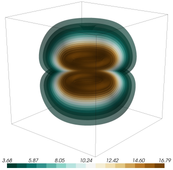
**Step 1:** Find instanton

(most probable velocity field $u$ in 3D space and time leading to given observable value $z$ starting from $u(\cdot, t=0) = 0$)

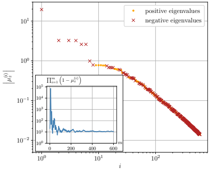



**Then:** Results for $\partial_3 u_3(0, T) = -25$

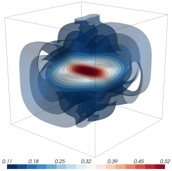
(including fluctuations)




instanton $\nabla \times u_z(\cdot, T)$




spectrum of second variation




dominant fluctuation

$\Rightarrow$ correct prediction $\mathbb{P}[\partial_3 u_3(0, T) \leq -25] \approx 1.5 \cdot 10^{-5}$ at $\mathrm{Re}_\lambda = 6.4$

# Conclusion

# Conclusion & Outlook

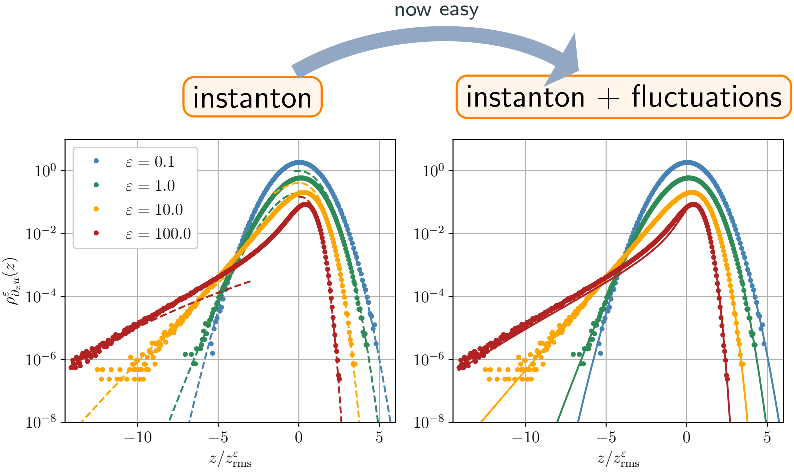
Results:

- Robust numerical optimization to find instantons
- Gaussian fluctuations computed by Riccati eq. or Fredholm det.
- Also for non-unique instantons w/ zero modes
- For non-eq. statistical field theories such as Navier–Stokes and KPZ, good agreement to Monte Carlo

What's next?

- Asymptotic analytical predictions/scaling of prefactor
- Higher-order fluctuations around instanton: loop expansion or FRG
- Extend to more general stochastic and dynamical systems
- Spectrum of Hessians: spectral gaps & asymptotic density of states?
- Incorporating renormalization of field theory into prefactor

# Thank you for your attention!




Read more e.g. in: T. Schorlepp and T. Grafke (2025) "**Scalability of the second-order reliability method for stochastic differential equations with multiplicative noise**" (arXiv:2502.20114)