Probability distribution function of the 2d Ising order parameter

Félix Rose
Laboratoire de Physique Théorique et Modélisation, CY Cergy Paris Université

Laboratoire de Physique Théorique de la Matière Condensée, Sorbonne Université
March 11th, 2025

A few words about me...

- **Studies:** CPGE ("prépa") → École polytechnique → Master ÉNS – ICFP, Quantum physics

- **Experience:**
  - PhD (2014 – 2017)
    direction : Nicolas Dupuis
    LPTMC, Sorbonne Université
  - Postdocs in Germany (2017 - 2023)
    Technical Univ. Munich
    Max-Planck Institute for Quantum Optics
    Univ. Heidelberg
  - **"Junior Chair" since 2023**
    LPTM, CY Cergy Paris University
    - 5y "tenure track", funding (1/2 PhD + expenses).
    - Teaching load ~ MCF.

Research interests

Strongly correlated quantum systems
E.g tuned to a phase transition; mixtures.

Platforms: cold atoms

Connection to classical strong correlations:
today's presentation.

Statistics of the sum of variables

Sums of random variables

Random variables $X_1, \dots, X_n$

$$\frac{1}{n}(X_1 + \dots + X_n) \to ?$$

- Arises in mathematics, physics, finance...
- Of particular interest: displays universality!

[Jona-Lasinio, Martin-Löf, Wilson, Hilfer, Jasnow, ...]

Central limit theorem

- Simplest case: X_i
  - Independent, identically distributed;
  - $\langle X_i^2 \rangle < \infty$.
- Central limit theorem.

$$X_1 + \cdots + X_n \to \mathcal{N}(n\langle X_i \rangle, n\langle X_i^2 \rangle_c)$$

Gaussian up to rescaling.

A NOTE ON THE RENORMALIZATION GROUP APPROACH TO THE CENTRAL LIMIT THEOREM

SÉBASTIEN OTT   2303.13905

Abstract. A proof of the Central Limit Theorem using a renormalization group approach is presented. The proof is conducted under a third moment assumption and shows that a suitable renormalization group map is a contraction over the space of probability measures with a third moment. This is by far not the most optimal proof of the CLT, and the main interest of the proof is its existence, the CLT being the simplest case in which a renormalization group argument should apply. None of the tools used in this note are new. Similar proofs are known amongst expert in limit theorems, but explicit references are not so easy to come by for non-experts in the field.

Mathematical renormalization group proof:

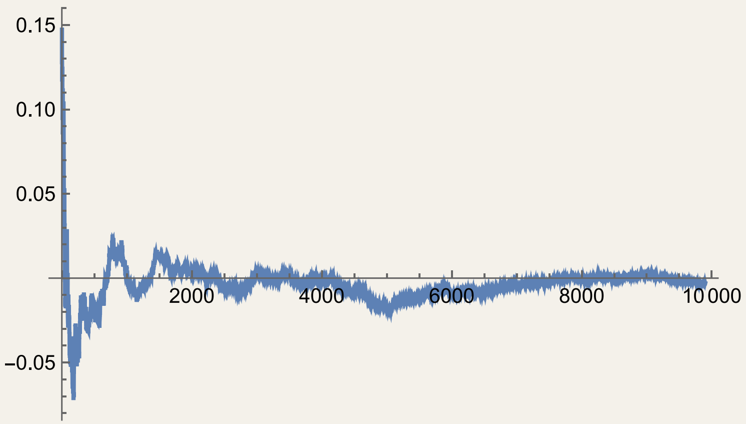
$X \mapsto 2^{-1/2}(X + X)$ = coarse graining + rescaling

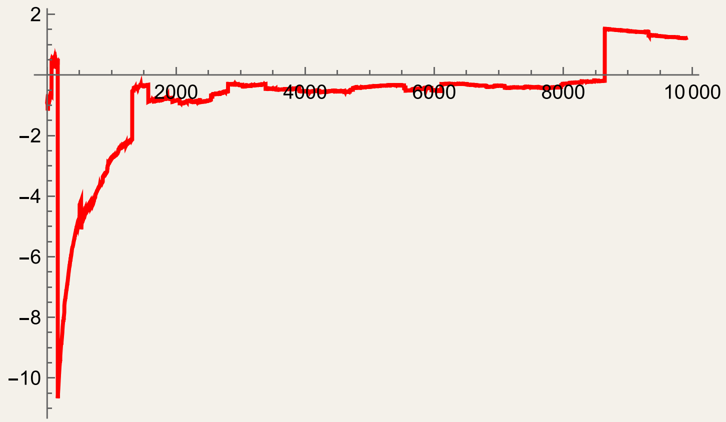
Decreases the distance to the Gaussian

→ Gaussian = fixed point.

Extension: Lévy stable distributions

- CLT hypothesis are crucial! Infinite variance → sum no longer Gaussian.





Cumulative average for Gaussian and Cauchy laws.

- Generalized central limit theorem:
  if $a_n(X_1 + \cdots + X_n) + b_n \to Z$ for some a_n, b_n, then Z is a Lévy-stable law.

Lévy-stable = stable under affine combinations.

Correlated variables

- Beyond CLT/Lévy distributions: independence?
  →strongly correlated variables, e.g Ising spins!

L: box size,
d: dimension.

$$\hat{s}_i = \pm 1, \qquad P(\{\hat{s}_i\}) \propto e^{-\beta H(\{\hat{s}_i\})}, \qquad H = -J \sum_{\langle ij \rangle} \hat{s}_i \hat{s}_j$$

$$\hat{s} = \frac{1}{L^d} \sum_i \hat{s}_i,$$

$P(\hat{s} = s) = ?$

- Crucial role of the bulk (L=∞) correlation length ξ_∞!

$$\langle \hat{s}_i \hat{s}_j \rangle \sim |i - j|^{-d+2-\eta} \exp(-|i - j|/\xi_\infty)$$

η: anomalous dimension.

- ξ_∞ finite: gather spins into weakly correlated blocks ~ξ_∞, central limit theorem applies.

Rate function

Critical Ising spins, correlation length ξ_∞.

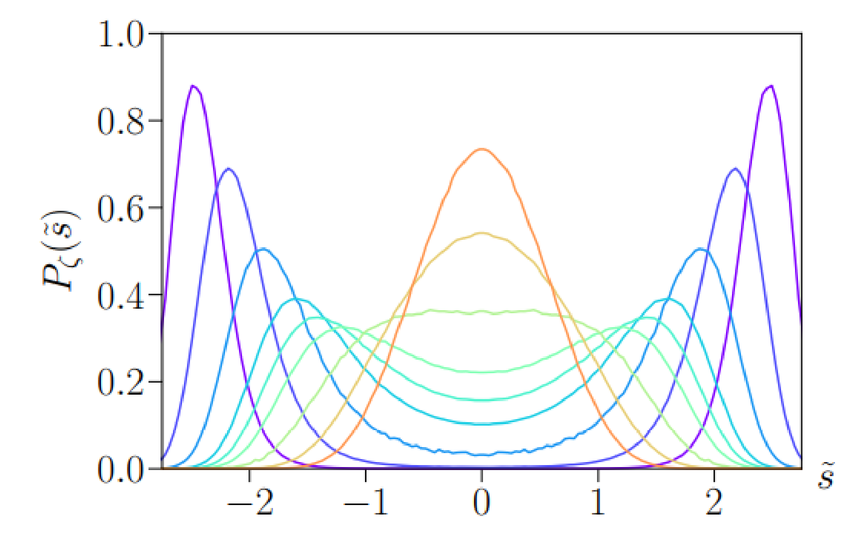
Rate function: $P_\zeta(\hat{s} = s) \simeq \exp\left(-L^d I(s, \xi_\infty, L)\right)$.

Scaling hypothesis: $L^d I(s, \xi_\infty, L) = I_\zeta(\tilde{s})$.

$\tilde{s} = L^{(d-2+\eta)/2} s$, $|\zeta| = L/\xi_\infty$.

$\mathrm{sign}(\zeta) = (-)1$ in the (broken) symmetry phase.

Remark: $P(s)$ depends on how the limit $\xi_\infty \to \infty$, $L \to \infty$ is taken.



$P_\zeta(\tilde{s})$ vs $\tilde{s}$ for ζ=-4,-3,...4.

3*d* Ising model, periodic boundaries.

Monte-Carlo simulations.

[Balog, Rançon, Delamotte, PRL '22]

Litterature

- Universality: connection to RG fixed points.

  [Jona-Lasinio, Martin-Löf, Wilson, Hilfer, Jasnow, ...]

- Theory in select models.

  [Dyson hierarchical model: Dyson, Bleher, Sinai, ...; KPZ: Sasamoto & Spohn]

- Experimental and numerical observation.

  [Numerics: Binder, Wilding, Tsypin, ...]
  Experiments: turbulence: Pinton, ..., liquid crystal growth: Sano, ...]

Field theoretical approach

Close to the transition, long range physics are described by a continuum field theory.

$$S = \int_x \frac{1}{2}(\nabla\hat{\phi})^2 + m^2\hat{\phi}^2 + g\hat{\phi}^4. \qquad \mathcal{Z} = \int \mathcal{D}[\hat{\phi}]e^{-S[\hat{\phi}]}.$$

Order parameter: magnetization $\langle\hat{\phi}\rangle$.

Sum of spins $\rightarrow \hat{s} = L^{-d}\int_x \hat{\phi}(x)$.

$$P(\hat{s} = s) = \mathcal{N}\int \mathcal{D}[\hat{\phi}]\delta(s - \hat{s})e^{-S[\hat{\phi}]}$$

How to compute P?

Effective action formalism

$$\mathcal{Z}[J] = \int \mathcal{D}[\hat{\phi}] \exp\left(-S[\hat{\phi}] + \int_x J\hat{\phi}\right),$$

Effective action

$$\Gamma[\phi] = -\ln \mathcal{Z}[J] + \int_x J\phi.$$

J: external source.

$\hat{\phi}$: fluctuating field, $\phi = \langle\hat{\phi}\rangle$: order parameter.

- Vertices $\Gamma^{(n)} = \delta^n\Gamma/\delta\phi^n$: physical information.
  - $\Gamma(\phi \to \text{const.}) = L^d U(\phi)$: effective potential → thermodynamics.
  - $\Gamma^{(2)} = [G]^{-1}$: inverse propagator.

Modified effective action

$$P(\hat{s} = s) = \lim_{M\to\infty} \mathcal{N} \int \mathcal{D}[\hat{\phi}] e^{-\frac{M^2}{2}(s-\hat{s})^2} e^{-S[\hat{\phi}]}.$$

$$S_M[\hat{\phi}] = S[\hat{\phi}] + \frac{M^2}{2}\left(\int_x (\hat{\phi}(x) - s)\right)^2$$

- $M = 0$: original action.
- $M \to \infty$: PDF, zero mode frozen!

- Modified Legendre transform

$$\Gamma_M[\phi] = -\ln \mathcal{Z}_M[J] + \int_x J_x \phi_x - \frac{M^2}{2}\left(\int_x (\phi_x - s)\right)^2.$$

- Γ_M defined to have a good limit for M large.
- Γ_M is independent of s!

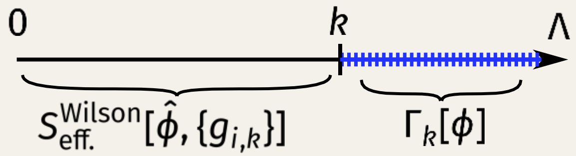
Constraint effective action

$\check{\Gamma} = \lim_{M\to\infty} \Gamma_M$: constraint effective action.

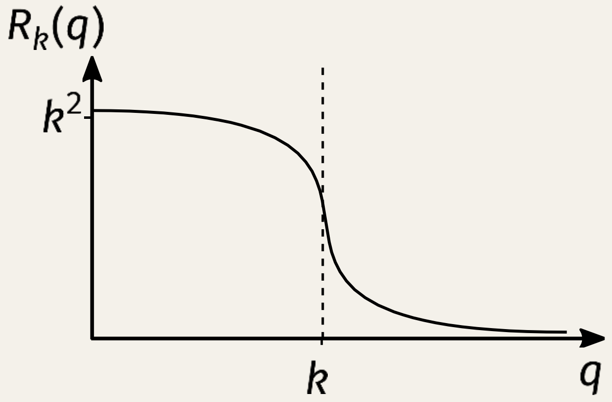
- Constraint: $M \to \infty$ ~ large mass only for the mode q = 0.
- RG approach is the the same with zero mode frozen
  → explicit box size dependency!

$$\check{\Gamma}(\phi \to \text{const.}) = L^d I_\zeta(\phi). \quad \Gamma(\phi \to \text{const.}) = L^d U(\phi).$$

The FRG in a nutshell



- Similar in concept to Wilsonian RG, degrees of freedom are progressively integrated out.
- Implemented by adding to S a "mass-like" term:



$$S \to S_k = S + \Delta S_k,$$

$$\Delta S_k[\hat{\phi}] = \frac{1}{2} \sum_q \hat{\phi}(q) R_k(q) \hat{\phi}(q).$$

R_k: modes at momenta $\lesssim k$ get a very large mass.

- k-dependent effective action $\Gamma \to \Gamma_k$.

$$\Gamma_{k=\Lambda} = S \xrightarrow{\text{RG flow}} \Gamma_{k=0} = \Gamma$$

Exact flow equation:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_k R_k \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right\}$$

[Wetterich, PLB '93; Morris, IJMP '94; Ellwanger, ZPC '94]

Constraint effective action flow

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_k R_k \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \right\}$$

Propagator $G_k(q) = (\Gamma_k^{(2)}(q) + R_k(q))^{-1}$.

$\Gamma_k \to \check{\Gamma}_k$: replace $U_k \to I_k$.

$$\partial_k U_k[\phi] = \frac{1}{2} \sum_q \partial_k R_k(q) G_k(q) \qquad \partial_k I_k[\phi] = \frac{1}{2} \sum_{q \neq 0} \partial_k R_k(q) G_k(q)$$

- Equations identical up to the removal of the zero mode.
- L^{-1} acts as an infrared cutoff: $I_{k \to 0}$ has a finite limit.

$$q = \frac{2\pi}{L}(n_1, \dots, n_d),$$
$$n_i \in \mathbb{Z}.$$

Going to $d = 2$

- $d = 3$: solved by LPA. [Balog et al.]

$d = 2$?

- $Z[h] = \langle e^{hs} \rangle$: moment generating functional of $P(s)$.

- Interest : stronger correlations; no exact results.

- LPA not enough! Need to include field corrections: $\eta^{\mathrm{LPA}} = 0$!

BMW approach

- Most approaches rely on a gradient expansion.
- Other idea: the celebrated Blaizot-Méndez-Galain-Wschebor (BMW) approximation. [Blaizot et coll., PRE ’06; Benitez et coll., PRE ’09]

  (Not a vertex truncation!)
- Close flow equations of $\check{\Gamma}_k^{(2)}(p_n)$:
  full momentum dependence!

Validity of gradient expansion

- Issue due to zero mode discrepancy between $p = 0$ and $p \neq 0$!

$$\partial_k \check{\Gamma}_k^{(2)}(p;\phi) = \frac{1}{2L^d} \sum_{q\neq 0} \partial_k \check{G}_k(q;\phi) \check{\Gamma}_k^{(4)}(p,-p,q,-q;\phi) - \frac{1}{2L^d} \sum_{q\neq 0,-p} \partial_k \left(\check{G}_k(q;\phi)\check{G}_k(q+p;\phi)\right) \check{\Gamma}_k^{(3)}(p,q,-q-p;\phi)\check{\Gamma}_k^{(3)}(-p,-q,p+q;\phi).$$

- Formally, for $p > 0$

$$\check{\Gamma}_k^{(2)}(p;\phi) - \check{\Gamma}_k^{(2)}(0;\phi) \simeq \Delta_{0,k}(\phi) + p^2 Z_k(\phi) + O(p^4).$$

(Recall $\check{\Gamma}_k^{(2)}(0;\phi) = I''_k(\phi)$.)

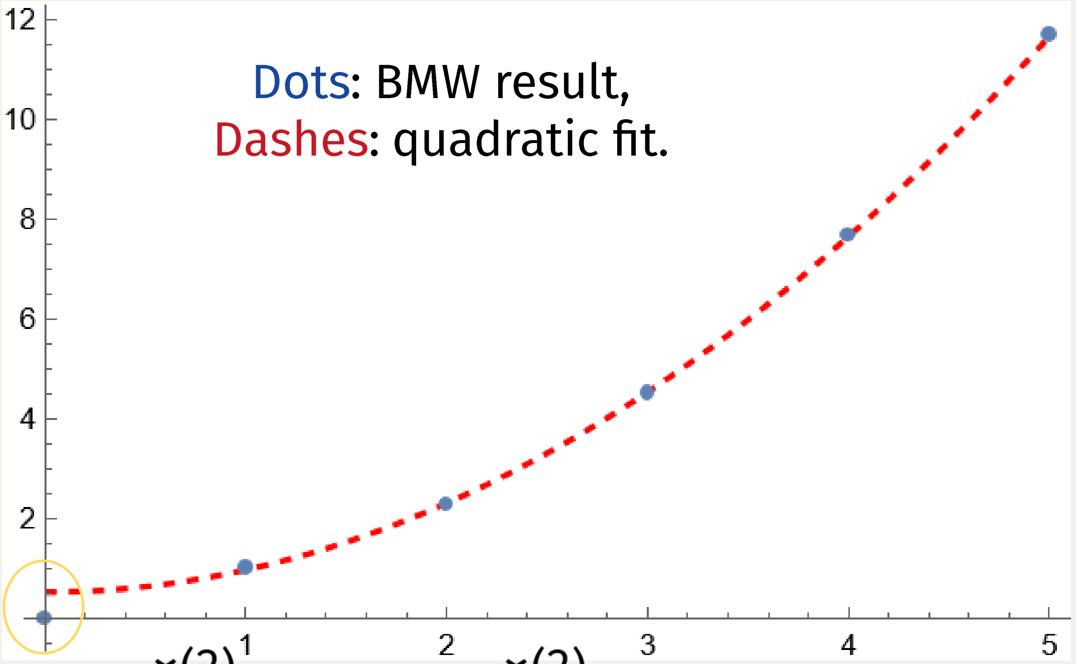
BMW results

- Blaizot-Méndez-Galain-Wschebor (BMW) approximation:
  Full momentum dependence of $\Gamma_k^{(2)}(p_n)$.

$$p_n = \frac{2\pi}{L}(n, 0, \ldots, 0).$$

- Discrepancy between $n = 0$ and $n > 0$!

- Informs derivative expansion.



$\check{\Gamma}_{k=0}^{(2)}(p_n;\phi) - \check{\Gamma}_{k=0}^{(2)}(0;\phi)$ vs. n.

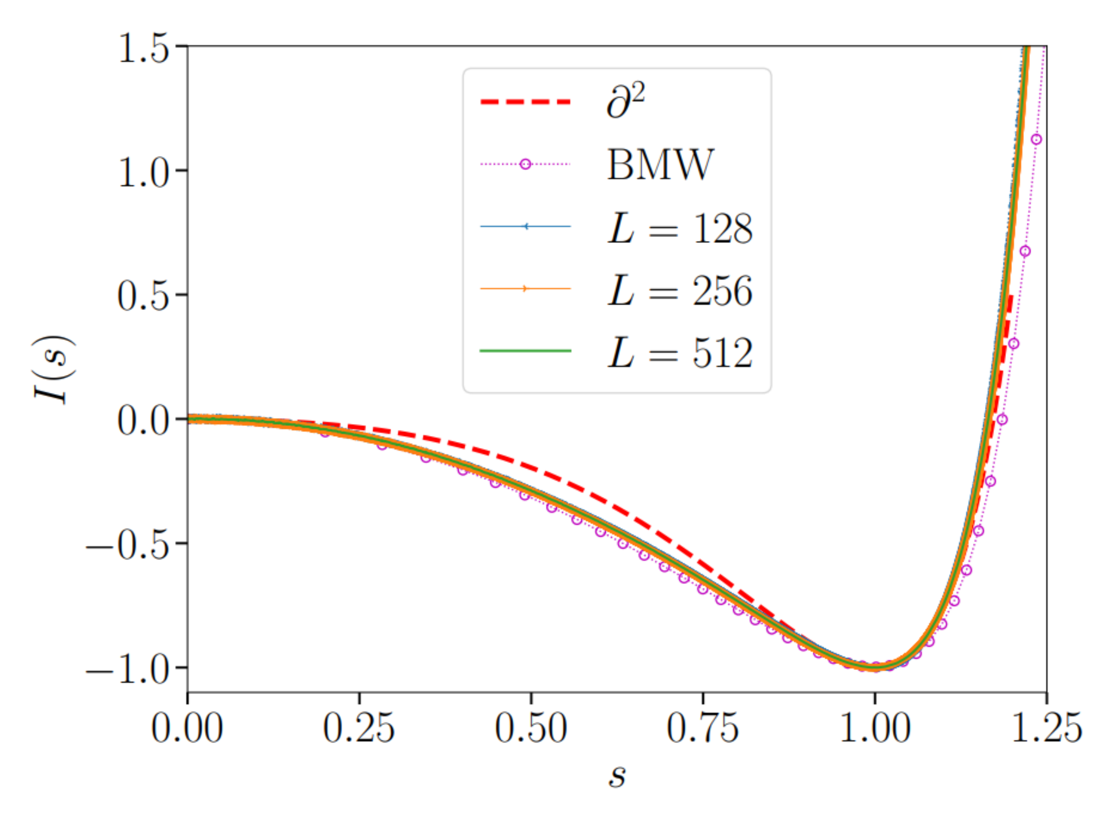
BMW results

- Rate function:
  DE$_2$ vs. BMW vs. MC.

- Shape: BMW sensible improvement over DE$_2$!

- Issue in the minimum value still not understood.

Collab.: A. Rançon, I. Balog



$I_{\zeta=0}(\tilde{s})$ vs. $\tilde{s}$.

Future projects

Bose-Bose mixtures

[Collab.: P. Jakubczyk, O. Stachowiak]

- Short term: rich $T > 0$ phase diagram by mean-field (MF) in $3d$: triple / quadruple points, multicritial points, by tuning inter/intraspecies interaction...

→ Fate in $2d$, beyond MF? FRG approach!

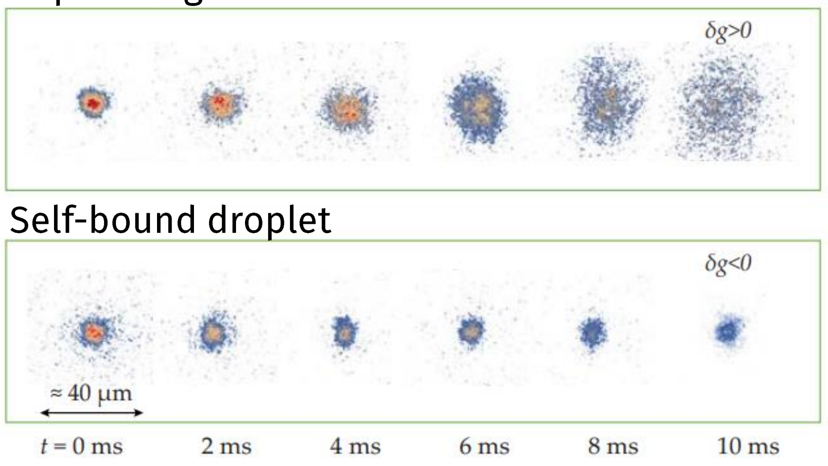
- Connection to frustrated magnets: internal symmetry group U(1)×U(1)

  [Delamotte Mouhanna Tissier PRB '04; Sánchez-Villalobos, Delamotte, Wschebor arXiv 24]

- Longer term: more complex phases, e.g self-stabilized liquid quantum "droplets"

MF is unstable: quantum correlations crucial to include!

Repulsive gas [Cabera, ..., Tarruell, Science '18]



Future projects

- **Planckian transport**

Strong correlations → transport properties.

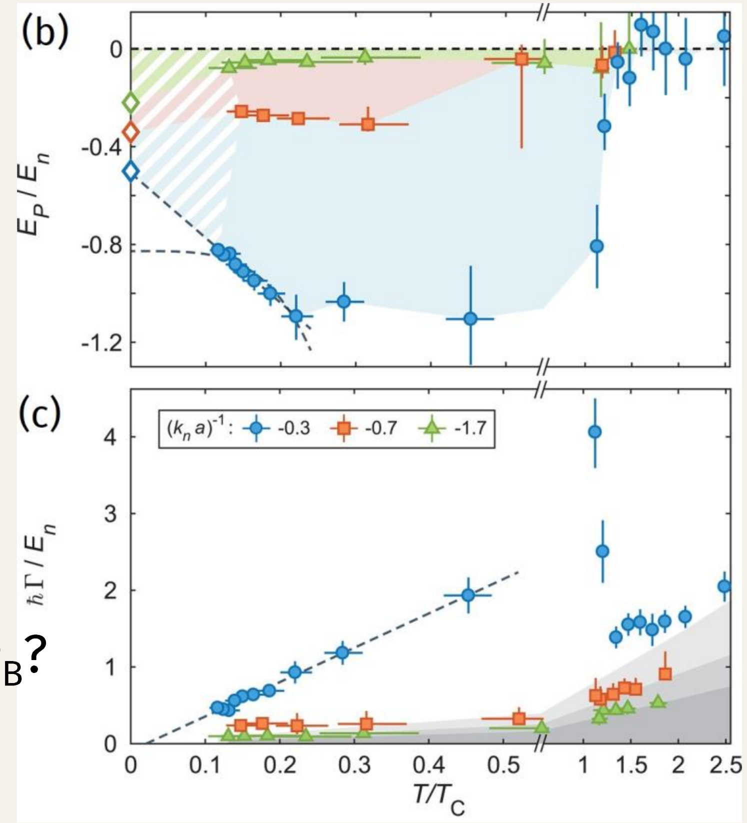
Quasiparticle description invalid
→ excitation lifetime τ as small as possible, $\tau \geq \tau_P = \hbar / k_B T$.

Seen e.g. in strange metals, SYK model, quantum criticality.

E.g. $O(N)$ model:

- Conductivity: insulator or conductor scenario?
- Viscosity: Kovtun-Son-Starinets conjecture $\eta / s \geq (1/4\pi)\, \hbar / k_B$?

Extension to other applications, e.g. Bose polaron.



[Yan, ..., Zwierlein Science '20]