Conformal invariance of Rigidity Percolation

Nina Javerzat

- **NJ**, M. Bouzid Phys. Rev. Lett. **130**, 268201 (2023)
- **NJ** Phys. Rev. Lett. **132**, 018201 (2024)

A short presentation

Critical phenomena & conformal field theory

Quantum entanglement

Amorphous systems

Glass theory

PhD | 1st postdoc | 2nd postdoc

LPTMS (Paris Saclay) | SISSA (Italy) | LIPhy (Grenoble)

R. Santachiara

E. Tonni

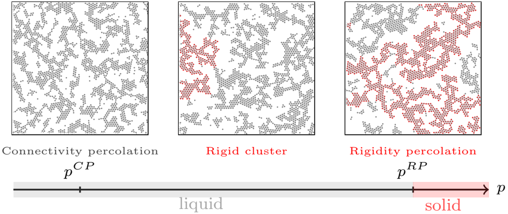
M. Bouzid
(3SR Grenoble)

M. Ozawa
E. Bertin

Rényi complexities
in disordered systems
→Tomorrow @ ENS

Connectivity Percolation:
How do bonds connect nodes ?

Rigidity Percolation:
How do bonds *constrain* nodes ?



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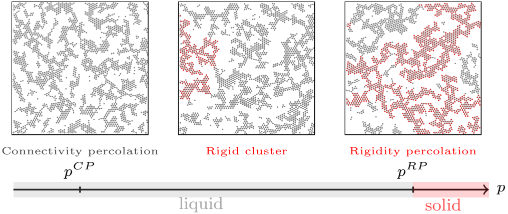
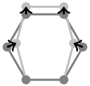


Imagine bonds as rigid bars:

- 1 particle = d dofs
- 1 bond = 1 constraint

$N_{\text{constraints}} \geqslant N_{\text{local dofs}}$ **rigid cluster**



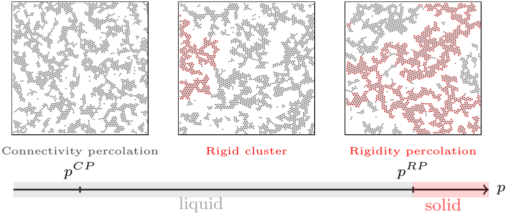
floppy cluster

$N_{\text{constraints}} < N_{\text{local dofs}}$
→ "soft" deformation modes

Rigidity percolation vs Connectivity percolation

Connectivity Percolation:
How do bonds connect nodes ?

Rigidity Percolation:
How do bonds *constrain* nodes ?



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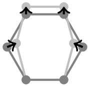
rigid cluster

deformation costs energy !



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$N_{\text{constraints}} < N_{\text{local dofs}}$

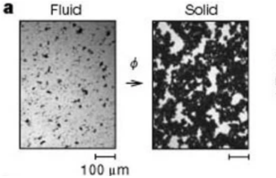
→ "soft" deformation modes

zero energy cost

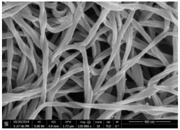
Generalize: bonds as springs, Lennard-Jones interactions...→central potentials

Challenge: understand how amorphous systems become solid

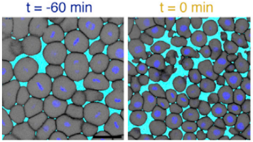
Non trivial: no long-range structural order



Colloidal gelation
Trappe et al. Nature 2001



Collagen network
Sharma et al. Nat.Phys. 2016



Living tissue
Petridou et al. Cell 2021

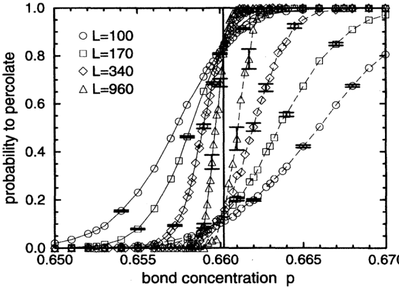
Rigidity Percolation: universal framework

for (athermal) fluid/solid transitions

Seminal work Jacobs, Thorpe 1995

→Implement smartly Laman's theorem: Pebble Game algorithm
("book-keeping" of dofs and constraints)

→Analyze rigidity of large graphs
+ standard scaling analysis of rigid clusters



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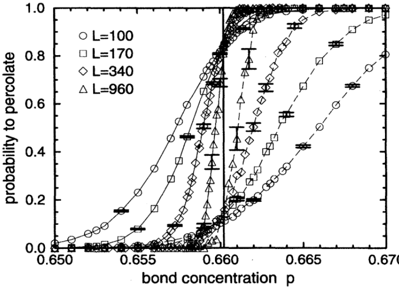
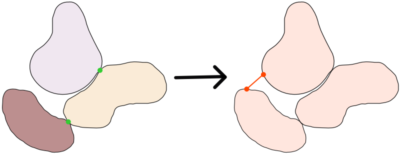


- 2nd order transition
- Critical threshold $p_{\rm RP} \approx 0.6602 > p_{\rm CP}$

- Critical exponents:
  - Correlation length $\nu = 1.21(6)$
  - Order parameter $\beta = 0.18(2)$
  - Fractal dimension $D_f = 1.86(2)$

RP is “long-range correlated” / non-local: **cascade events**



3 rigid clusters → *1 large rigid cluster*

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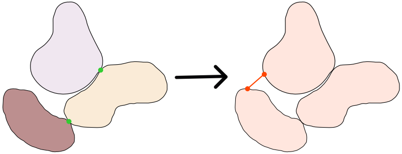
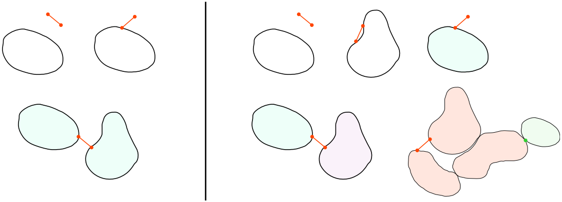


Clusters mechanisms unique to RP !



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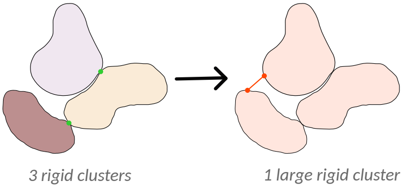

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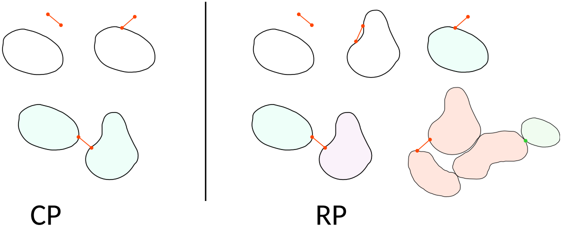
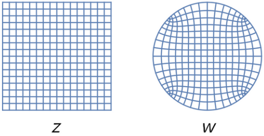


What is the partition function of the rigid subgraphs ?....

Some motivations:

Many critical phenomena are *conformal invariant*:
observables transform covariantly
under angle-preserving maps



$$\mathrm{Obs.}(\Lambda(x_1), \Lambda(x_2), \cdots) \stackrel{\text{in law}}{=} f(x_1, x_2, \cdots)\mathrm{Obs.}(x_1, x_2, \cdots).$$

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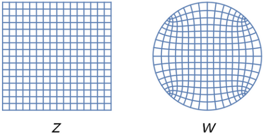

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Exploiting the constraints from conformal invariance is very powerful:

- Exact critical exponents: e.g. the backbone exponent of percolation Nolin et al 2023
- Solve 2d unitary models (Ising, 3−state Potts,...)
- And non-unitary ones: e.g. 2d random clusters $Q-$state Potts model Di Francesco, Saleur, Zuber, Viti, Delfino, Picco, Ribault, Santachiara, Ikhlef, Estienne, Jacobsen, He, ...
- Conformal bootstrap in $d > 2$: exponents of the 3d Ising model El Showk et al 2012
- In other fields too: e.g. entanglement entropy in quantum physics

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we can use conformal invariance to make new predictions

would be cool to find a scale but not conformal percolation model

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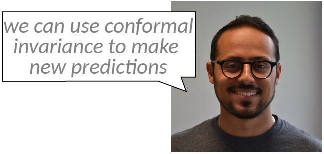
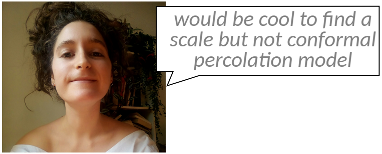
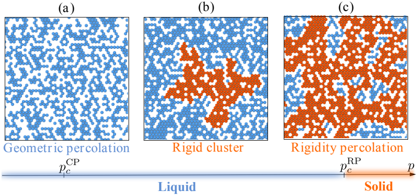




The model: rigidity percolation on the triangular lattice



Our observables: the **connectivities**

$$p_{12\cdots n}(z_1, \cdots, z_n) \equiv \mathrm{Prob}\left[z_1, \cdots, z_n \in \text{same rigid cluster}\right]$$

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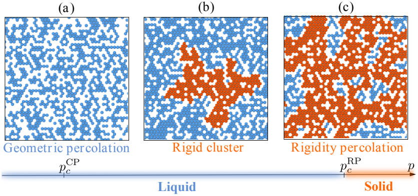

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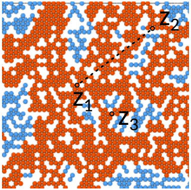

Assume: $p_{12\cdots n}(z_1, \cdots, z_n) \xrightarrow{\text{scaling lim.}} \langle \Phi(z_1) \cdots \Phi(z_n) \rangle$
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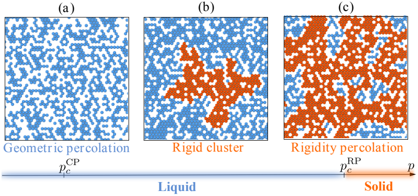

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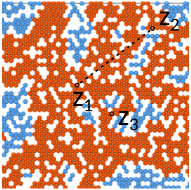

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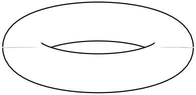
Strategy: test if connectivities are compatible with conformal invariance

We measured

$p_{12} = \mathrm{Prob}\left[z_1, z_2 \in \text{same rigid cluster}\right]$

on $L_1 \times L_2$ torii



$\mathrm{r} = |\mathrm{z}_1 - \mathrm{z}_2|$

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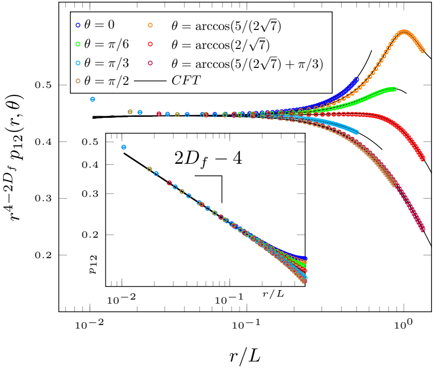

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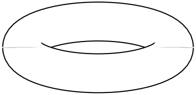

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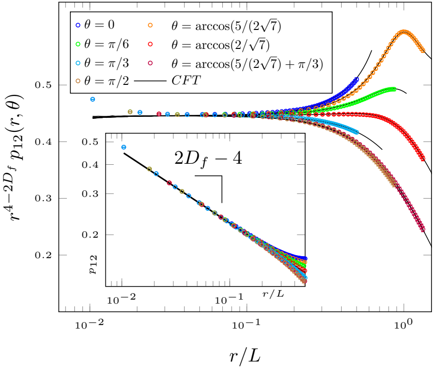

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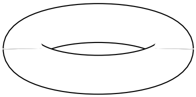

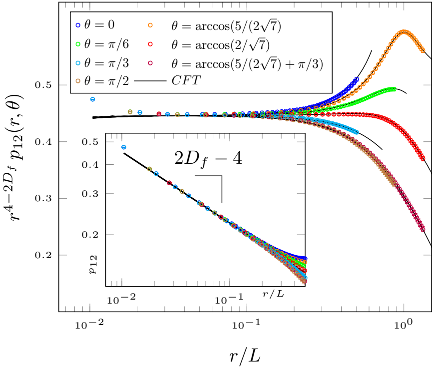

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p_{12}^{cylinder} follows the CFT prediction:

$$p_{12}^{\text{cylinder}}(z_1, z_2) = \mathrm{cst}\,\langle\Phi(\mathrm{z}_1)\Phi(\mathrm{z}_2)\rangle_{\mathrm{cyl}} = \mathrm{cst}\,\left(\frac{2\pi}{\mathrm{L}}\right)^{4-2\mathrm{D_f}}\left[4\sinh\left(\frac{\pi}{\mathrm{L}}\mathrm{z}_{12}\right)\sinh\left(\frac{\pi}{\mathrm{L}}\bar{\mathrm{z}}_{12}\right)\right]^{-2+\mathrm{D_f}}$$

$\rightarrow\Phi$ transforms as a Virasoro primary ✓

We also measured p_{12} on **finite torii** (elliptic nome $q \neq 0$)

$p_{12}(z_{12}) = \frac{\mathrm{cst}}{|z_{12}|^{4-2D_f}} \sum_{\Phi_a} C_a \langle \Phi_a \rangle_q \left(\frac{z_{12}}{L_2}\right)^{h_a} \left(\frac{\bar{z}_{12}}{L_2}\right)^{\bar{h}_a}$ Operator product expansion

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For Connectivity Percolation: **NJ** PhD thesis

$$p_{12}^{CP}(r,\theta) = \underbrace{\frac{\mathrm{cst}}{r^{4-2D_f}}}_{\text{plane limit}} \left(1 + \underbrace{C_\nu \langle \Phi_\nu \rangle_q \left(\frac{r}{L_2}\right)^{2-1/\nu} + \frac{4-2D_f}{c} \langle T \rangle_q \cos 2\theta \left(\frac{r}{L_2}\right)^2 + \cdots}_{\text{finite}-\text{size/torus corrections}} \right)$$

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Torii with $L_1 \neq L_2$ → get $\langle T \rangle_q = (2\pi)^2 \left(\frac{c}{24} - \sum_{\Phi \in \mathrm{CFT}} \mathcal{N}_\Phi \, h_\Phi \, q^{h_\Phi} \bar{q}^{\bar{h}_\Phi} + \cdots \right)$

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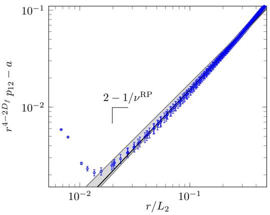

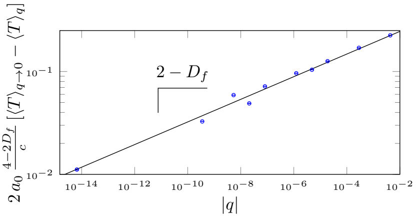

→Same low-lying fields in CP and RP

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Episode 1: "Bulk" of rigid clusters consistent with conformal invariance ✓

Episode 2: Analyze the clusters' *interfaces* with Schramm-Loewner Evolution.

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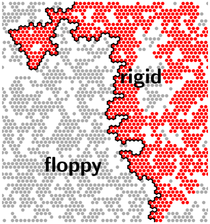
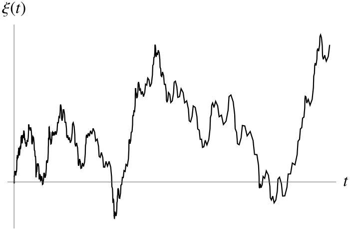


(random) curve γ_t *in* $\mathbb{H}$

conformal maps $g_t(\mathbf{z})$ ⟶

$$\begin{cases} \frac{dg_t(\mathbf{z})}{dt} = \frac{2}{g_t(\mathbf{z}) - \xi_t} \\ g_0(\mathbf{z}) = \mathbf{z} \end{cases}$$



(stochastic) real function

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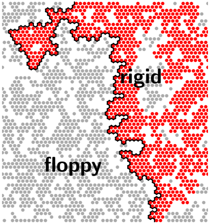

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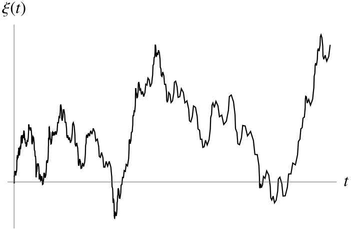


(stochastic) real function

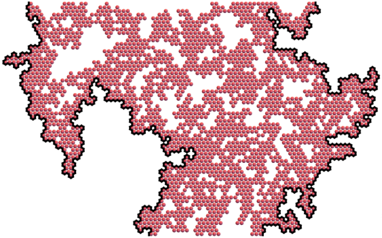
Theorem (Schramm 2000):

Conformally invariant curves map to Brownian motions: $\xi_t = \sqrt{\kappa} B_t$.

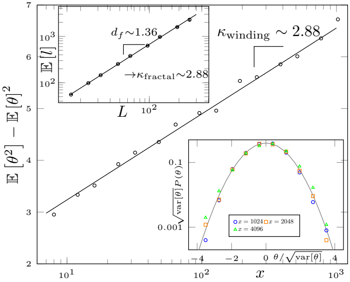
→Universal behaviour of the curve parametrized by κ.

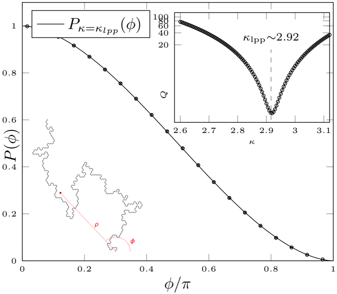
SLE analysis of RP in a nutshell:

1. Identify spanning perimeters of rigid clusters (in a strip)
2. Obtain instances of ξ_t from instances of the discrete curves $\{z_0, z_1, \cdots, z_l\}$, by iterative mapping via g_t
3. Test consistency of ξ_t with a Brownian motion
   $\rightarrow$independent evidence of conformal invariance of RP ✓



4. Estimate κ from SLE's observables:
   - fractal dimension $\ell \sim L^{d_f}$: $d_f = 1 + \frac{\kappa}{8}$ Beffara 2008
   - winding angle $\theta(\ell)$: $\mathrm{var}\left[\theta(\ell)\right] = a + \frac{2\kappa}{8+\kappa} \log \ell$
     Duplantier Saleur 1988, Schramm 2000
   - Left-passage probability:
     $P_\kappa(\phi) = \frac{1}{2} + \frac{\Gamma(\frac{4}{\kappa})}{\sqrt{\pi}\Gamma(\frac{8-\kappa}{2\kappa})} \cot(\phi)\, {}_2F_1\left[\frac{1}{2}, \frac{4}{\kappa}, \frac{3}{2}, -\cot^2(\phi)\right]$ Schramm 2000





$\rightarrow \kappa_{\text{RP}} \approx 2.90(2)$.

Can we say something new about RP ?

SLE reinterpreted in CFT: *bulk* critical exponents in terms of κ

Bauer & Bernard 2002, Cardy 2004, Doyon & Cardy 2007

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Exponents of Rigidity Percolation consistent with:

[Jacobs, Thorpe 1995 $\nu = 1.21(6)$, $D_f = 1.86(2)$]

$$D_f^{\mathrm{RP}} = 1 + 3/(2\kappa) + \kappa/8 \overset{\kappa\approx 2.9}{\approx} 1.88$$

$$\nu^{\mathrm{RP}} = \kappa/(3\kappa - 6) \overset{\kappa\approx 2.9}{\approx} 1.1$$

$$\Rightarrow D_f^{\mathrm{RP}} = 2 + \frac{1 - 2\nu^{\mathrm{RP}}}{4\nu^{\mathrm{RP}}(3\nu^{\mathrm{RP}} - 1)}$$

Not yet the exact exponents,
but a new relation from conformal invariance :)

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  3. From SLE/CFT, **a new relation** for RP: $D_f^{\mathrm{RP}} = 2 + \frac{1-2\nu^{\mathrm{RP}}}{4\nu^{\mathrm{RP}}(3\nu^{\mathrm{RP}}-1)}$

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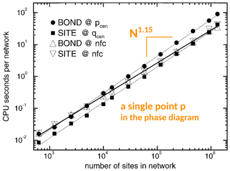
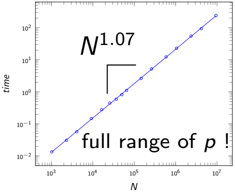


Original Pebble Game [Jacobs Hendrickson 97]

→max. size ≈ 1000 × 1000



Our scaling

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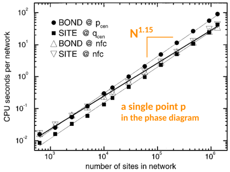

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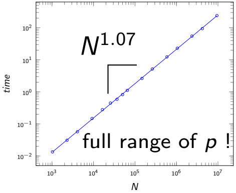

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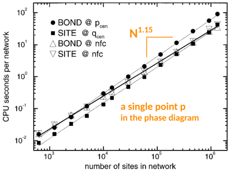

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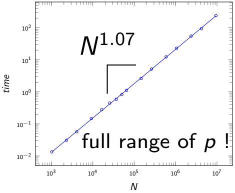

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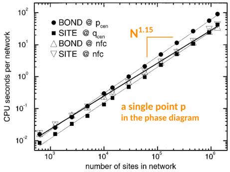

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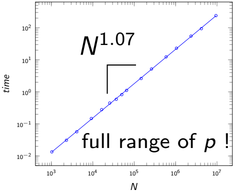

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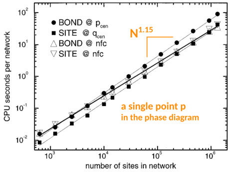

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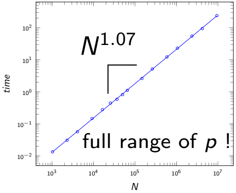

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◦ Starting collaborations with mathematicians:

[V. Beffara and H. Vanneuville, Institut Fourier, Grenoble]

Maybe find the **exact value** of κ ?...

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  →**new RP critical points** ?..

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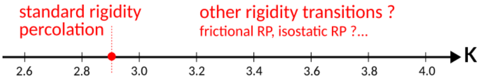

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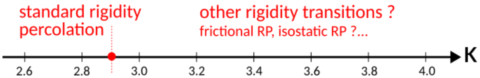


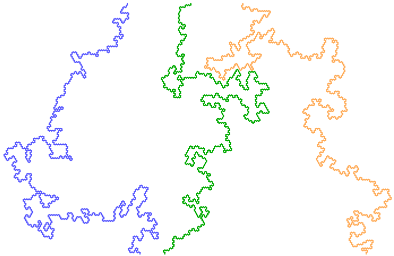
Thank you for your attention !

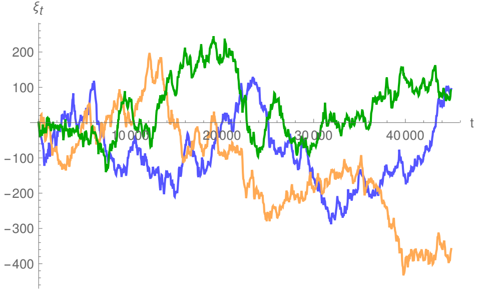
1– Extract ξ_t from a discrete curve $\left\{z_0^0, z_1^0, \cdots, z_l^0\right\}$

Compute iteratively t_j and ξ_{t_j} by applications of g_{t_j};

$$\left\{z_j^{j-1},\cdots,z_l^{j-1}\right\}\to\left\{z_{j+1}^{j}=g_{t_j}(z_{j+1}^{j-1}),\cdots,z_l^{j}=g_{t_j}(z_l^{j-1})\right\},\qquad g_{t_j}(z)=\xi_{t_j}+\frac{2L_y}{\pi}\cosh^{-1}\left[\frac{\cosh\left[\frac{\pi}{2L_y}(z-\xi_{t_j})\right]}{\cos\Delta_j}\right]$$

$$\xi_{t_j}=\mathrm{Re}(z_j^{j-1})\qquad t_j=t_{j-1}-2\left(\frac{L_y}{\pi}\right)^2\log\left(\cos\Delta_j\right)\qquad \Delta_j=\frac{\pi}{2L_y}\mathrm{Im}(z_j^{j-1})$$





2– Test if $\xi(t)$ is a Brownian motion

Increments $X_j \equiv \xi_{t_j+\delta} - \xi_{t_j}$ must be independent and normal

→Test the joint distribution of $(X_1, X_2, \cdots, X_n)$: Kennedy 2008

- define $m = 2^n$ cells: possible sign sequences, eg. $(+, -, \cdots, -)$
- count the number O_j of instances falling in each cell j
- compare to expectation E_j for indep. normal variables using

$$\chi^2 = \sum_{j=1}^{m} \frac{(O_j - E_j)^2}{E_j}$$

and associated p-value

n	5	7	9
p-value	0.95	0.78	0.81

p-values not small:

ξ_t = Brownian motion ✓