Generalised Hydrodynamics of the KdV Soliton Gas.

Laboratoire de Physique Théorique de la Matière Condensée

Thibault Bonnemain, 18th February 2025

[Based off joint work with B. Doyon and G. El]

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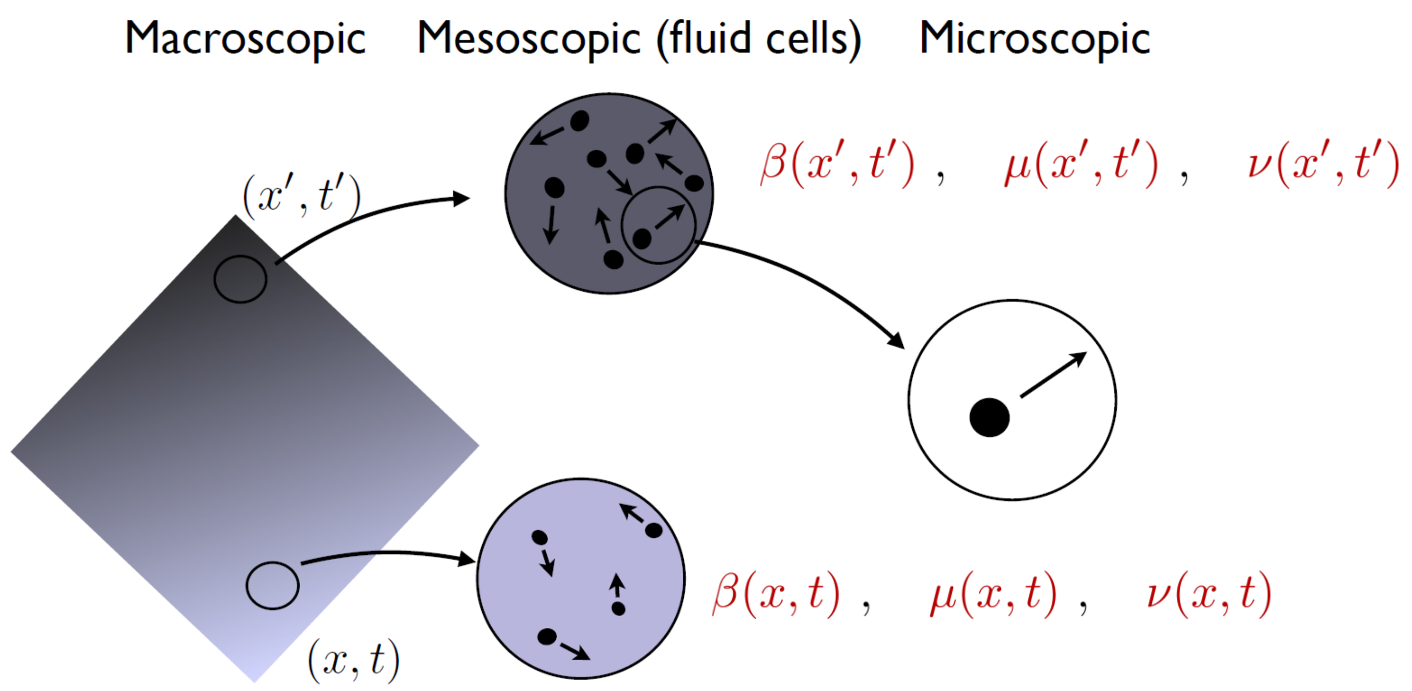

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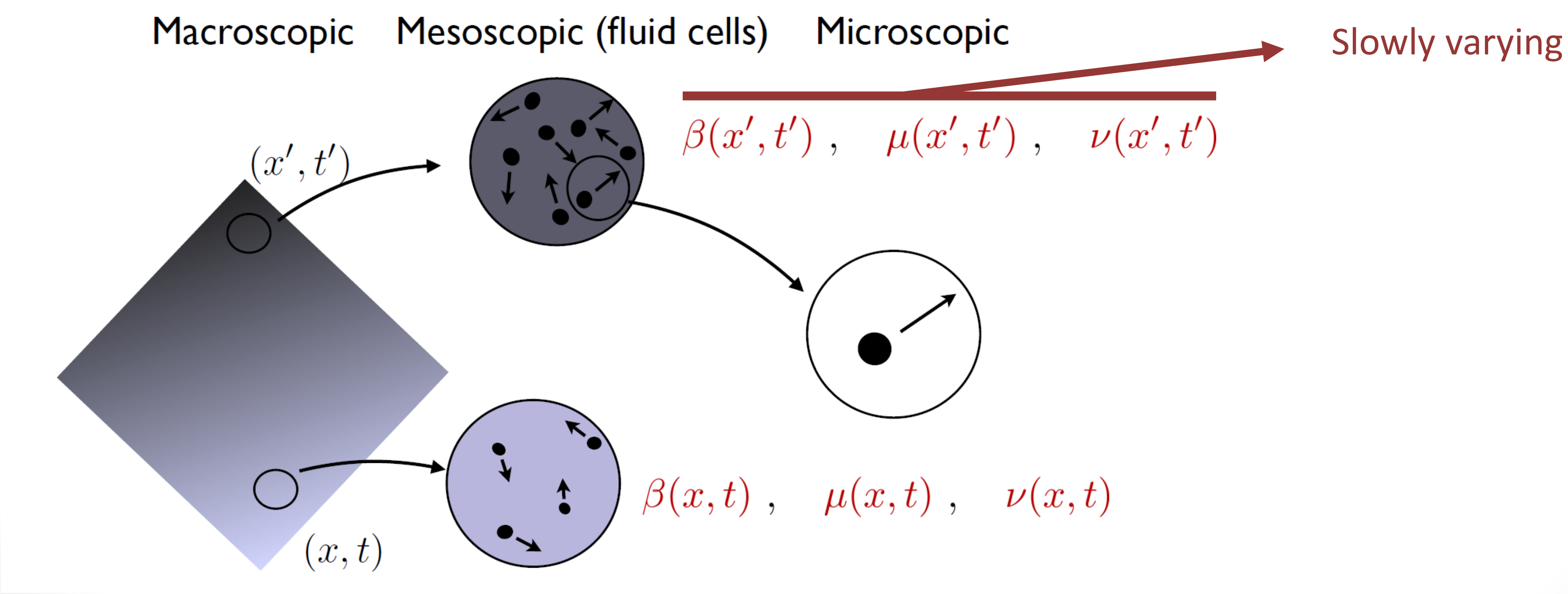

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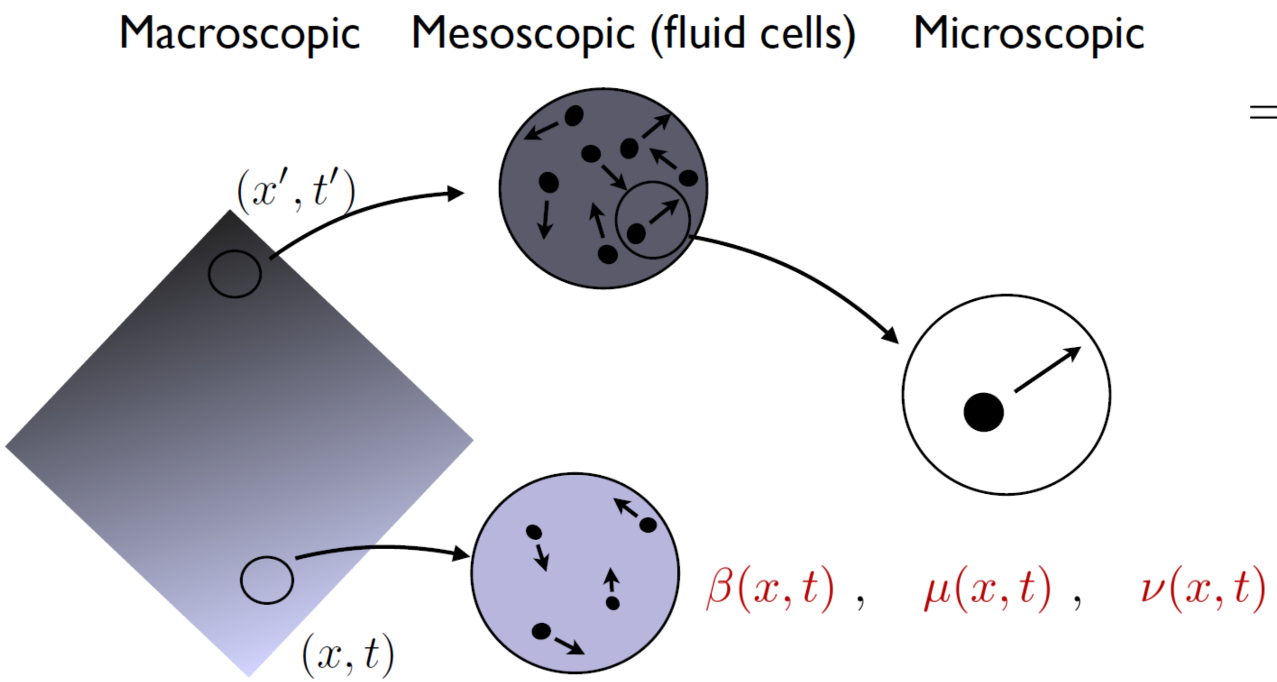

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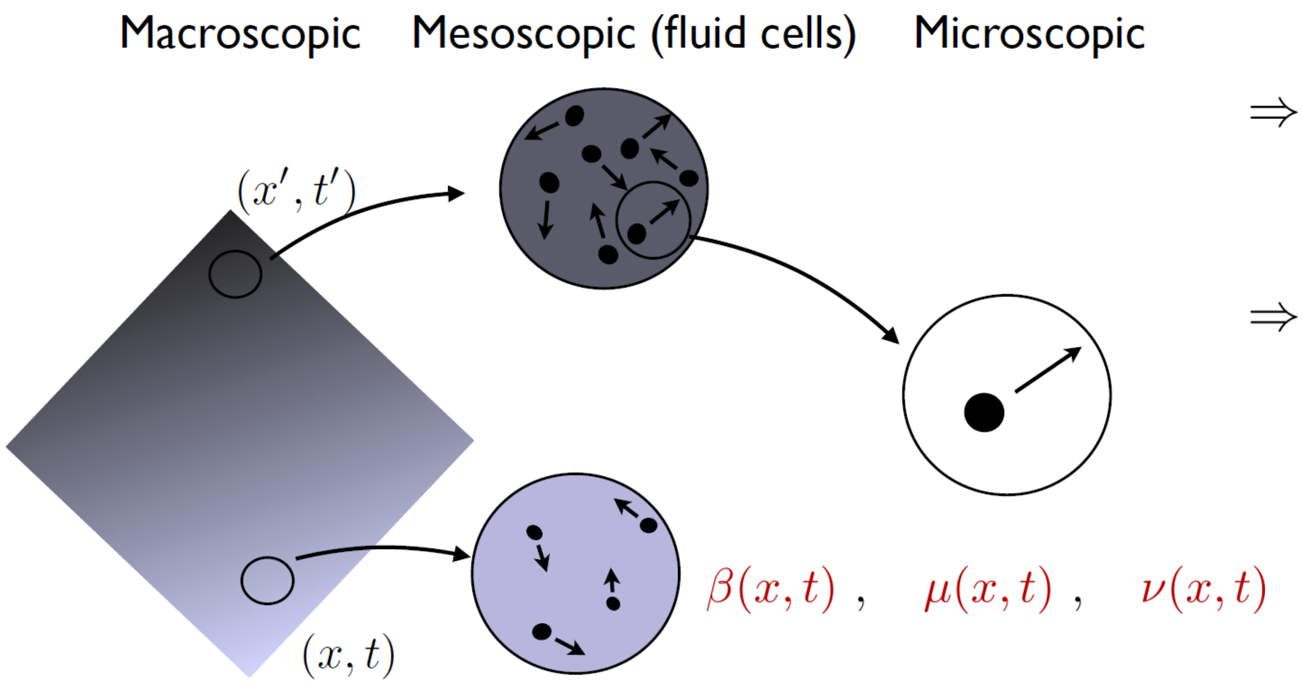

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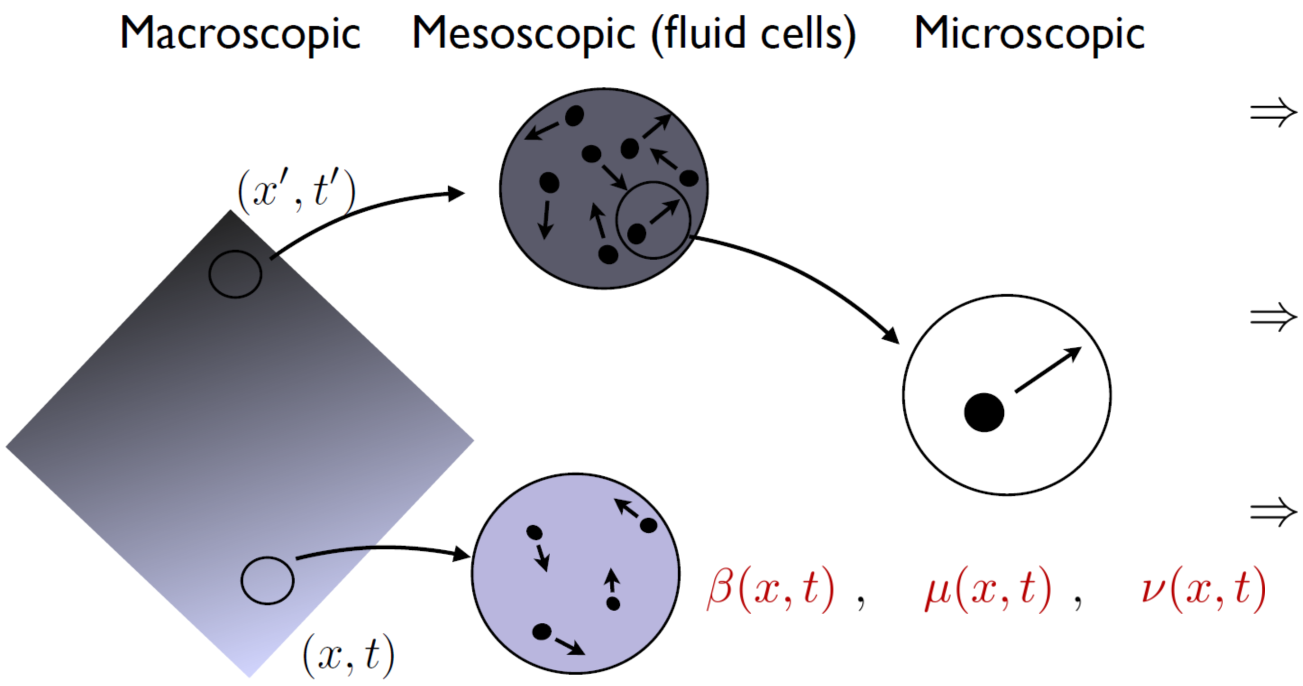

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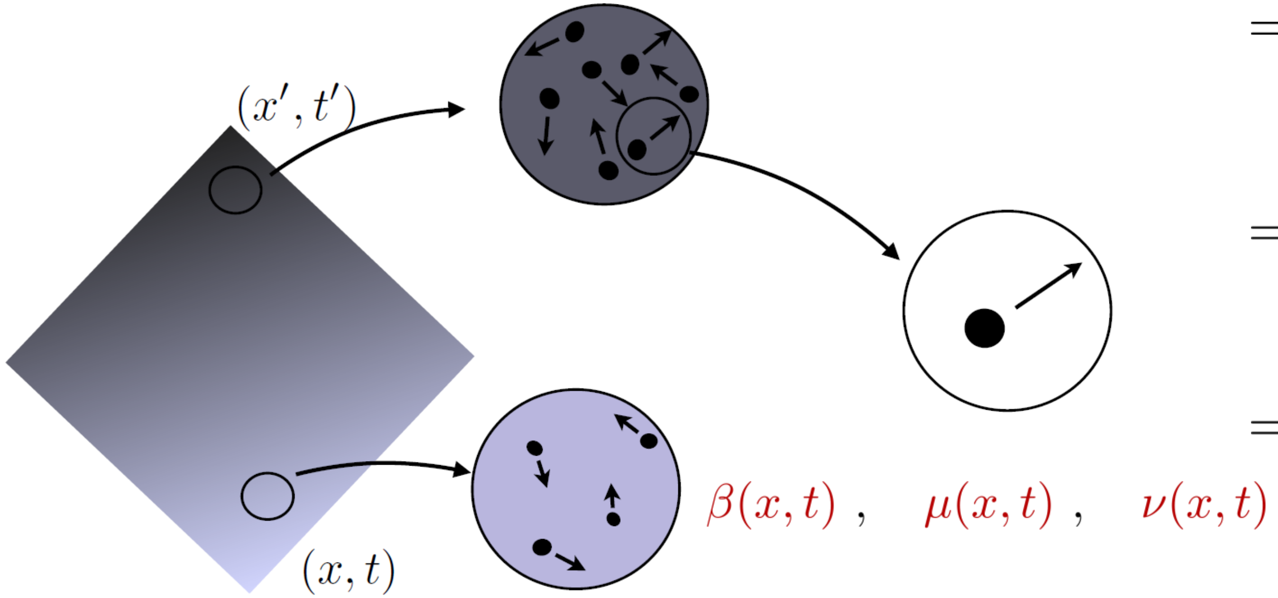

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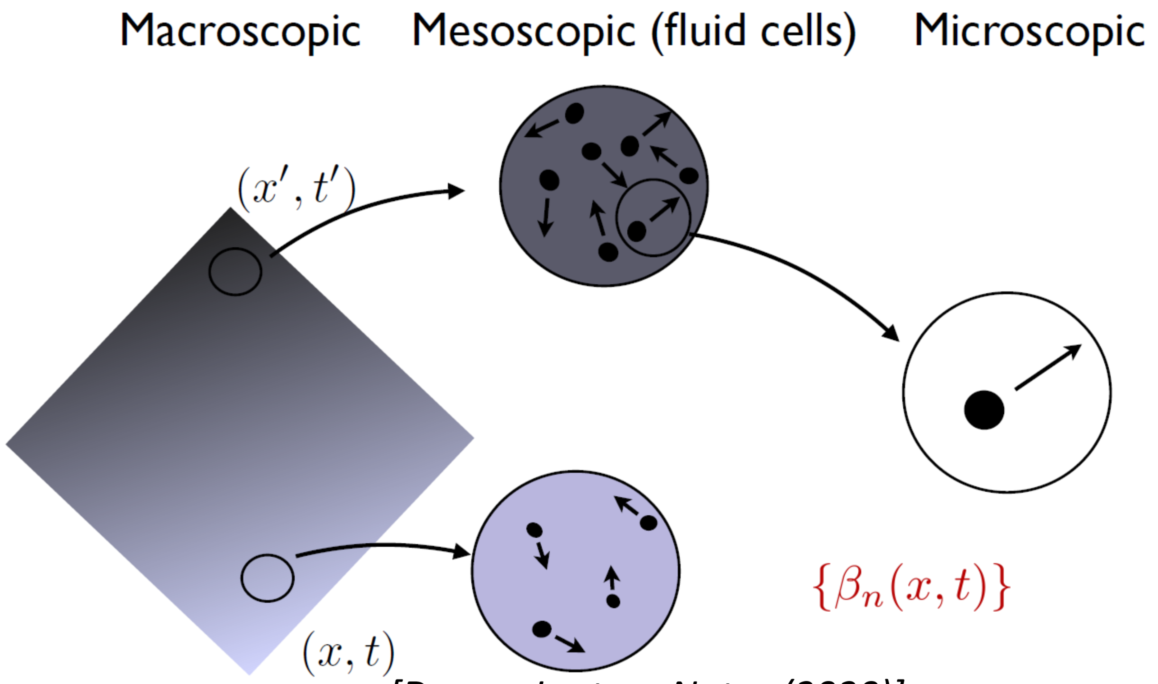

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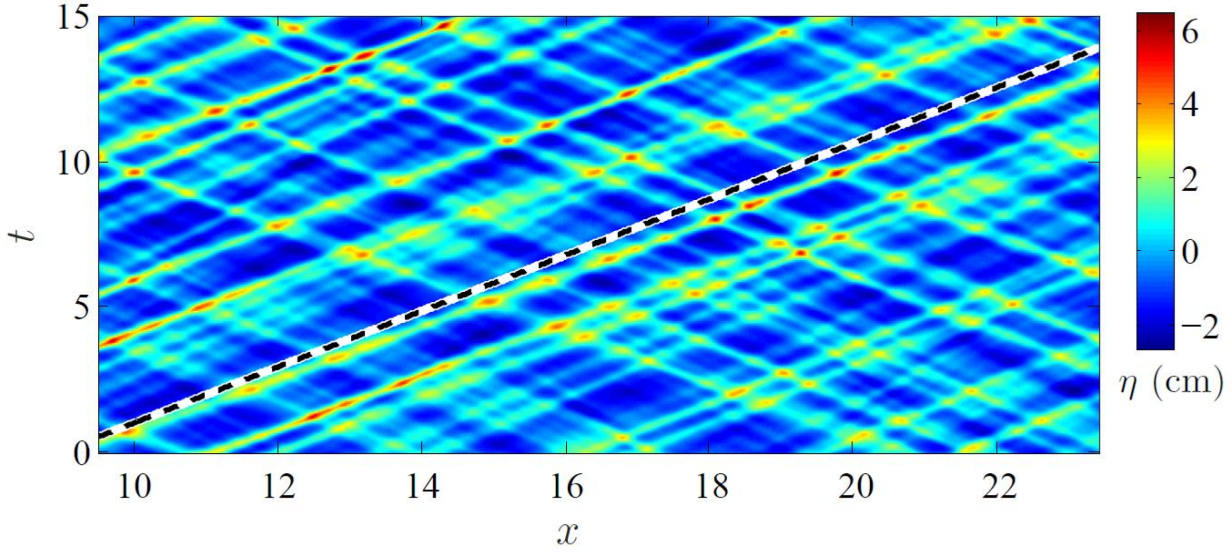
Soliton gases in experiments

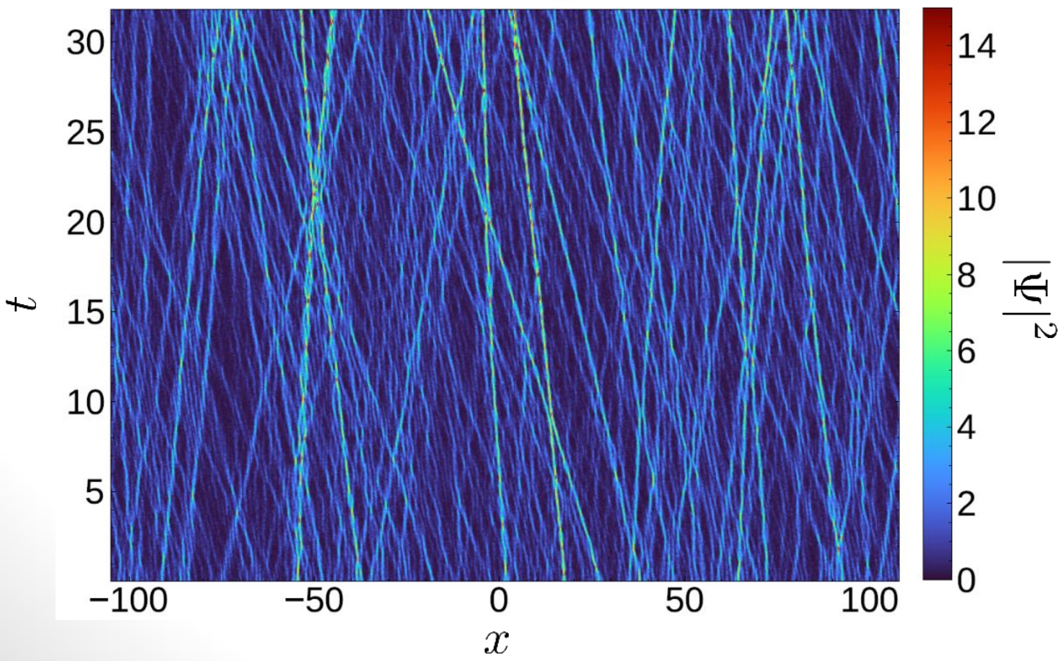


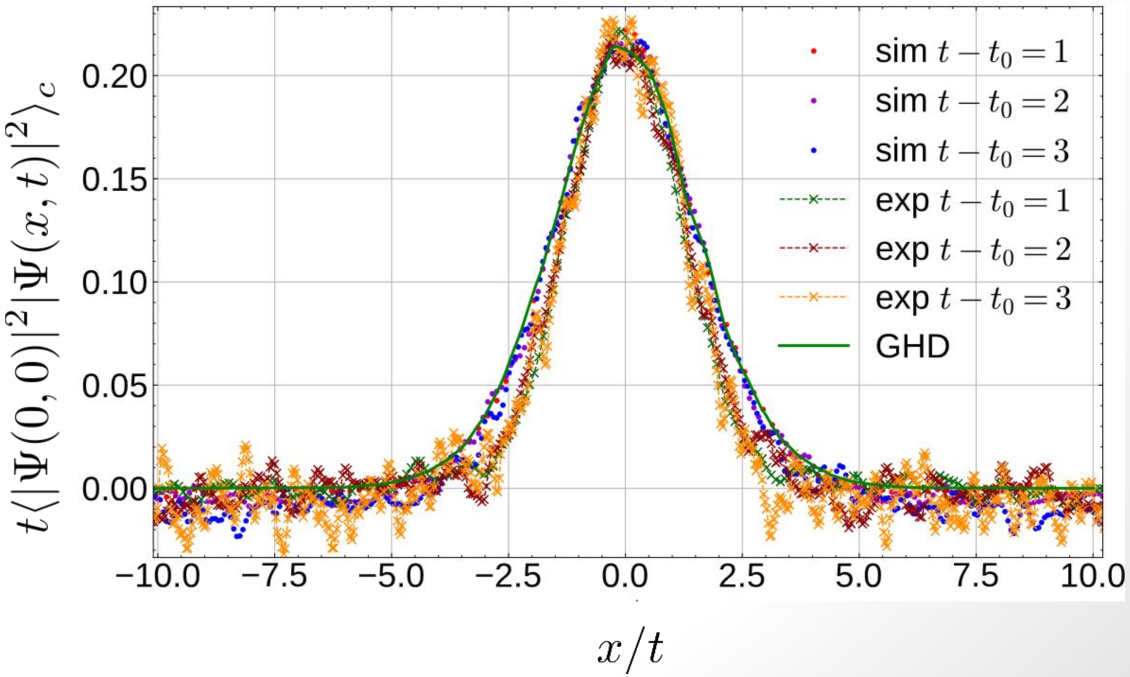
Left: Soliton gas in a water tank.

[Redor et al. (2019)]

Below: Soliton gas in an optical fiber and intensity correlations.

[Curtesy of Elias Charnay]





The Korteweg-de Vries equation

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- Infinite set of conservation laws

Time conserved "charges"

$$Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad \text{and} \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ ,$$

Space conserved "currents"

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- Exactly solvable via Inverse Scattering Transform (IST).

[Gardner, Greene, Kruskal, Miura (1967)]

Some properties of N-soliton solutions

- Long time asymptotics of $N-$soliton solutions

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

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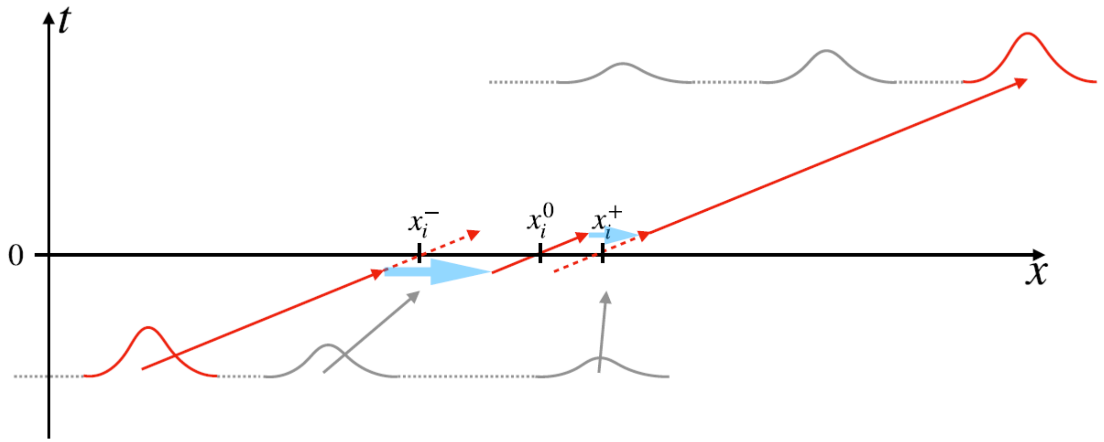
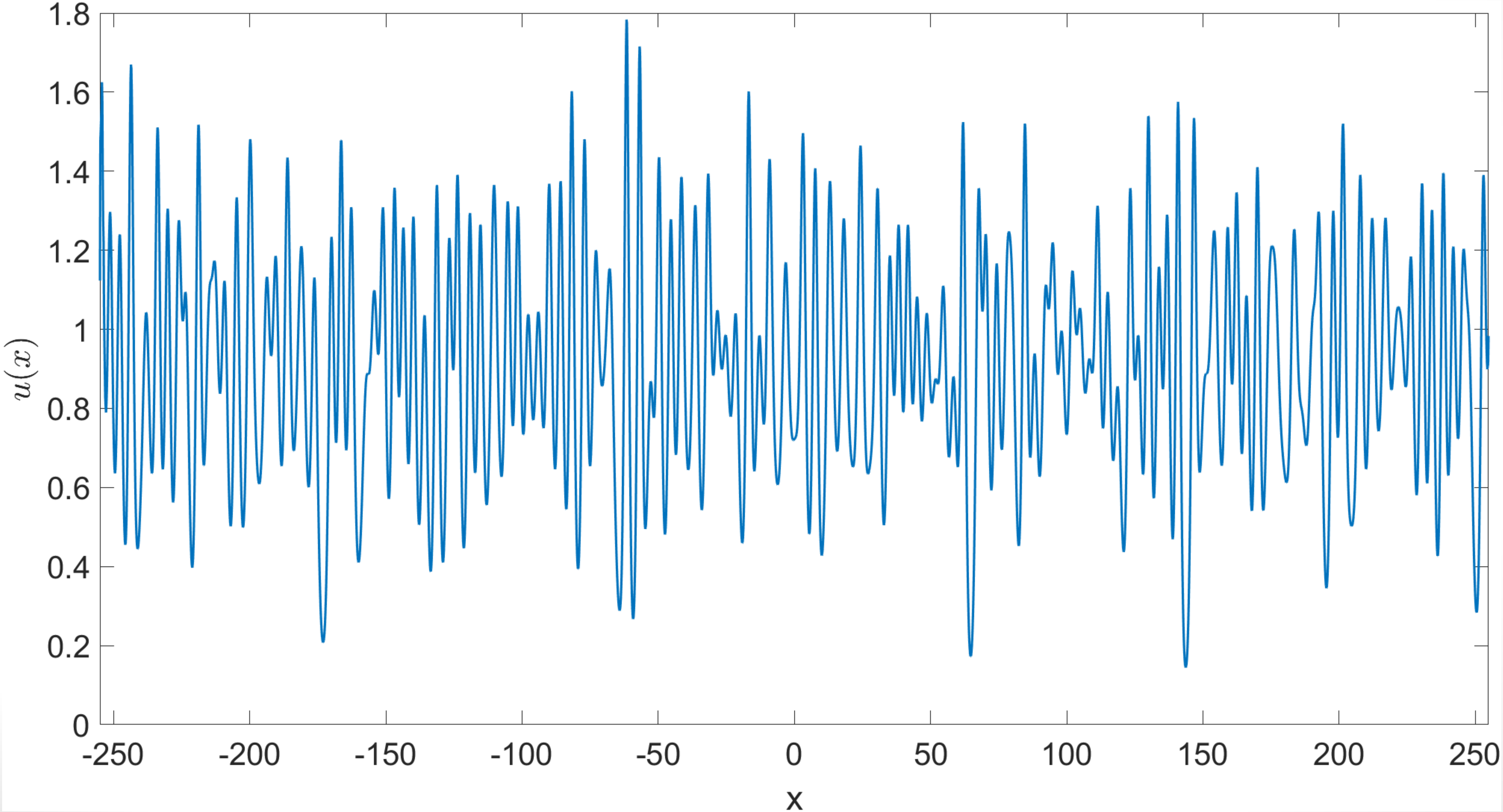
- Relation between asymptotic states given by scattering shift



$$x_i^+ - x_i^- = \sum_j \frac{\text{sgn}(\eta_i - \eta_j)}{\eta_i} \ln \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| .$$

Soliton gas: basic idea and motivations



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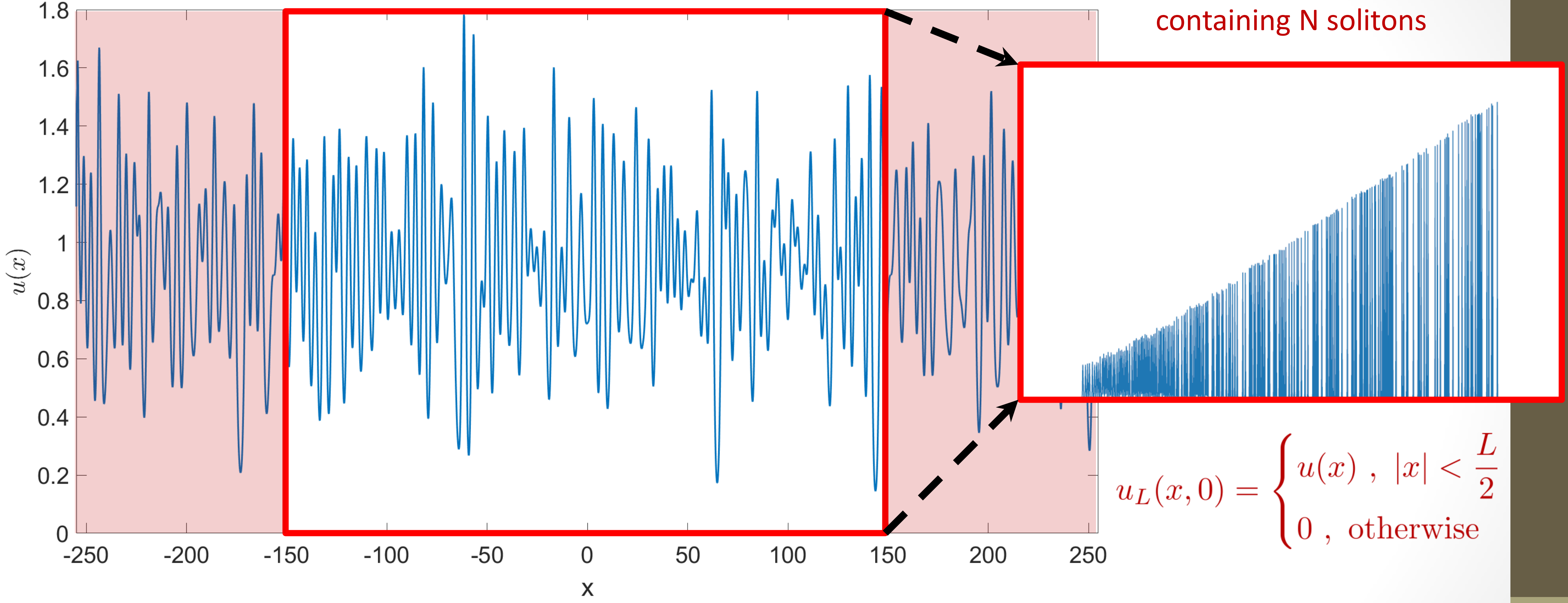

Fluid cell of size L containing N solitons

$$u_L(x,0) = \begin{cases} u(x)\ , & |x| < \frac{L}{2} \\ 0\ , & \text{otherwise} \end{cases}$$

Asymptotically: $$u_L(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

Thermodynamics

- Partition function

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x, t=0) < \epsilon_x, x \notin [0, L]\right)$$

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Soliton bare velocity

$$p(\eta) = 4\eta^2$$

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$$\text{e.g.} \quad w(\eta) = \sum_k \beta_k h_k(\eta)$$

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Constraint / Entropy

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- Thermodynamic limit $L \to \infty$: large deviations theory *[Varadhan (1966), Touchette (2009)]*

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- Interpretation

$$\eta \, n(\eta) \, \mathrm{d}x^-(\eta) = \rho(\eta) \, \mathrm{d}x$$

Change of metric

Some more (maybe) familiar relations

- Occupation function

$$\log n(\eta) = -w(\eta) + \int_\Gamma \mathrm{d}\mu n(\eta) \log \left| \frac{\eta - \mu}{\eta + \mu} \right| , \qquad \mathcal{F} = -\int_\Gamma \eta \, n(\eta) \, \mathrm{d}\eta \, ,$$

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	GHD	Simulations
$\mathsf{C}_{00}^{\mathrm{DC}}$	0.0235	0.022 ± 0.003
$\mathsf{C}_{01}^{\mathrm{DC}}$	0.027	0.024 ± 0.004
$\mathsf{C}_{11}^{\mathrm{DC}}$	0.042	0.039 ± 0.005
$\mathsf{C}_{00}^{\mathrm{U}}$	0.22	0.2 ± 0.03
$\mathsf{C}_{01}^{\mathrm{U}}$	0.28	0.23 ± 0.04
$\mathsf{C}_{11}^{\mathrm{U}}$	0.39	0.36 ± 0.05
$\mathsf{C}_{00}^{\mathrm{L}}$	0.2	0.2 ± 0.01
$\mathsf{C}_{01}^{\mathrm{L}}$	0.25	0.23 ± 0.01
$\mathsf{C}_{11}^{\mathrm{L}}$	0.36	0.34 ± 0.02

From thermodynamics to hydrodynamics

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⇨

$$\partial_t \rho(\eta;x,t) + \partial_x \left[v^{\mathrm{eff}}(\eta;x,t)\rho(\eta;x,t)\right] = 0 \ .$$

?

Derivation of GHD equations

[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

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$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

- Change of metric: $\eta\, n(\eta)\, \mathrm{d}x^-(\eta) = \rho(\eta)\, \mathrm{d}x$

$$\partial_t n(\eta; x, t) + v^{\mathrm{eff}}(\eta; x, t) \partial_x n(\eta; x, t) = 0 \ .$$

$$v^{\mathrm{eff}}(\eta; x, t) = 4\eta^2 + \frac{1}{\eta} \int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta; x, t)[v^{\mathrm{eff}}(\eta; x, t) - v^{\mathrm{eff}}(\mu; x, t)]\mathrm{d}\mu \ .$$

Derivation of GHD equations

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- Continuity equation for the DOS

$$\partial_t \rho(\eta; x, t) + \partial_x \left[\rho(\eta; x, t) v^{\mathrm{eff}}(\eta; x, t) \right] = 0 \ .$$

Concluding remarks and perspectives

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