Topological pathways to two-dimensional quantum turbulence

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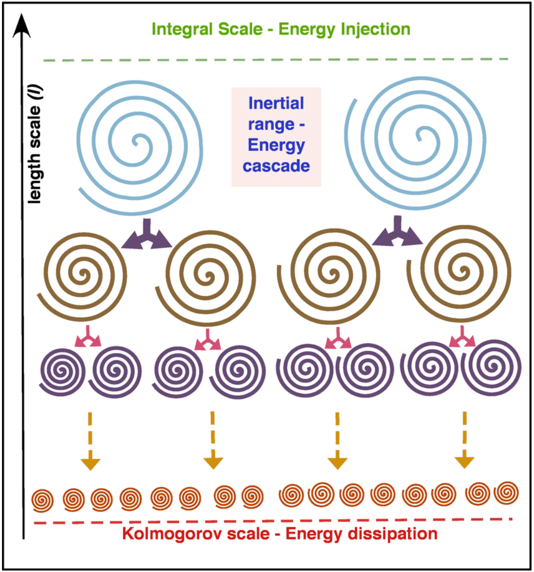
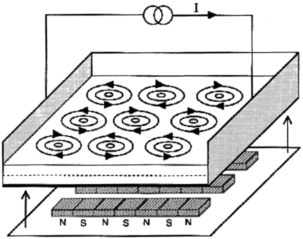
Breakdown of large eddies into progressively smaller ones

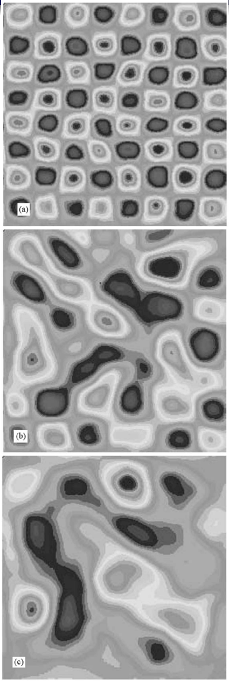


Richardson 1922:

«Big whirls have little whirl, that feed on their velocity,
and little whirls have lesser whirls and so on to viscosity.»

2D: Tabeling group, LPS de l'ENS 90's



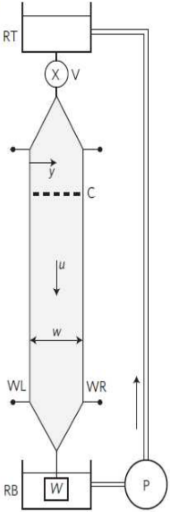


Thin layer of electrolyte

vortices driven by electromagnetical forces

P. Tabeling, Phys. Rep. (2002)

Soap films ... and more

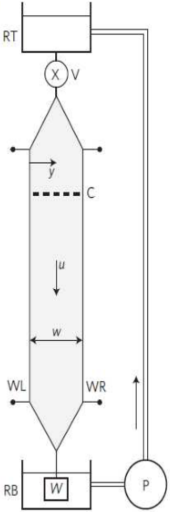


Tuan Tran et al., Nat. Phys. 2010

M. A. Rutgers, PRL 1998

Y. Couder, J.M. Chomaz, M. Rabaud, Physica D 1989

Soap films ... and more



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Great Red Spot

smaller vortices and storm clusters can be seen being engulfed by the GRS
but formation dynamics is disputed
Sánchez-Lavega et al.
Geophys. Res. Lett. 2024
1939: formation of white ovals which merged to form "Red Jr." in 2020

velocity field $\vec{v}(\vec{r}, t)$, vorticity: $\vec{\nabla} \times \vec{v} = \omega(\vec{r}, t)\, \vec{e}_z$
point vortex i, position $\vec{r}_i(t)$, circulation Γ_i: $\omega(\vec{r}, t) = \sum_i \Gamma_i\, \delta(\vec{r} - \vec{r}_i(t))$

$$H = -\frac{1}{2\pi} \sum_{i>j} \Gamma_i\, \Gamma_j \ln |\vec{r}_i - \vec{r}_j|\ , \quad \text{writing } \vec{r}_i(t) = (x_i, y_i) \text{ yields } \begin{cases} \Gamma_i\, \dot{x}_i = \partial H / \partial y_i \\ \Gamma_i\, \dot{y}_i = -\partial H / \partial x_i \end{cases}$$

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Onsager 1949

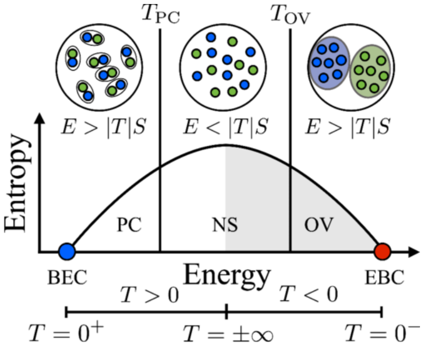
Phase space has a finite volume:
$\int d\Omega = (\int dxdy)^N = A^N$.

$$\Phi(E) \stackrel{def}{=} \int_{H<E} d\Omega = \int_{-\infty}^{E} \Phi'(E) dE.$$

$\Phi'(E)$ is positive with $\Phi'(\pm\infty) = 0$
(since $\Phi(-\infty) = 0$, $\Phi(+\infty) = A^N$)

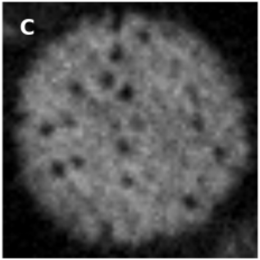
→ negative temperatures
(since $T^{-1} = (\partial S/\partial E) = \Phi''/\Phi'$)

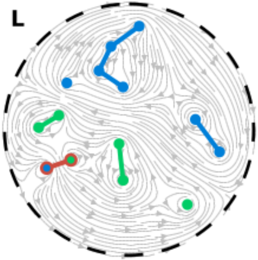
« If $T < 0$ vortices of the same sign will tend to cluster, so as to use up excess energy at the least possible cost in terms of degrees of freedom.»



Simula, Davis, Helmerson PRL 2014

Vortex clustering in two dimensional superfluids

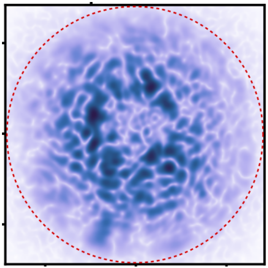


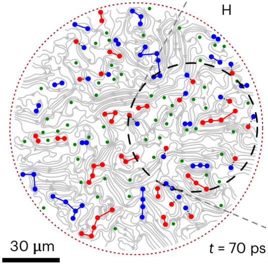


group of Helmerson and Johnstone, Monash University (Science 2019)

see also group of Neely, Davis, and Rubinsztein-Dunlop, University of Queensland

^{87}Rb atoms

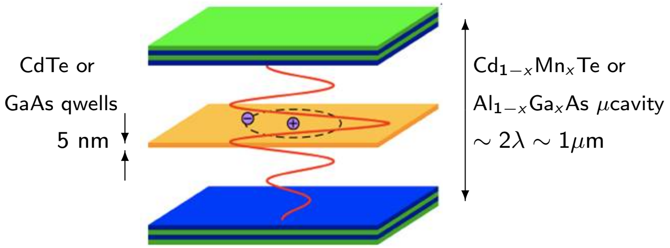




group of Sanvitto and Ballarini, University of Lecce (Nat. Photon. 2023)

exciton-polaritons (hybrid light–matter quasiparticles)

Cavity polaritons



interacting bosons

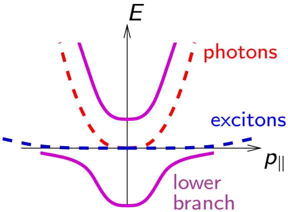
$m_{\mathrm{eff}} \lesssim 10^{-4}\, m_{\mathrm{e}}$

$T_{\mathrm{BEC}} \sim 10$ K

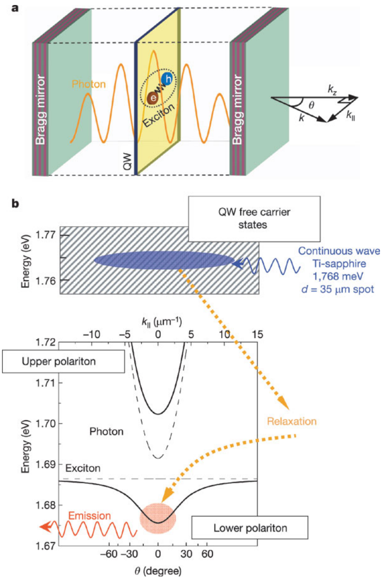
lifetime $\lesssim 50$ ps

optical detection

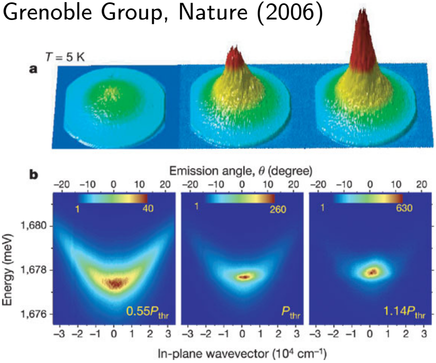
polarization degree of freedom

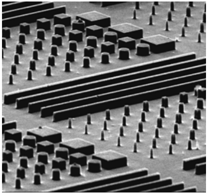


BEC of polaritons



Grenoble Group, Nature (2006)





Grenoble: institut Néel

Paris-Saclay: C2N

Paris: LKB

What is so special about superfluids ?

Superfluid order parameter: $\psi(\vec{r}, t) = A \exp(i\, S)$

$\rightarrow$ ground state: stationnary and uniform: $A(\vec{r}, t) = A_0$, $S(\vec{r}, t) = -\mu t/\hbar$

$\rightarrow$ constant uniform velocity $\vec{v}_0$: $S(\vec{r}, t) = -\mu t/\hbar + (m\, \vec{v}_0 \cdot \vec{r} - \frac{1}{2} m v_0^2 t)/\hbar$

$\rightarrow$ non uniform velocity: $S(\vec{r}, t)$ with $\vec{v}(\vec{r}, t) = \frac{\hbar}{m} \vec{\nabla} S$

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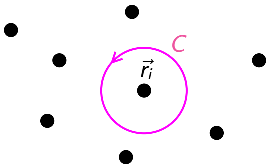
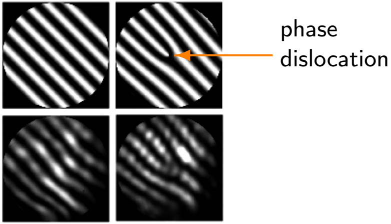
$\rightarrow$ non uniform velocity: $S(\vec{r},t)$ with $\vec{v}(\vec{r},t) = \frac{\hbar}{m}\vec{\nabla}S$

This impose $\vec{\nabla}\times\vec{v} = 0$: irrotational flow. Except if there are points $\vec{r}_i$ where S is ill defined, i.e. where $A(\vec{r}_i) = 0$. Then $\omega(\vec{r}) = \sum_i \Gamma_i\,\delta(\vec{r}-\vec{r}_i)$.

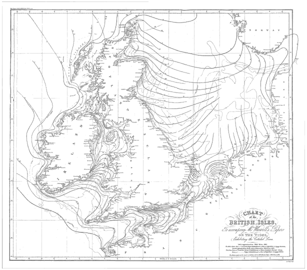
$$\oint_C \vec{v}\cdot d\vec{r} = \Gamma_i = \frac{\hbar}{m}\oint_C dS = \frac{\hbar}{m}2\pi n_i \quad (n_i \in \mathbb{Z},\ \text{typically}\ \ n_i = \pm 1)$$

Onsager-Feynman quantization condition



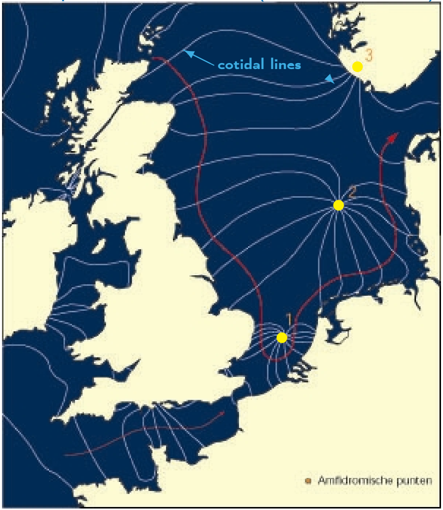


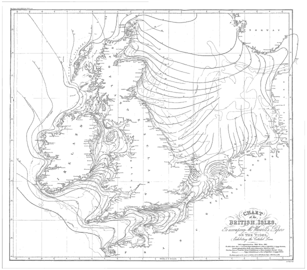
Rev. Sci. Instrum. **84**, 026103 (2013)



W. Whewell
Phil. Trans. R. Soc. 1836

Principal tidal constituent (M2: semi-diurnal)





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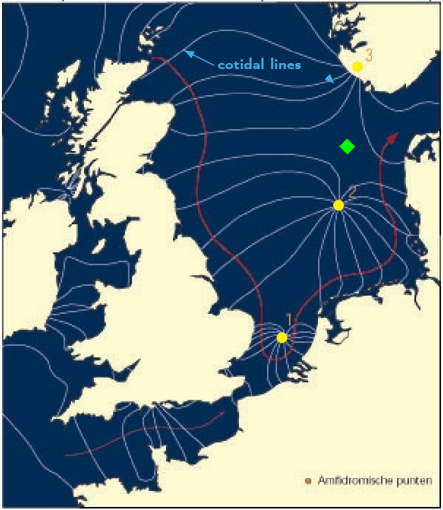

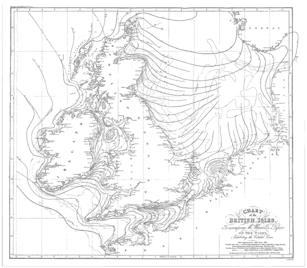

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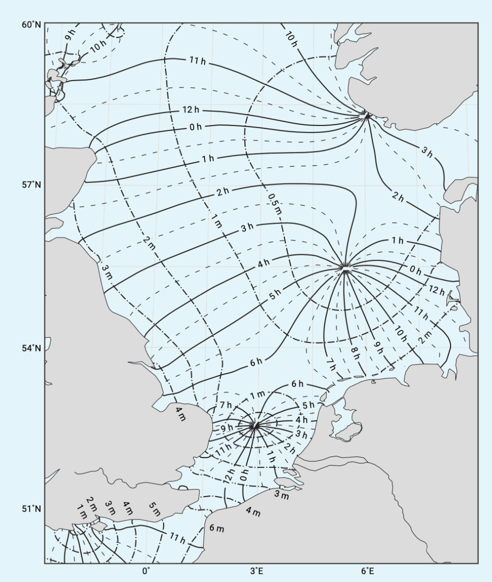
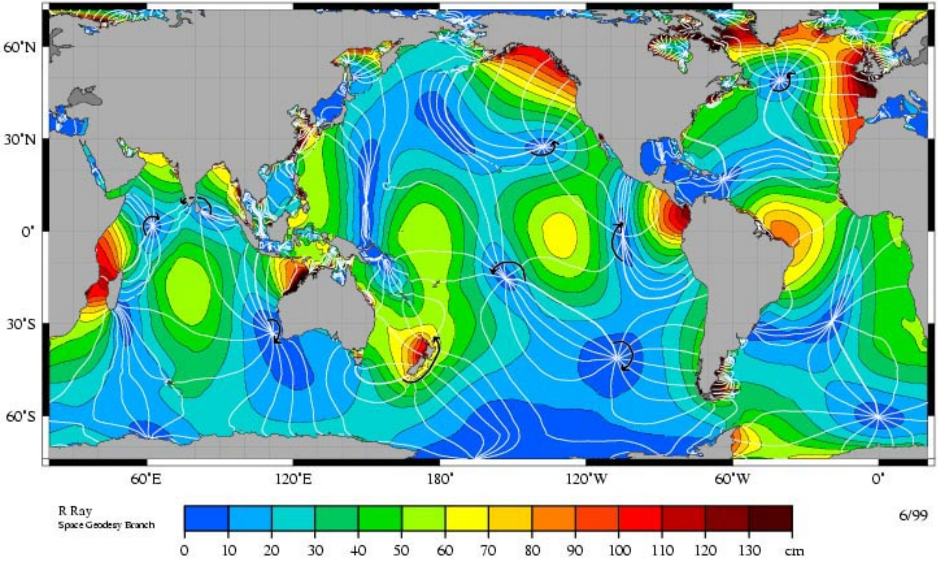
$$\psi = h(\vec{r})\exp\{i(S(\vec{r}) - \omega t)\}$$



co-range and cotidal lines

Amphidromic M2 points



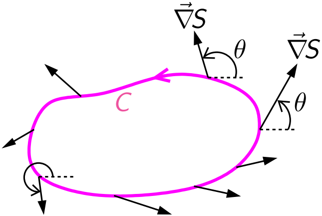
$T_{M2} = 12$ h 25 min, $\Delta t_{cotidal} = 1$ h 2 min, $\omega_{M2} \times \Delta t_{cotidal} = \pi/6$

$\Longrightarrow$ quantized circulation of tidal current of the M2 component

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r})\exp\{iS(\vec{r})\}$

$\vec{v} = \frac{\hbar}{m}\vec{\nabla} S$



$$I_{\mathrm{V}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}S \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}S}{2\pi}$$

$$I_{\mathrm{P}}(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}\theta \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}\theta}{2\pi}$$

I_{V} : vorticity, I_{P} : Poincaré index

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

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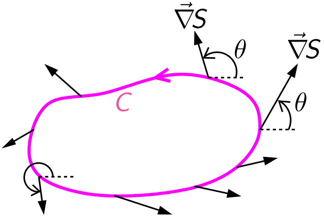

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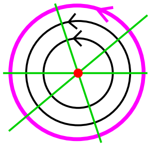
vortex



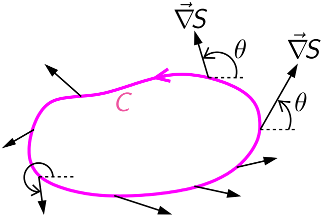
$I_{\mathrm{V}} \in \mathbb{Z}$

$I_{\mathrm{P}} = +1$

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r}) \exp\{iS(\vec{r})\}$

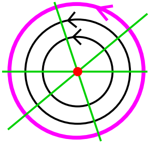
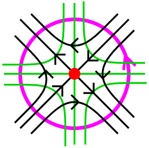
$\vec{v} = \frac{\hbar}{m}\vec{\nabla} S$



$$I_V(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}S \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}S}{2\pi}$$

$$I_P(C) = \frac{1}{2\pi}\oint_C \vec{\nabla}\theta \cdot \mathrm{d}\vec{\ell} = \oint_C \frac{\mathrm{d}\theta}{2\pi}$$

I_V : vorticity, I_P : Poincaré index

vortex		$I_V \in \mathbb{Z}$ $I_P = +1$
saddle		$I_V = 0$ $I_P = -1$

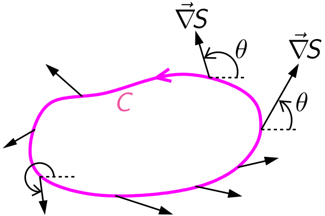
Two topological indices

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)

$\vec{r} = x\vec{e}_x + y\vec{e}_y$

$\Psi(\vec{r}) = A(\vec{r})\exp\{iS(\vec{r})\}$

$\boldsymbol{\vec{v} = \frac{\hbar}{m}\vec{\nabla} S}$



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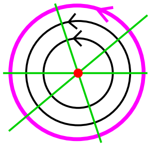
I_{V} : vorticity, I_{P} : Poincaré index

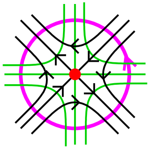
vortex



$I_{\mathrm{V}} \in \mathbb{Z}$

$I_{\mathrm{P}} = +1$

saddle

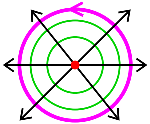


$I_{\mathrm{V}} = 0$

$I_{\mathrm{P}} = -1$

node

min/max of the phase



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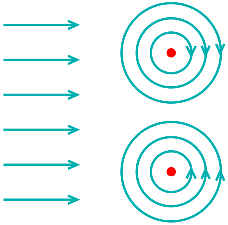
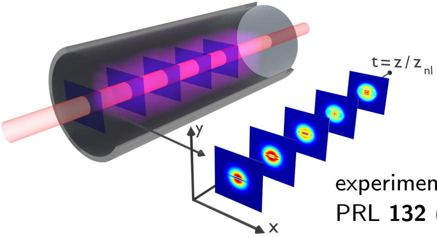
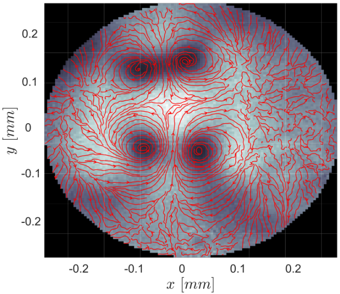



CLAC!

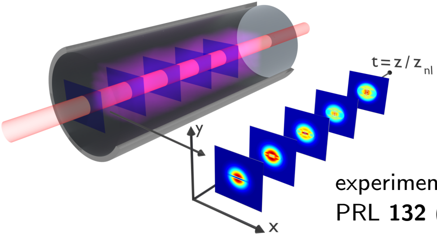


experiment in Nice
PRL **132** (2024)

output intensity and streamlines



Experimental streamlines



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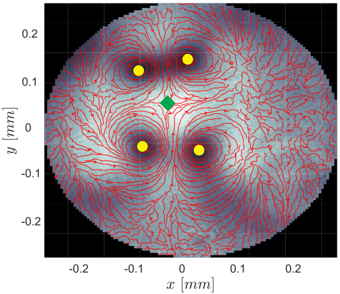

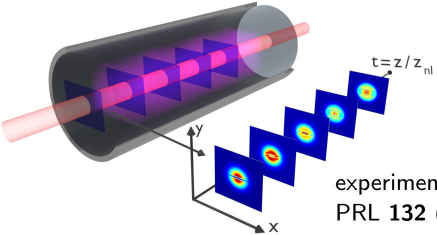

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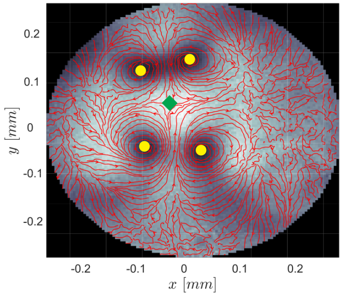
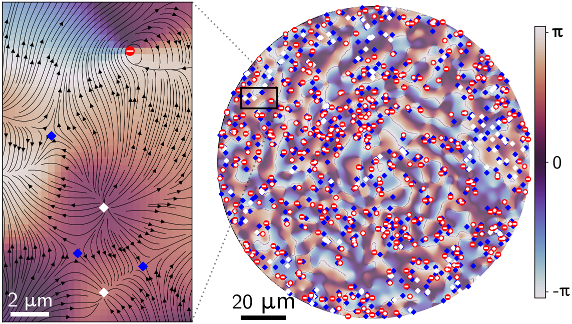


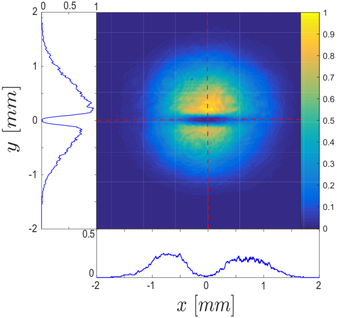


experiment in Lecce
arXiv:2411.11671

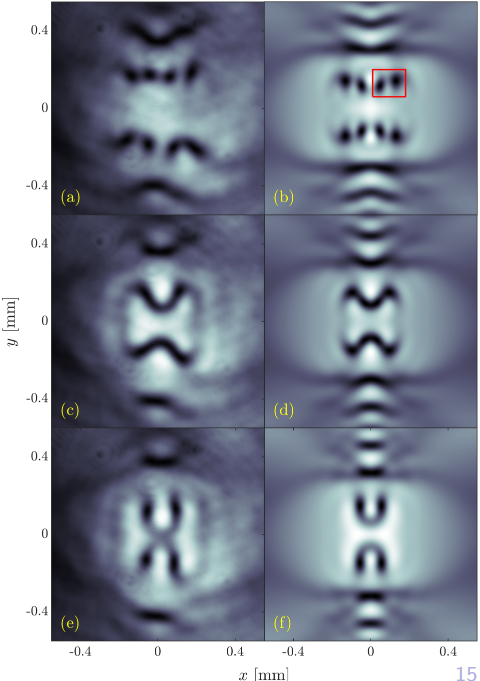
direct measure of the (large) number of vortices, saddles and nodes as a function of time

$$\psi(\vec{r},0) = \sqrt{I_1}\exp\left(-\frac{r^2}{w_{\mathrm{G}}^2}\right) + \sqrt{I_2}\exp\left(-\frac{x^2}{w_x^2} - \frac{y^2}{w_y^2}\right) e^{i\Phi_2}$$

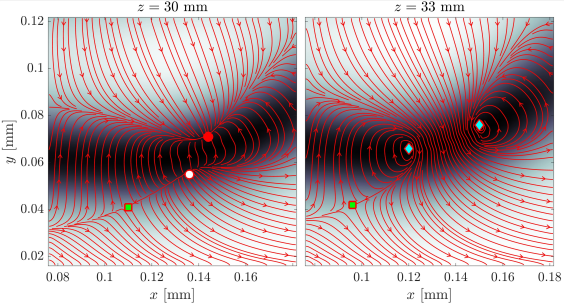
$I_1 = I_2 \qquad \Phi_2 \simeq \pi$



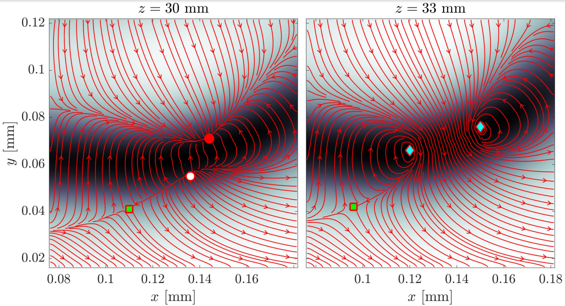
(a,b) : $\Phi_2 = 0.96\,\pi$
(c,d) : $\Phi_2 = \pi$
(e,f) : $\Phi_2 = 1.05\,\pi$



Mechanism of vortex formation/annihilation: fold-Hopf bifurcation



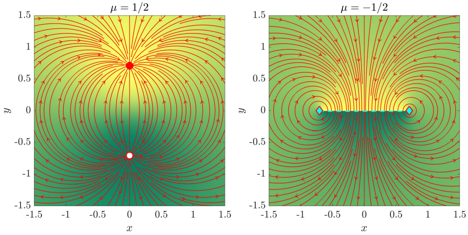
Mechanism of vortex formation/annihilation: fold-Hopf bifurcation

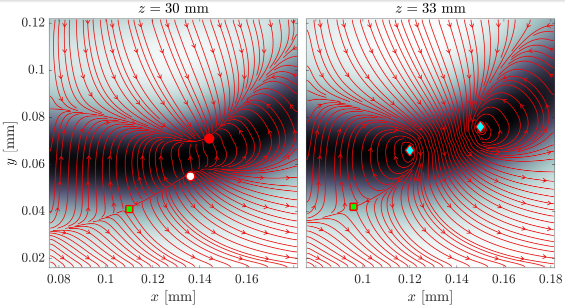


fold-Hopf bifurcation:

$$\begin{cases} v_x = -2\sigma xy \\ v_y = \mu + \sigma x^2 - y^2 \end{cases}$$

$\sigma = 1, \ \mu \in \mathbb{R}$





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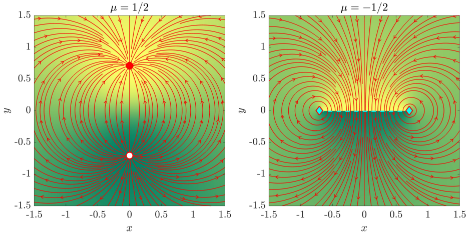
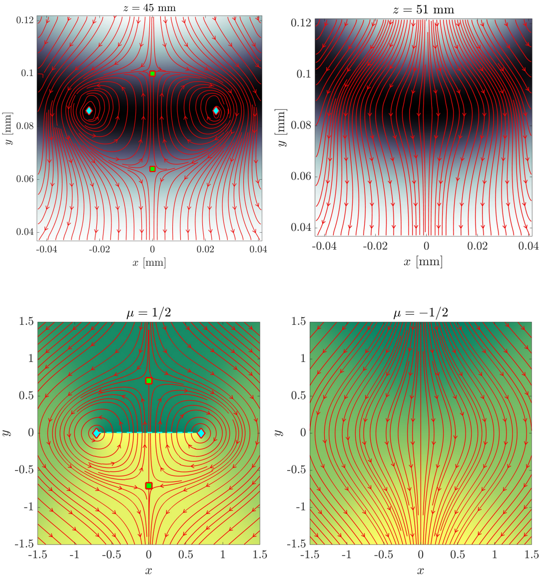


orbitally equivalent system: $\vec{v} = \vec{\nabla} S_{\mathrm{fH}}$, $S_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$.

✓ gradient system

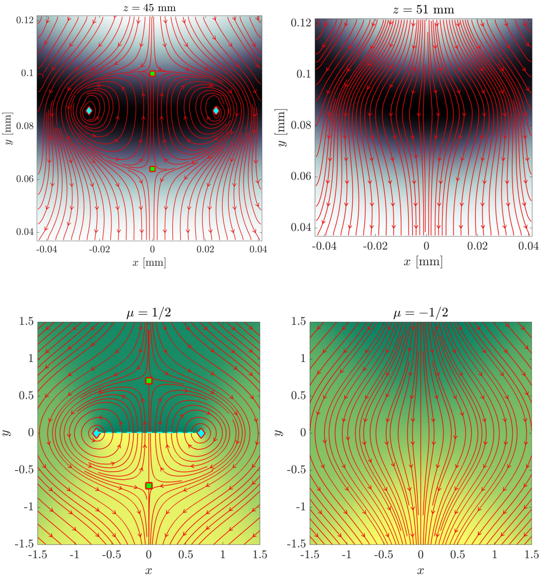
✓ verifies Onsager-Feynman quantization condition



$$\vec{v} = \vec{\nabla} S_{\mathrm{fH}} \qquad \text{where} \qquad S_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$$

$$\text{with} \qquad \sigma = -1 \quad \text{and} \quad \mu \in \mathbb{R}$$

vortex formation/annihilation: Bristol mechanism ... also fold-Hopf



close to the bifurcation point

the phase of the model wave function (solution of Helmoltz equation) matches $S_{\mathrm{fH}}(\vec{r})$

$$\vec{v} = \vec{\nabla} S_{\mathrm{fH}} \quad \text{where} \quad S_{\mathrm{fH}}(\vec{r}) \equiv \arg\left[x^2 + \sigma(y^2 + \mu) + i\sigma y\right]$$

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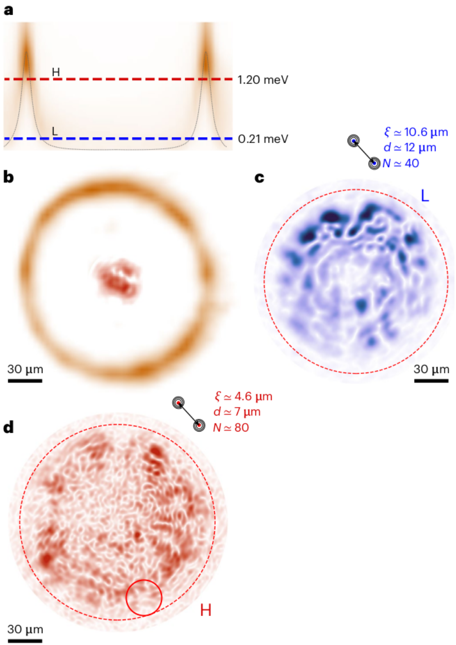
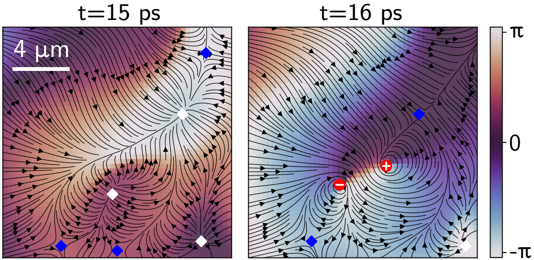
Experiment at Lecce



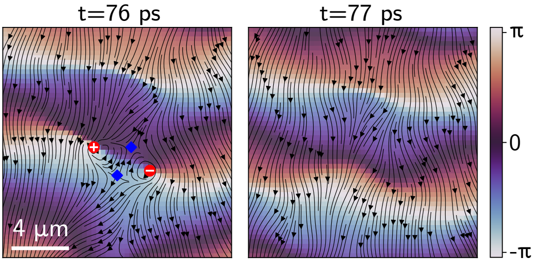
Node collision

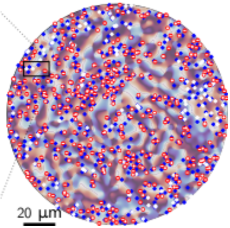
(fold-Hopf with $\sigma = 1$)



Bristol mechanism

(fold-Hopf with $\sigma = -1$)





simplest possible processes

$$\text{node} + \text{node} \underset{b}{\overset{a}{\rightleftharpoons}} \text{vortex}_{(+)} + \text{vortex}_{(-)}$$

$$\varnothing \underset{d}{\overset{c}{\rightleftharpoons}} \text{node} + \text{saddle}$$

The (positive) quantities a, b, c, and d are the reaction rates.

Rate equations:

$$\frac{\mathrm{d}V_\pm}{\mathrm{d}t} = aN^2 - bV_+V_-,$$
$$\frac{\mathrm{d}S}{\mathrm{d}t} = c - dNS,$$
$$\frac{\mathrm{d}N}{\mathrm{d}t} = c - 2aN^2 + 2bV_+V_- - dNS,$$

where $N(t)$ denotes the number of nodes, $S(t)$ the number of saddles, and $V_+(t)$ [$V_-(t)$] the number of vortices with positive [negative] vorticity.

In the following $b = 0$

Rate equations of elementary chemical reactions

$$(b=0) \qquad \frac{\mathrm{d}V_\pm}{\mathrm{d}t} = aN^2, \qquad \frac{\mathrm{d}S}{\mathrm{d}t} = c - dNS, \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = c - 2aN^2 - dNS,$$

No imparted angular momentum:

$$V_+(t) - V_-(t) = C^{st} \ll typical\left(V_+(t) + V_-(t)\right) \quad \rightsquigarrow \quad V_+(t) = V_-(t) \equiv \frac{V(t)}{2}$$

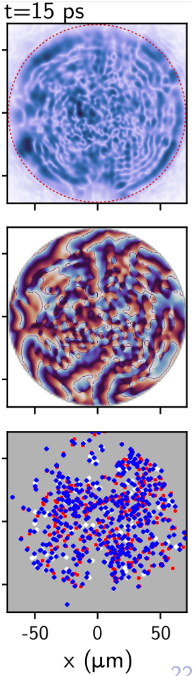
rescaled quantities

$\tau = t/t_0$, $n = N/N_0$, $v = V/N_0$, $s = S/N_0$

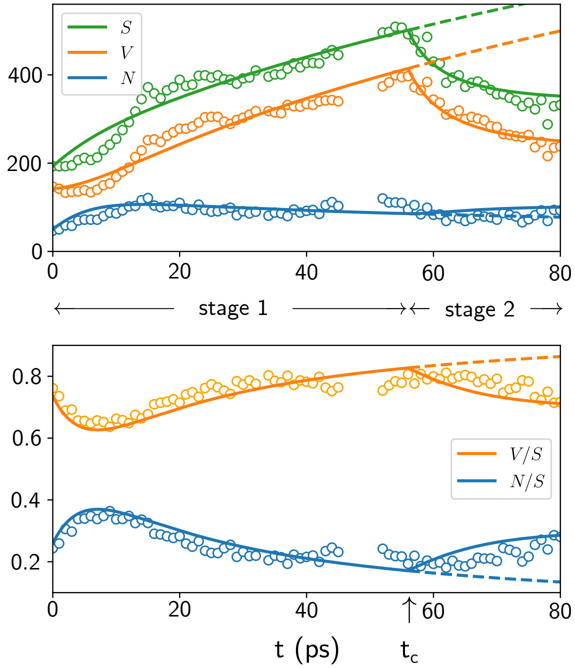
with $t_0 = 1/\sqrt{2ac}$ and $N_0 = \sqrt{c/2a}$

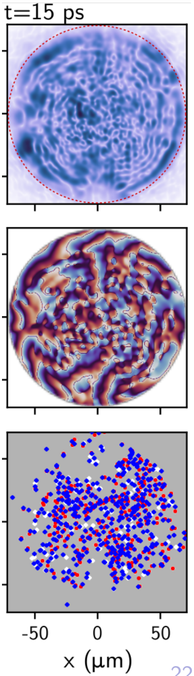
$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}\tau} = n^2, \qquad \frac{\mathrm{d}s}{\mathrm{d}\tau} = 1 - \gamma ns, \qquad \frac{\mathrm{d}n}{\mathrm{d}\tau} = 1 - n^2 - \gamma ns,}$$

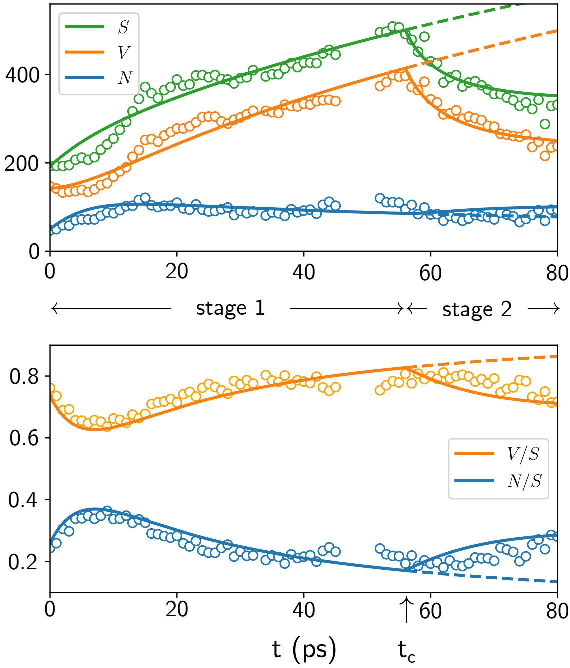
where the single parameter $\gamma \equiv d/2a$ governs the qualitative features of the dynamical system.


t=15 ps
-50
0
50
x (µm)

experimental results







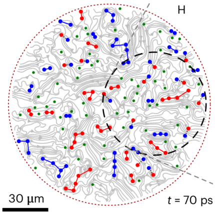
$$I_P = N(t) + V(t) - S(t) = C^{st} \simeq 0$$

End of clustering

The vortices are no longer grouped in packs of the same vorticity.
Nodes don't seem to play a significant role.
$\rightsquigarrow$ Bristol mechanism:

$$\text{vortex}_{(+)} + \text{vortex}_{(-)} + \text{saddle} + \text{saddle} \underset{f}{\overset{e}{\rightleftharpoons}} \varnothing$$

for simplicity, take $f = 0$.



$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = n^2 - \varepsilon\, v^2 s^2, \quad \frac{\mathrm{d}s}{\mathrm{d}\tau} = 1 - \gamma ns - \varepsilon\, v^2 s^2 \quad \frac{\mathrm{d}n}{\mathrm{d}\tau} = 1 - n^2 - \gamma ns,$$

where $\varepsilon = \frac{1}{2} e N_0^3 t_0 = ec/(8a^2)$ is the rescaled rate of annihilation of saddles and vortices through the Bristol mechanism.

$V \approx S \rightsquigarrow$ Bristol is effectively a 4 vortices mechanisms.

Conclusion and perspectives

2D quantum vortices

carry **2** topological indices

Two mechanisms of vortex formation/annihilation:

- Experimentally relevant
- fulfill **topological** and **quantum** constrains

Involve critical points $\neq$ vortices:
candidate observables for studying the transition to turbulence and its decay

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➪ effective kinetic description $\approx$ Landau mean field

➥ Topological constrains $\rightarrow$ non trivial spatial correlations between (different types of) critical points?

☞ Can one think of other nonlinear phenomena that are topologically constrained?

➪ Thermodynamic counterpart in BKT transition ?

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Thank you for your attention

refs: Congy, Azam, Kaiser, Pavloff, PRL **132**, 033804 (2024), Panico et al. arXiv:2411.11671

Berezinskii-Kosterlitz-Thouless transition

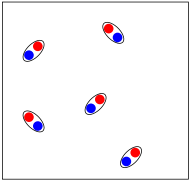
T_{BKT}

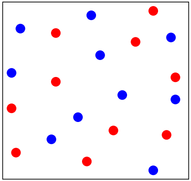
superfluid — normal → T

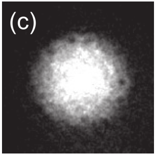
superfluid	normal
$\rho = \rho_s + \rho_n$	$\rho = \rho_n$
$\langle \psi^\dagger(\vec{r})\psi(\vec{0}\,)\rangle \sim r^{-\alpha}$	$\langle \psi^\dagger(\vec{r})\psi(\vec{0}\,)\rangle \sim \exp(-r/\ell)$

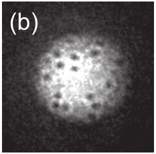
$\alpha_{crit} = \frac{1}{4}$

$$\alpha = \frac{1}{\rho_s \lambda_{th}^2} \leq \frac{1}{4}$$

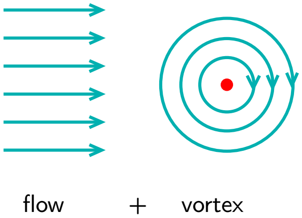




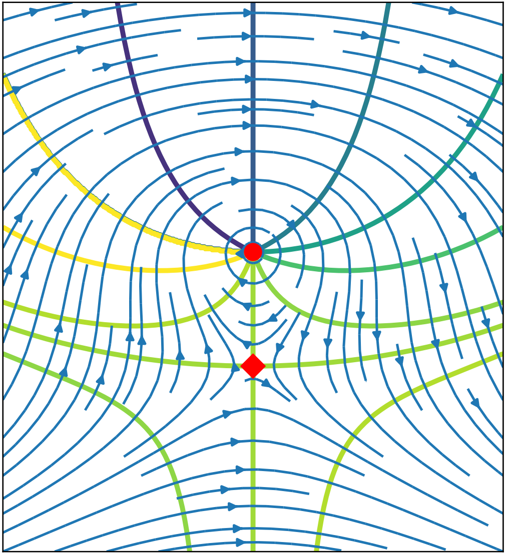


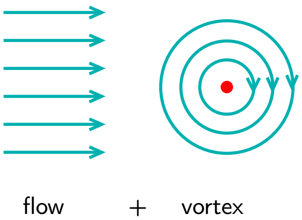


Shin group (Seoul)
PRL 2013

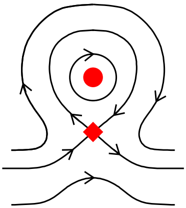


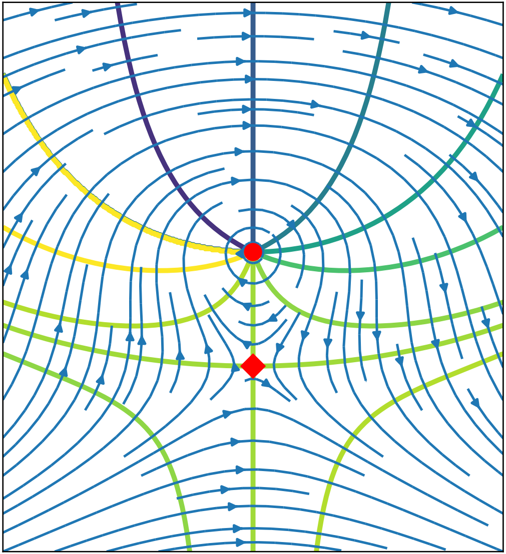
$$\psi = e^{\mathrm{i}kx}\,(x - \mathrm{i}y)$$

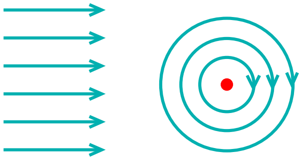




$$\psi = e^{\mathrm{i}kx}\,(x - \mathrm{i}y)$$

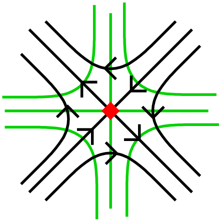


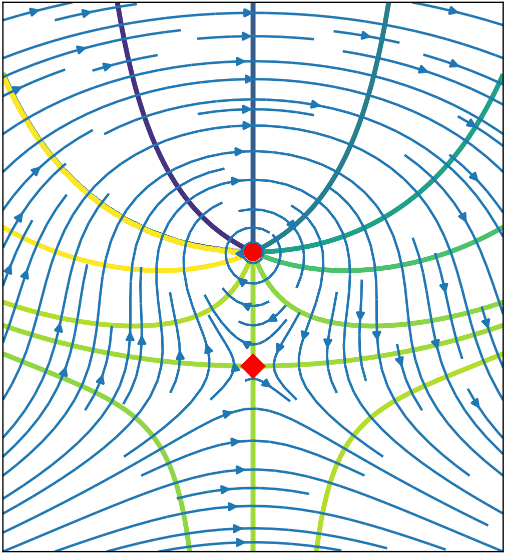




flow + vortex

$$\psi = e^{ikx}(x - iy)$$





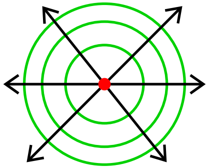
In a quantum fluid

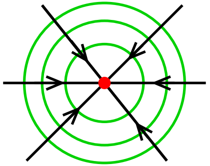
$$\Psi(\vec{r}) = A(\vec{r}) \exp\{iS(\vec{r})\}$$

$$\vec{v} = \vec{\nabla} S$$

phase S: velocity potential

min or max of S:





Other points with zero velocity: (elusive) nodes

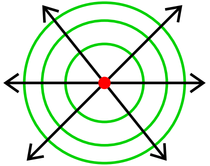
In a quantum fluid

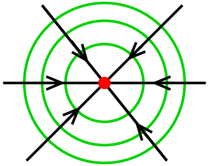
$$\Psi(\vec{r}) = A(\vec{r}) \exp\{iS(\vec{r})\}$$

$$\vec{v} = \vec{\nabla} S$$

phase S: velocity potential

min or max of S:





Nodes are forbidden:

- in an incompressible fluid:

$$\vec{\nabla} \cdot \vec{v} = 0 \Longrightarrow \vec{\nabla}^2 S = 0$$

Hence not seen, e.g., in a Coulomb gas model

- in a stationnary configuration:

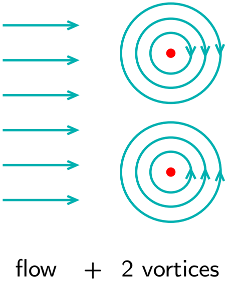
No sink nor source !

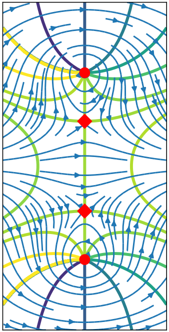
$$\vec{\nabla}^2 \psi + [f(A) + U(\vec{r}) - \mu]\psi = 0 \qquad (1)$$

where $\psi = A\exp(iS)$:

$$\vec{\nabla}^2 \psi = \left\{ \vec{\nabla}^2 A - A|\vec{\nabla} S|^2 + i(A\vec{\nabla}^2 S + 2\vec{\nabla} A \cdot \vec{\nabla} S) \right\} e^{iS}$$

at a stagnation point where $\vec{\nabla} S = 0$,
the imaginary part of (1) yields $\vec{\nabla}^2 S = 0$





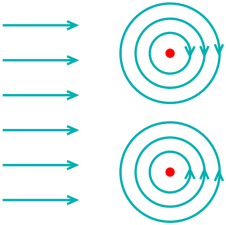
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{\mathrm{i}kx}\left[x - \mathrm{i}k(y^2 - b)\right]$$

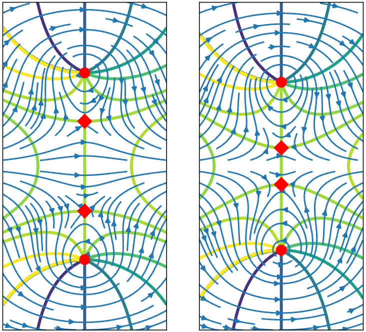
$k^2 b = 1.3$

Danse of the saddles

Nye, Hajnal and Hannay, Proc. R. Soc. (1988)



flow + 2 vortices



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1.3	1.05	

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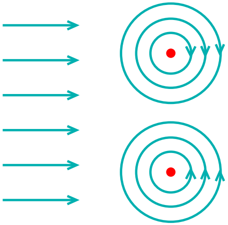
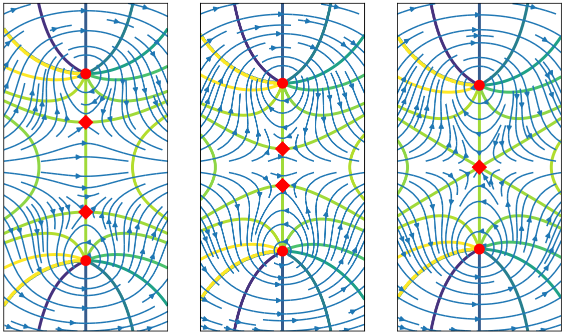


flow + 2 vortices



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$k^2 b =$

1.3	1.05	1

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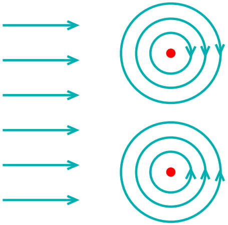
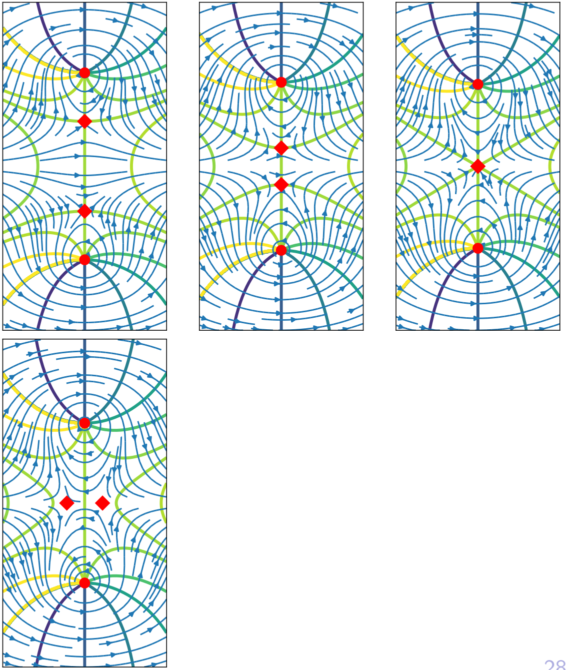


flow + 2 vortices

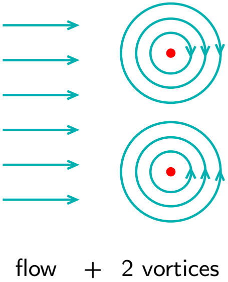


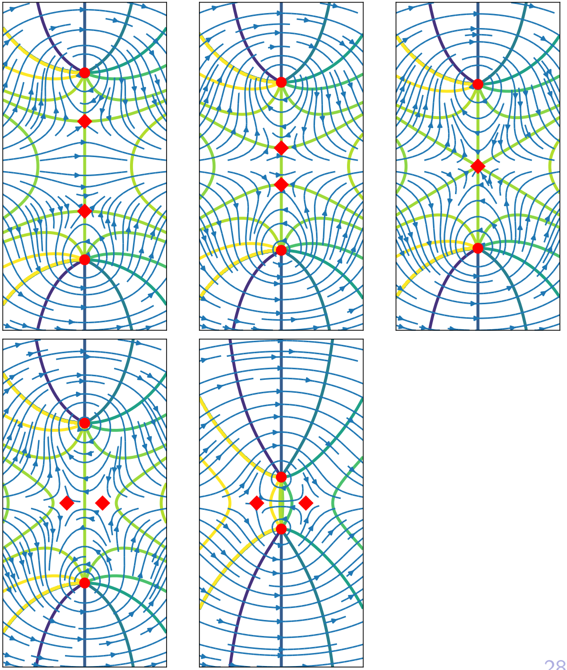
solution of $(\Delta + k^2)\psi = 0$

$$\psi = e^{ikx}\left[x - ik(y^2 - b)\right]$$

$k^2 b =$

1.3	1.05	1
0.95		





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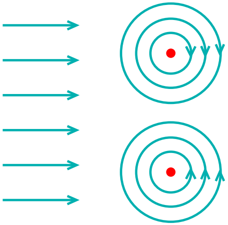

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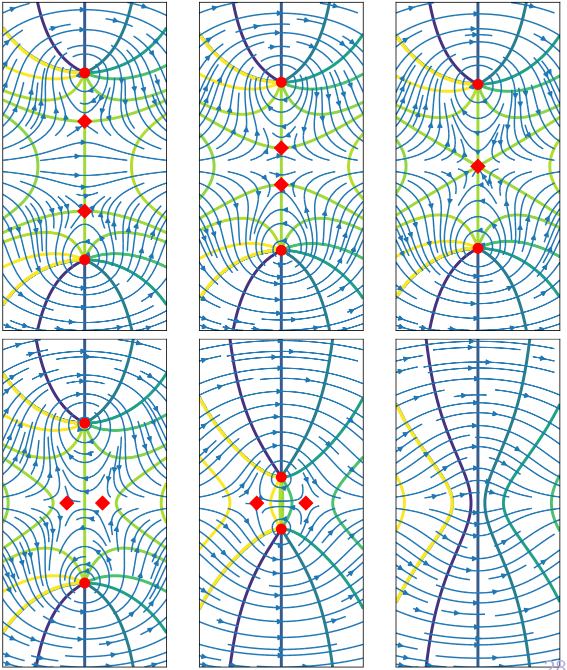

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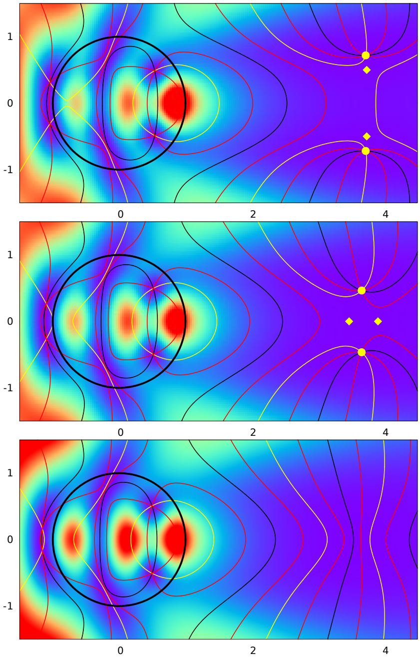
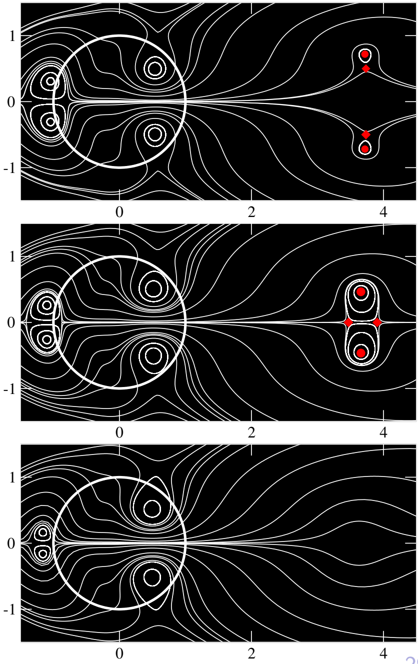
$k^2 b =$

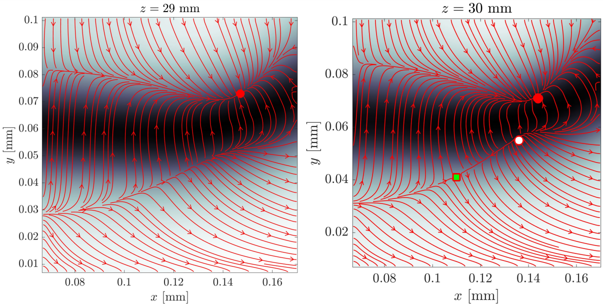
1.3	1.05	1
0.95	0.1	-0.1

2D scattering on an attractive cylinder



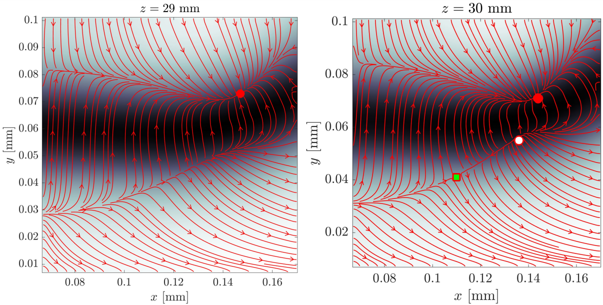


Formation of saddles and nodes: saddle-node bifurcation



$\Phi_2 = 0.96\,\pi$

Formation of saddles and nodes: saddle-node bifurcation

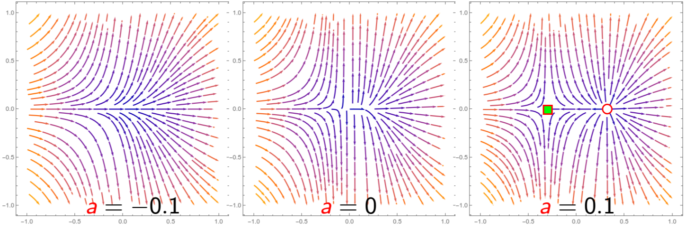


$\Phi_2 = 0.96\,\pi$

Saddle-node:

$$\begin{cases} v_x = x^2 - a \\ v_y = y \end{cases}$$

$a \in \mathbb{R}$



$\vec{v} = \vec{\nabla}\left(\frac{1}{3}x^3 - ax + \frac{1}{2}y^2\right)$

orbitally equivalently: $\vec{v} = \vec{\nabla} S$ where

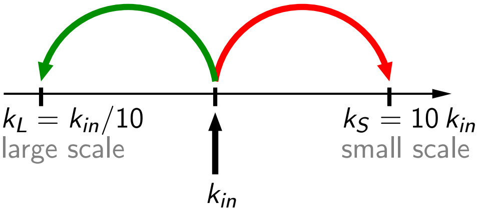
$S(\vec{r}) = (almost)any\ fct\ of\ Z = \frac{1}{3}x^3 - ax + \frac{1}{2}y^2$

velocity $\vec{u}(\vec{r},t)$ vorticity $\vec{\nabla} \times \vec{u} = \omega(\vec{r},t)\,\vec{e}_z$

Navier-Stokes $\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}P + \nu\Delta\vec{u}$ incompressible flow $\vec{\nabla} \cdot \vec{u} = 0$

kinetic energy $E = \frac{1}{2}\rho\int v^2\, d^2r = \int_0^\infty E(k)dk$

enstrophy $\Omega = \frac{1}{2}\int \omega^2\, d^2r = \int_0^\infty \Omega(k)dk$ $\Omega(k) = k^2 E(k)$



$$\begin{cases} E_{in} = E_L + E_S \\ \Omega_{in} = \Omega_L + \Omega_S \Leftrightarrow k_{in}^2 E_{in} = k_L^2 E_L + k_S^2 E_S \end{cases}$$

$E_L/E_S = 10^2$: energy transfered to large scales (inverse cascade)

$\Omega_S/\Omega_L = 10^2$: enstrophy transfered to small scales (direct cascade)