Gas dynamics of solitons in integrable systems

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Plan before lunch:

1. Introduction: turbulence in dispersive hydrodynamics
2. Kinetic description of soliton gas
3. Solutions: polychromatic gas and condensate
4. Conclusion and perspectives

Dispersive hydrodynamics

- Nonlinear wave phenomena in media dominated by dispersive effects.

- Shallow water regime: water level $u(x,t)$ governed by Korteweg-de Vries (KdV) eq.

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0.$$

- This equation is universal and has been derived in other contexts: plasma, Bose-Einstein condensate, electrical line, etc.

- KdV equation can be written as compatibility condition of Lax pair:

$$-\partial_x^2\psi - u\,\psi = \zeta\,\psi, \qquad \partial_t\psi = -\,\partial_x^3\psi + 3(\zeta - u)\psi\,.$$

First equation is Schrödinger eq. with potential $-u(x,t) \in \mathbb{R}$, eigenfunction $\psi(x,t) \in \mathbb{C}$ and **time-independent** eigenvalue $\zeta \in \mathbb{R}$ (also termed isospectrality).

Inverse scattering transform

- Lax pair: $-\partial_x^2\psi - u\,\psi = \zeta\,\psi, \quad \partial_t\psi = -\,\partial_x^3\psi + 3(\zeta - u)\psi\,.$
- Initial value problem can be solved via inverse scattering transform (or "nonlinear Fourier transform"):

initial condition: $u(x,0) \xrightarrow[x\to\pm\infty]{} 0$

$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0$

solution $u(x,t)$

direct scattering

inverse scattering

scattering data:

$\zeta > 0$ free states

$\zeta_n < 0$ bound states

linear evolution of scattering data

scattering data at $t > 0$

- Drastically simplifies for **reflectionless potential:** $u(x,t)$ can be reconstructed with bound states scattering data only.

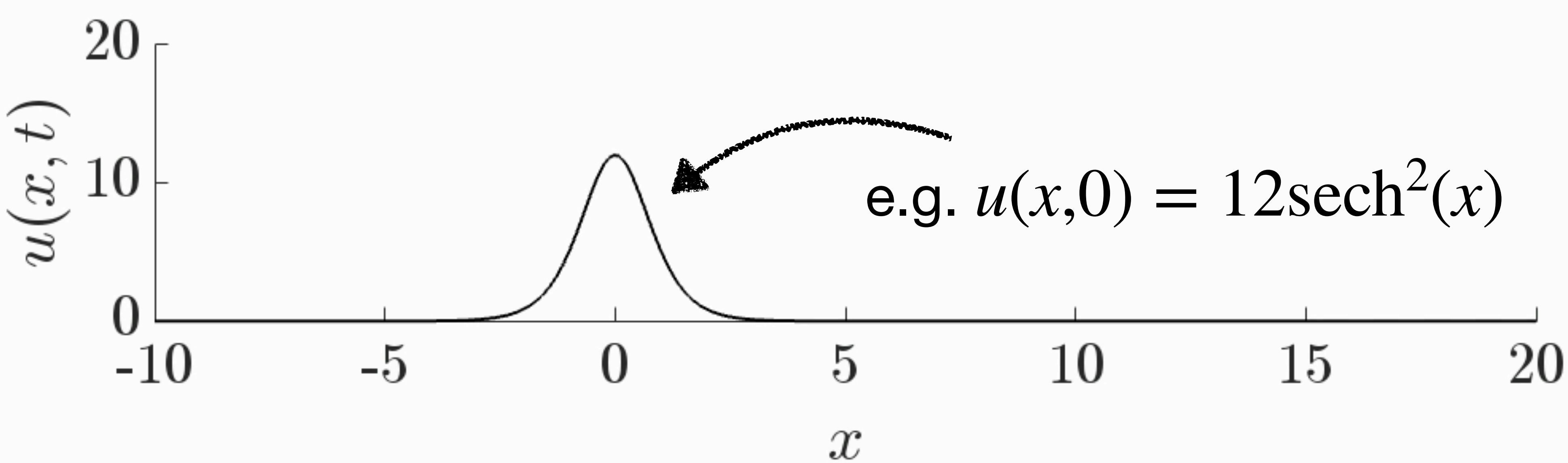
N-soliton solution

- Reflectionless potentials have the asymptotic behaviour:

$$u(x,t) \sim \sum_{i=1}^{N} u_s(x - x_i^0, t; \eta_i) \text{ when } t \to \infty,$$

where $\zeta_i = -\eta_i^2$ and u_s is called soliton solution:

$$u_s(x,t;\eta) = 2\eta^2 \operatorname{sech}^2\left[\eta(x - 4\eta^2 t)\right].$$

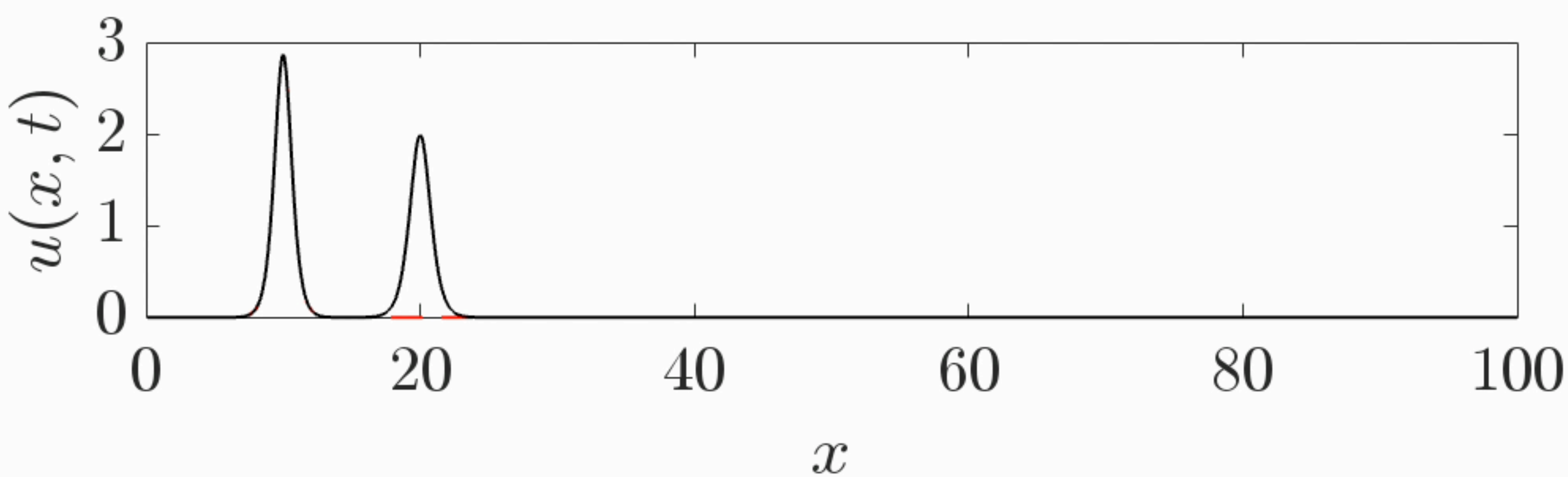


- $u(x,t)$ is entirely parameterised by $\{(\eta_i, x_i^0)\}_{1\le i\le N}$, and often called N-soliton solution.

Soliton: a wave that behaves like a particle

- Free soliton parameterised by $\eta > 0$ propagates at the constant speed $4\eta^2$ without deformation.

- Collision between two solitons parameterised by η and $\mu < \eta$:



  - In asymptotic regimes $t \to \pm\infty$ shape of solitons is preserved.
  - After collision of η-soliton is shifted by $G(\eta, \mu) = \dfrac{1}{\eta} \ln \left| \dfrac{\eta + \mu}{\eta - \mu} \right|$.

- Interaction between multiple solitons **always factorises to 2-soliton interaction.**

Soliton gas within integrable turbulence context

- Solve initial value problem where initial condition $u(x,0)$ is generated by random process:

random $u(x,0)$ (reflectionless) $\xrightarrow{\partial_t u + 6u\partial_x u + \partial_x^3 u = 0}$ statistics of $u(x,t)$ e.g. moments, PDF

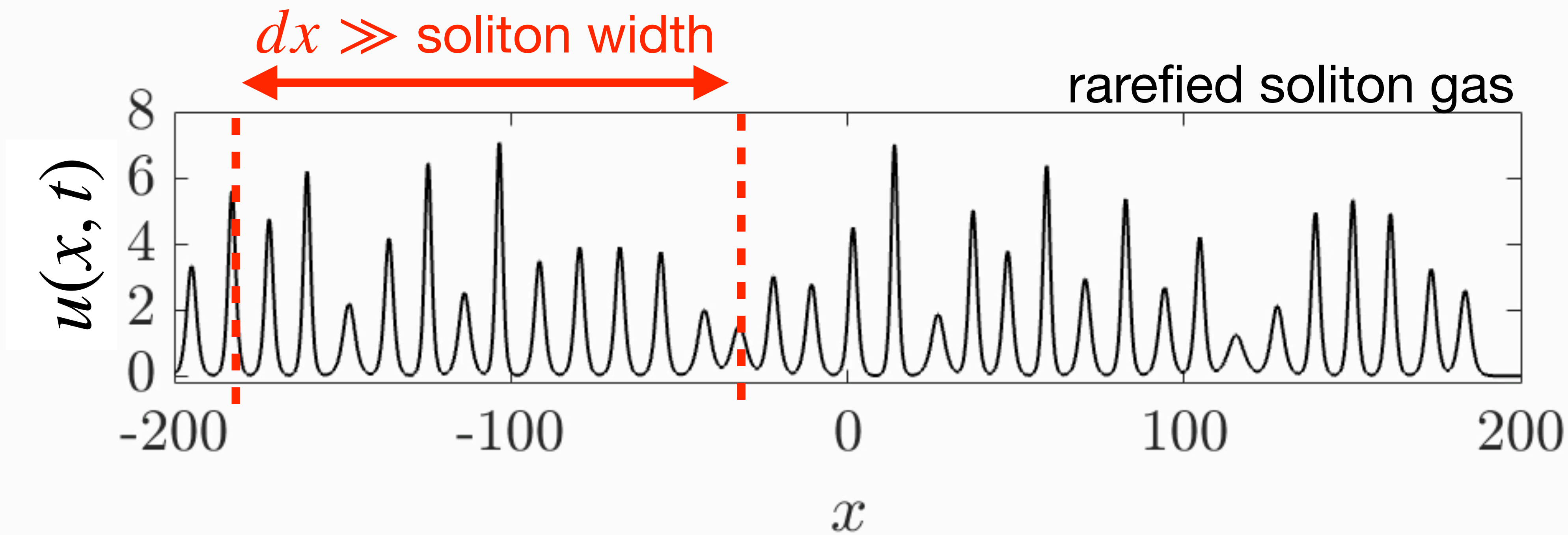
direct scattering ↓ ↑ inverse scattering

statistics of “soliton content” $\xrightarrow{\text{asymptotics } t \gg 1}$ **soliton gas**

- Random process generating (approximately) reflectionless potentials: partially coherent wave (Meiss & Horton ’82, Congy *et al* 2024), modulational stability (Gelash *et al* 2019, Agafontsev *et al* 2024).

- Fits within integrable turbulence framework introduced by Zakharov in 2009 where $u(x,0)$ can be any random potential.

Density of states



- Infinite randomised "soliton lattice" characterised by density of states $f(\eta; x, t) > 0$ of support Γ (Zakharov '71):

$$\#\text{solitons in } [x, x + dx] \text{ with parameters in } [\eta, \eta + d\eta] \subset \Gamma$$
$$= f(\eta; x, t)\, dx\, d\eta\,.$$

- Rigorous definitions of $f(\eta; x, t)$ for "dense" soliton gas via thermodynamic limit $N \to \infty$ of N-gap potentials (El 2003).

- Alternative construction with N-soliton solution (Girotti *et al* 2021, equivalence between two approaches in Jenkins & Tovbis 2024).

Kinetic description

- Effective velocity field of η-soliton in a gas (heuristic approach in Zakharov '71, El & Kamchatnov 2005):

$$s(\eta;x,t) = \underbrace{4\eta^2}_{\text{free velocity}} + \int_\Gamma \underbrace{G(\eta,\mu)}_{\text{position shift}} \overbrace{\underbrace{[s(\eta;x,t) - s(\mu;x,t)]}_{\text{relative mean velocity}} f(\mu;x,t)d\mu}^{\text{rate of collisions with } \mu\text{-solitons}} \,.$$

- Evolution of soliton gas is isospectral ⇒ number of solitons is conserved and density of states is governed by conservation law:

$$\partial_t f(\eta;x,t) + \partial_x \underbrace{[s(\eta;x,t)f(\eta;x,t)]}_{\text{current of solitons}} = 0 \qquad \text{(kinetic equation)}$$

- Modulation equation: $\partial_x f \ll \dfrac{f}{\text{soliton width}}$.

- $f(\eta;x,t) =$ statistics of soliton content i.e. scattering data,
what about the statistics of the wavefield $u(x,t)$?

Moments of the wavefield

- Infinite number of conservation laws $\partial_t P_k(x,t) + \partial_x Q_k(x,t) = 0$:

$$k = 1: \ \partial_t\left(\frac{u}{2}\right) + \partial_x\left(\frac{3u^2}{2} + \frac{\partial_x^2 u}{2}\right) = 0,$$

$$k = 2: \ \partial_t\left(\frac{u^2}{8}\right) + \partial_x\left(\frac{u^3}{2} + \frac{u\partial_x^2 u}{4} - \frac{(\partial_x u)^2}{8}\right) = 0,$$

$k \geq 3$: see Wadati *et al* '75.

- Ensemble-averages (El 2003):

$$\langle P_k \rangle(x,t) = \frac{2}{2k-1} \int_\Gamma \eta^{2k-1} f(\eta; x, t)\, d\eta,$$

$$\langle Q_k \rangle(x,t) = \frac{2}{2k-1} \int_\Gamma \eta^{2k-1} s(\eta; x, t) f(\eta; x, t)\, d\eta,$$

$$\Rightarrow \langle u \rangle(x,t) = \int_\Gamma 4\eta\, f(\eta; x, t)\, d\eta, \quad \langle u^2 \rangle(x,t) = \int_\Gamma \frac{16}{3} \eta^3 f(\eta; x, t)\, d\eta\, .$$

Polychromatic soliton gas

- Ansatz: $f(\eta; x, t) = \sum_{j=1}^{P} w_j(x,t)\delta(\eta - \eta_j), \quad \eta_j < \eta_{j+1}\,.$
- Kinetic equation becomes a quasi-linear system of dimension P

$$\partial_t w_j + \partial_x(s_j w_j) = 0, \quad j = 1\ldots P,$$
$$s_j = 4\eta_j^2 + \sum_{k\neq j} G_{jk} w_k (s_j - s_k), \quad G_{jk} = G(\eta_j, \eta_k)\,.$$

- $P = 2: s_1 = 4\eta_1^2 - \dfrac{G_{12}w_2(4\eta_2^2 - 4\eta_1^2)}{1 - \left(G_{12}w_2 + G_{21}w_1\right)} < 4\eta_1^2,$

$$s_2 = 4\eta_2^2 + \frac{G_{21}w_1(4\eta_2^2 - 4\eta_1^2)}{1 - \left(G_{12}w_2 + G_{21}w_1\right)} > 4\eta_2^2\,.$$

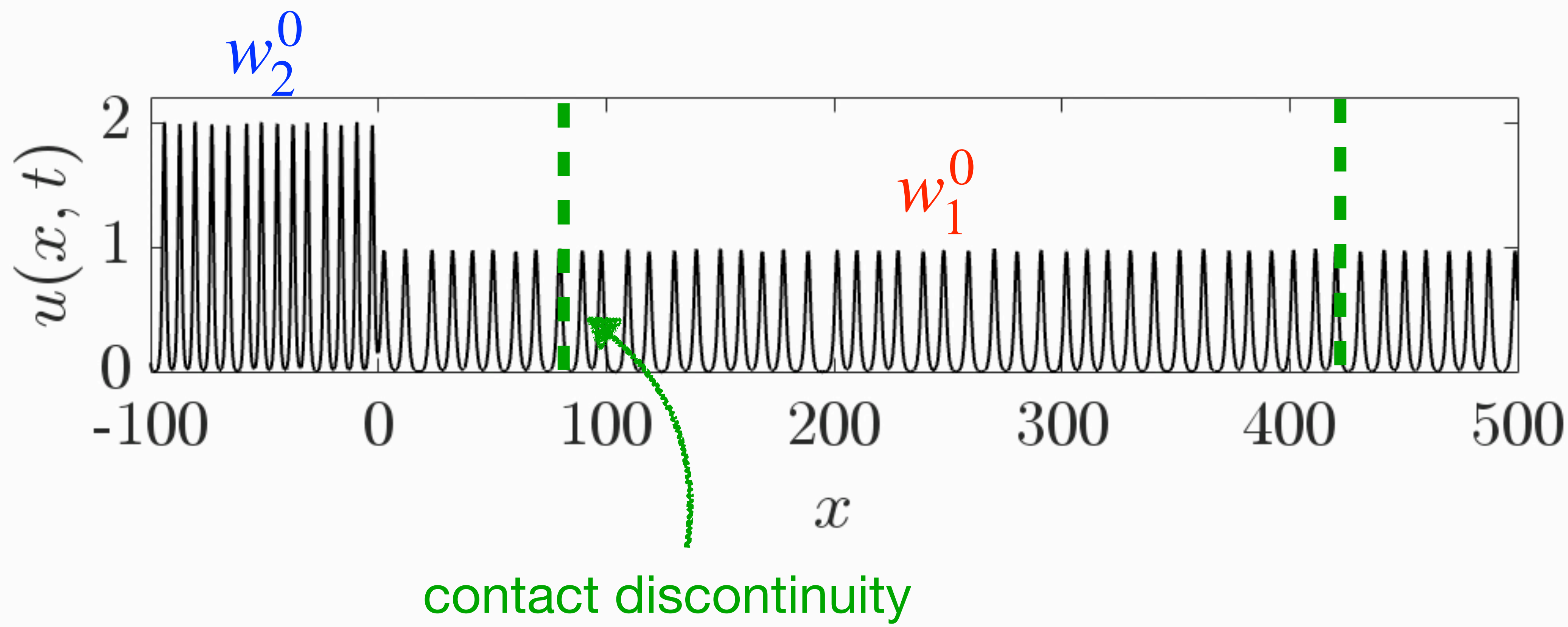
- This quasi-linear system is integrable and **linearly degenerate**, see El *et al* 2011 ⇒ no rarefaction wave solution.

Shock tube problem

- Riemann problem: step discontinuity for soliton densities (El & Kamchatnov 2005, Carbone *et al* 2016, Congy *et al* 2024)

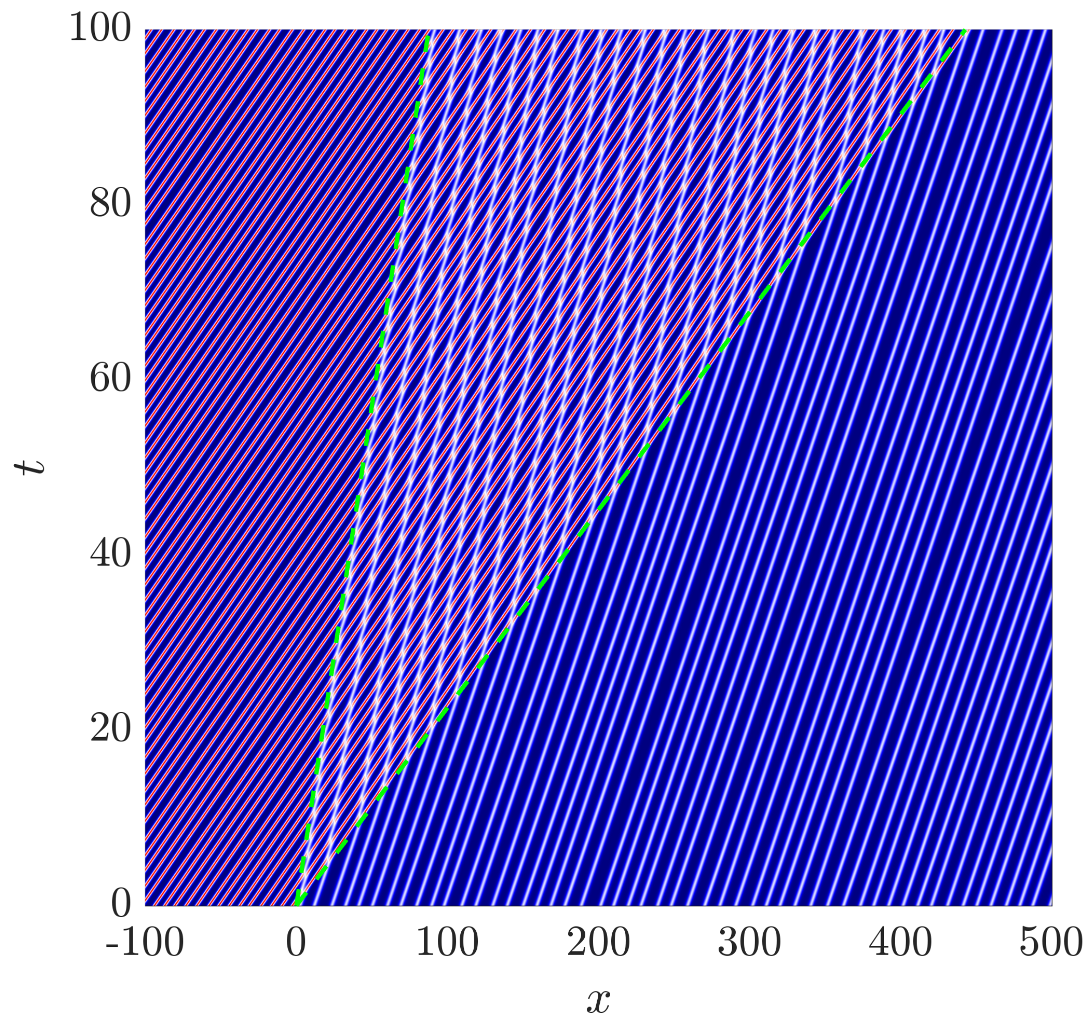
$$w_1(x,0) = \begin{cases} 0, & x < 0, \\ w_1^0, & x > 0, \end{cases} \qquad w_2(x,0) = \begin{cases} w_2^0, & x < 0, \\ 0, & x > 0. \end{cases}$$

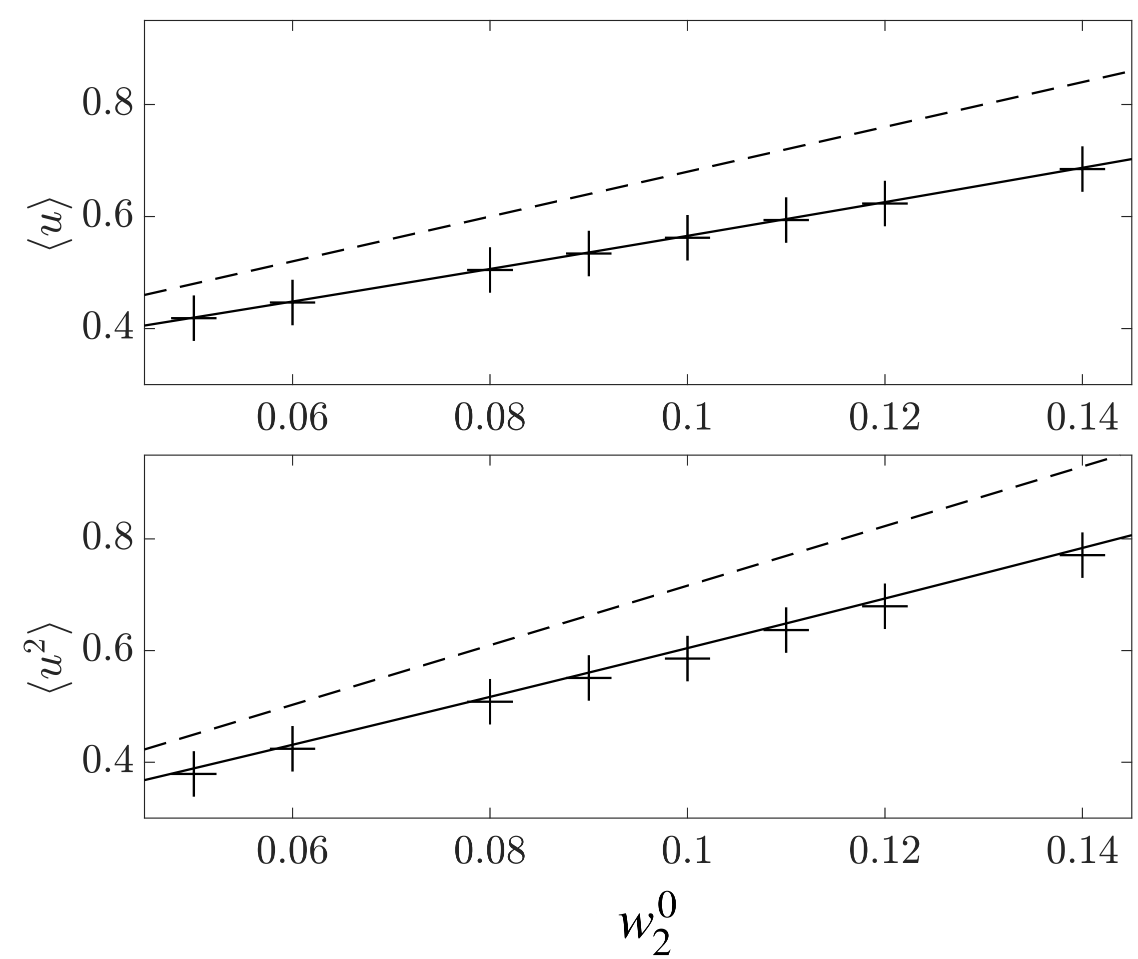
- Evolution of one soliton gas realisation:



$$w_1(x,t) = \begin{cases} 0, & \frac{x}{t} < c_1, \\ \kappa_1, & c_1 < \frac{x}{t} < c_2, \\ w_1^0, & c_2 < \frac{x}{t}, \end{cases} \qquad w_2(x,t) = \begin{cases} w_2^0, & \frac{x}{t} < c_1, \\ \kappa_2, & c_1 < \frac{x}{t} < c_2, \\ 0, & c_2 < \frac{x}{t}, \end{cases}$$

$$\kappa_1 = \frac{\left(1 - G_{12}w_2^0\right) w_1^0}{1 - G_{12}w_2^0 G_{21}w_1^0}, \quad \kappa_2 = \frac{\left(1 - G_{21}w_1^0\right) w_2^0}{1 - G_{12}w_2^0 G_{21}w_1^0}, \quad c_1 = \frac{4\eta_1^2 - 4\eta_2^2 G_{12}w_2^0}{1 - G_{12}w_2^0}, \quad c_2 = \frac{4\eta_2^2 - 4\eta_1^2 G_{21}w_1^0}{1 - G_{21}w_1^0}.$$





Soliton condensate

- "Densest possible" soliton gas (El & Tovbis 2020, Girotti *et al* 2021, Kuijlaars & Tovbis 2021):

$$\int_\Gamma G(\eta,\mu) f(\mu) d\mu = 1, \quad \forall \eta \in \Gamma = [0,\lambda_1] \cup \ldots \cup [\lambda_{2P}, \lambda_{2P+1}] \, .$$

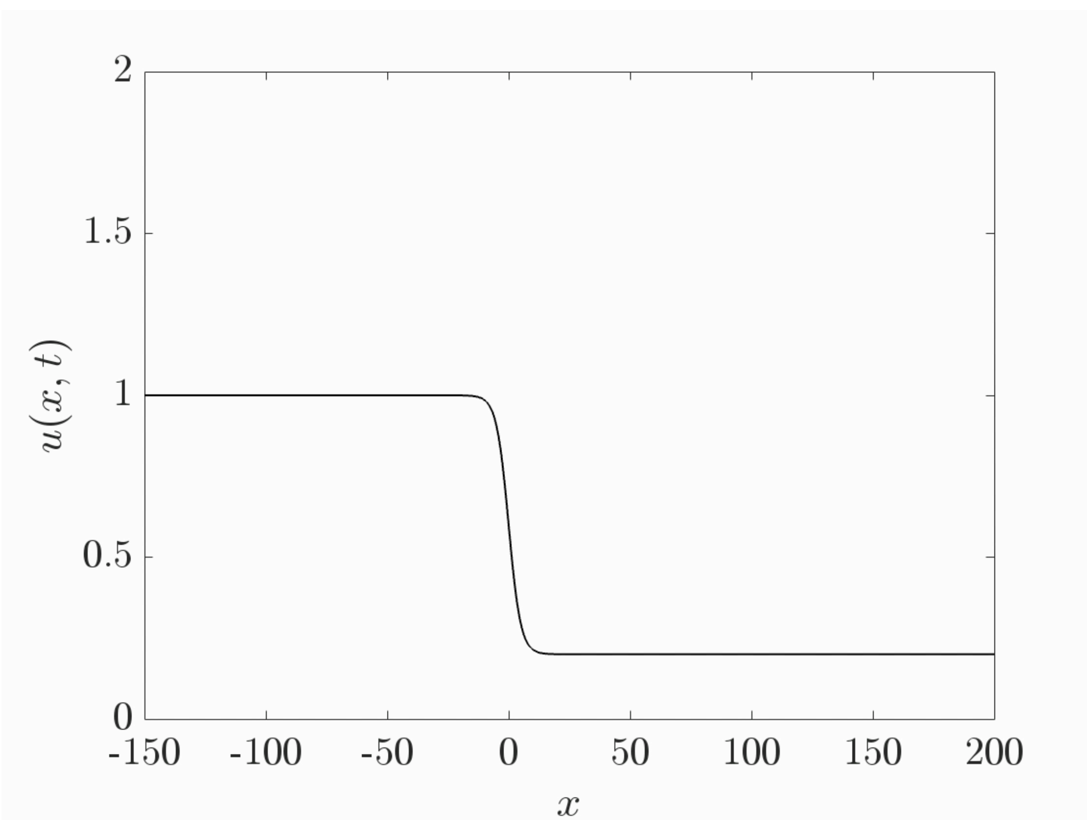
- e.g. $P = 0: \; f(\eta) = \dfrac{\eta}{\pi\sqrt{\lambda_1^2 - \eta^2}}, \quad s(\eta) = 6(2\eta^2 - \lambda_1^2) \, .$
  - So many interactions that soliton velocity $s(\eta)$ can be negative!
  - $\langle u^2 \rangle = \langle u \rangle^2 = \lambda_1^4 \; \Rightarrow \; u(x,t) = \lambda_1^2$ **with probability 1.**
- Suppose $\lambda_j(x,t)$ slowly varies: kinetic equation reduces to **genuinely nonlinear,** KdV-Whitham modulation system (Congy *et al* 2023)

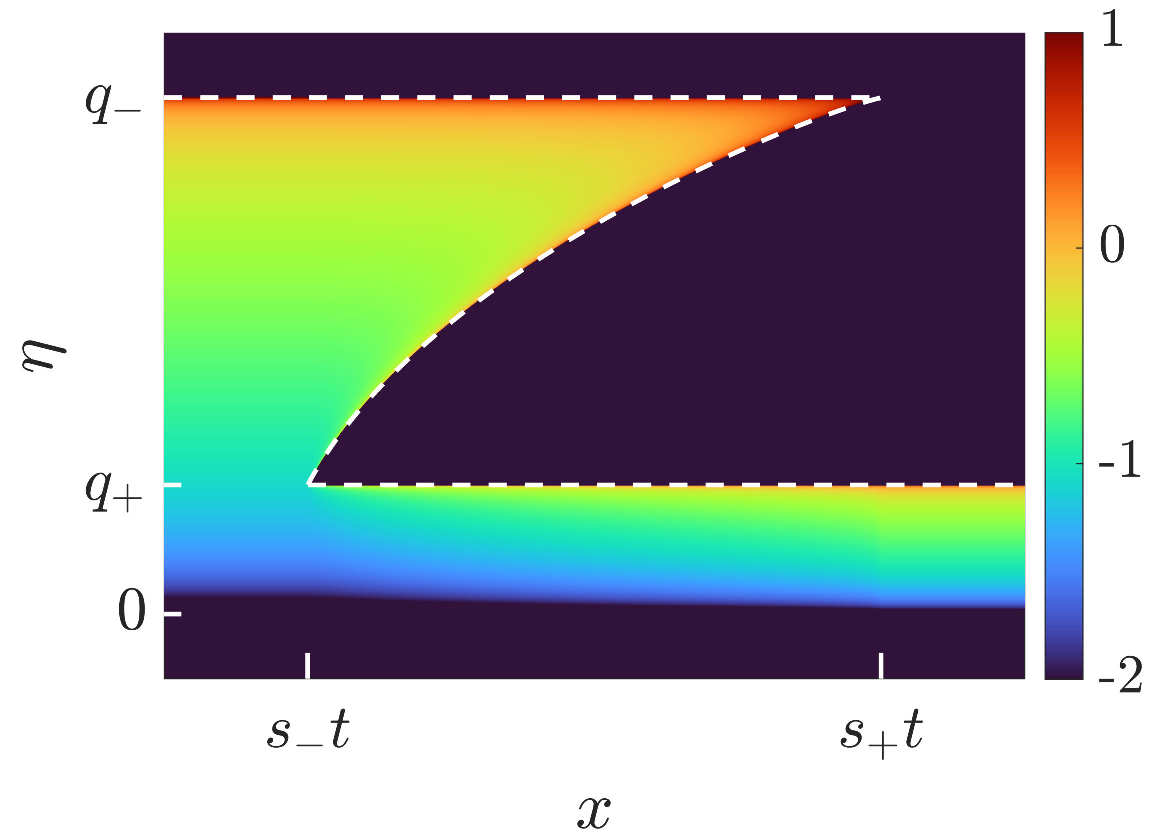
$$\partial_t \lambda_j + v_j(\boldsymbol{\lambda}) \partial_x \lambda_j = 0, \quad j = 1 \ldots P, \qquad \text{e.g. } P = 0: \; v_1(\lambda_1) = 6\lambda_1^2 \, .$$

Gurevich-Pitaevskii problem

Consider the following Riemann problem:

$$f(\eta;x,0) = \begin{cases} \eta/\pi\sqrt{q_-^2-\eta^2}, & x<0, \\ \eta/\pi\sqrt{q_+^2-\eta^2}, & x>0, \end{cases} \Leftrightarrow u(x,0) = \begin{cases} q_-^2, & x<0, \\ q_+^2, & x>0. \end{cases}$$





Conclusion and perspectives

- Evolution of random, reflectionless potentials described by

$$\partial_t f + \partial_x (sf) = 0, \quad s(\eta) = 4\eta^2 + \int_\Gamma G(\eta,\mu) f(\mu)[s(\eta) - s(\mu)] d\mu,$$

$$\langle u \rangle = \int_\Gamma 4\eta\, f(\eta)\, d\eta, \quad \langle u^2 \rangle = \int_\Gamma \frac{16}{3} \eta^3 f(\eta)\, d\eta\,.$$

- Other well-studied soliton gas: focusing nonlinear Schrödinger equation where $\zeta \in \mathbb{C}$.

- What “remains” to be done for KdV soliton gas?
    - Determination of $\mathrm{PDF}(u; x, t)$.
    - Other analytical solutions of kinetic equation.
    - Thermodynamics of soliton gas (Bonnemain *et al* 2022).
    - Addition of dissipative effects (Fache *et al* 2024).

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