Sample-size dependence of the ground-state energy in a one-dimensional localization problem

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We study the sample-size dependence of the ground-state energy in a one-dimensional localization problem, based on a supersymmetric quantum mechanical Hamiltonian with a random Gaussian potential. We determine, in the form of bounds, the precise form of this dependence and show that the disorder-averaged ground-state energy decreases with an increase of the size R of the sample as a stretched-exponential function $\exp(-R^z)$ where the characteristic exponent z depends merely on the nature of correlations in the random potential. In the particular case where the potential is distributed as a Gaussian white noise we prove that $z=1/3$. We also predict the value of z in the general case of Gaussian random potentials with correlations. [S1063-651X(96)01607-8]

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I. INTRODUCTION

The one-dimensional Schrödinger Hamiltonian

$$\hat{H}=-\frac{d^2}{dx^2}+\phi^2(x)+\frac{d\phi(x)}{dx} \tag{1}$$

arises in such diverse areas of quantum mechanics as studies of solitons in conjugated polymers (polyacetylene) [1,2], in which $\phi(x)$ describes the dimerization pattern of the carbon-hydrogen sequence, or of electrons dynamics in two-dimensional systems subjected to random magnetic fields [3]. It also represents a celebrated toy model of supersymmetric quantum mechanics introduced by Witten [4].

In case of nonrandom potentials $\phi(x)$ the properties of the Hamiltonian in Eq. (1) have been thoroughly studied within the last two decades and several interesting results have been obtained. In particular, exact solutions of the Schrödinger equation have been derived for a class of the so-called shape invariant potentials [5]. Besides, such a form of the Hamiltonian inspired a different method of semiclassical quantization [6,7].

More recent studies of $\hat{H}$ have focused on situations in which $\phi(x)$ is random [3,8–11]. Here Eq. (1) describes a localization problem in which, remarkably, the average density of states, the localization length and the Lyapunov exponent can be computed exactly [8–10]. In fact, this problem represents an addition to the two known [12–14] examples of localization problems which can be solved exactly in the continuum.

In the present paper we study a different aspect of the one-dimensional localization problem associated with the Hamiltonian in Eq. (1) in which the potential $\phi(x)$ is a random function of the space variable x. We focus on the physically interesting situation where $\hat{H}$ is defined on a finite interval $[-R,R]$ and analyze the sample-size dependence of the ground-state energy $E_0(R,\{\phi\})$, which may be expressed in terms of nonlocal exponential functionals of the random potential $\phi(x)$. We consider mostly throughout the paper the case where $\phi(x)$ is Gaussian, δ-correlated white noise with the moments

$$\langle\phi(x)\rangle=0 \tag{2}$$

and

$$\langle\phi(x)\phi(x')\rangle=\sigma\delta(x-x'), \tag{3}$$

where the brackets denote the average with respect to realizations of $\phi(x)$. The relevance of the short-range correlations in the distribution of $\phi(x)$ and their influence on the sample-size dependence are also succinctly addressed and several predictions are made. We find that for such potentials the disorder-averaged ground-state (DAGS) energy

$$E_0(R)\equiv\langle E_0(R,\{\phi\})\rangle$$

decreases as the size R grows as a stretched-exponential function, $E_0(R)\propto\exp(-R^z)$, where the characteristic exponent z depends only on the correlation properties of the random potential. In the particular case where fluctuations in $\phi(x)$ are δ correlated, as in Eq. (3), we prove that $z=1/3$. The value of the exponent z is also predicted, on heuristic grounds, for the general case of the random Gaussian potentials with correlations. We show that such a stretched-exponential dependence on R stems from the atypical realizations of $\phi(x)$, which are reminiscent of the representative trajectories supporting long-time anomalous decay laws of the survival probability for diffusion in the presence of randomly placed traps [15,16] or of the Lifschitz singularities [17] in the low-energy spectrum of an electron in the presence of randomly dispersed scatterers. Our results are presented in the form of lower and upper bounds on $E_0(R)$, which show the same dependence on R and thus define this dependence exactly. The method of the derivation of bounds,

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which we invoke here, has been previously discussed in [18,19] and is based on the statistics of the extremes of the random potential $\phi(x)$.

The paper is structured as follows: In Sec. II we present a simple derivation of the ground-state energy $E_0(R,\{\phi\})$ in the case of the deterministic potentials $\phi(x)$. We recover in a straightforward way an explicit formula for $E_0(R,\{\phi\})$, which coincides when the appropriate notations are introduced with the one found in [20], where the influence of the finite-size effects on the ground-state energy in conventional quantum mechanics has been examined. We then present some arguments showing that this formula is still valid in the case of a random potential $\phi(x)$. In Sec. III we estimate the sample-size dependence of the ground-state energy considering only typical realizations of the random potential $\phi(x)$. Employing the standard Jensen inequality we show then that such an estimate constitutes a lower bound on the average ground-state energy. Additionally, we illustrate that such an estimate allows us to recover the correct low-energy behavior of the integrated density of states of the Hamiltonian (1). Further on, in Sec. IV, we devise a more accurate approach and derive a lower bound (subsection A) and an upper bound (subsection B) on $E_0(R)$, which exactly determine its behavior in the large-R limit. In subsection C we address the question of the DAGS behavior in situations, in which the fluctuations of the random potential $\phi(x)$ are correlated, and also discuss some similar features between the realizations of the disorder supporting the large-R behavior of the DAGS and the realizations of random walks which support anomalous long-time tails of the survival probability for the diffusion in the presence of traps. Finally, in Sec. V we conclude with a summary of our results.

II. CALCULATION OF THE ENERGY SHIFT IN A FINITE SAMPLE

The structure of the Hamiltonian in Eq. (1) is such that, for an arbitrary function $\phi(x)$, two independent solutions of the Schrödinger equation

$$\hat{H}\varphi_0^{(1,2)}(x)=0, \tag{4}$$

may be explicitly expressed as functionals of $\phi(x)$

$$\varphi_0^{(1)}(x)=\exp\left(\int_0^x dx'\,\phi(x')\right) \tag{5}$$

and

$$\varphi_0^{(2)}(x)=\varphi_0^{(1)}(x)\int_0^x\frac{dx}{[\varphi_0^{(1)}(x)]^2}. \tag{6}$$

A. Deterministic potentials $\phi(x)$

In this subsection we consider the case of the deterministic potentials $\phi(x)$, which decay fast enough when $x\to\pm\infty$ to make $\varphi_0^{(1)}$ normalizable on the whole axis, i.e.,

$$\int_{-\infty}^{\infty}dx[\varphi_0^{(1)}(x)]^2<\infty.$$

Therefore, $\varphi_0^{(1)}$ is a zero-energy wave function of the Hamiltonian $\hat{H}$ defined on the whole axis.

Consider now how the situation will be changed if one assumes that the Schrödinger equation with the Hamiltonian in Eq. (1) is defined not on the whole x axis, but only on a finite interval $[-R,R]$. The new ground-state wave function $\hat{\Psi}_0(x)$ on this finite interval satisfies the Schrödinger equation

$$\left[-\frac{d^2}{dx^2}+\phi^2(x)+\frac{d\phi(x)}{dx}\right]\hat{\Psi}_0(x)=E_0(R,\{\phi\})\hat{\Psi}_0(x), \tag{7}$$

with an *a priori* unknown energy $E_0(R,\{\phi\})$ that will depend on the explicit form of the potential $\phi(x)$. Equation (7) has to be supplemented by the following Dirichlet boundary conditions at the ends of the interval, i.e., at points $x=-R$ and $x=R$,

$$\hat{\Psi}_0(x=-R)=\hat{\Psi}_0(x=R)=0. \tag{8}$$

Our goal will be to estimate the energy shift $E_0(R,\{\phi\})$ of the ground state caused by the introduction of the boundary conditions imposed on a finite interval.

Multiplying both sides of Eq. (7) by $\varphi_0^{(1)}(x)$ and integrating from $-R$ to R, we get the following identity

$$E_0(R,\{\phi\})\int_{-R}^{R}dx\,\varphi_0^{(1)}(x)\hat{\Psi}_0(x)=\int_{-R}^{R}dx\,\varphi_0^{(1)}(x)\left[-\frac{d^2}{dx^2}+\phi^2(x)+\frac{d\phi(x)}{dx}\right]\hat{\Psi}_0(x). \tag{9}$$

Integrating by parts the kinetic term on the right-hand side (rhs) of Eq. (9) yields

$$E_0(R,\{\phi\})\int_{-R}^{R}dx\,\varphi_0^{(1)}(x)\hat{\Psi}_0(x)=\varphi_0^{(1)}(-R)\left.\frac{d\hat{\Psi}_0(x)}{dx}\right|_{x=-R}-\varphi_0^{(1)}(R)\left.\frac{d\hat{\Psi}_0(x)}{dx}\right|_{x=R}. \tag{10}$$

To proceed further on, we assume that R is sufficiently large and estimate the behavior of the derivative of $\hat{\Psi}_0(x)$ in this limit. Since $E_0(R,\{\phi\})$ vanishes when $R\to\infty$, one expects that the ground-state wave function on the finite (large) interval may be well approximated by a suitable linear combination of the two independent zero-energy solutions, $\varphi_0^{(1)}(x)$ and $\varphi_0^{(2)}(x)$. In the vicinity of $x=R$ one has

$$\hat{\Psi}_0(x)\approx\varphi_0^{(1)}(x)\left[1+\alpha\int_0^x\frac{dx}{[\varphi_0^{(1)}(x)]^2}\right], \tag{11}$$

where α is determined through the boundary condition at $x=R$, which gives

$$\alpha=-\left\{\int_0^R\frac{dx}{[\varphi_0^{(1)}(x)]^2}\right\}^{-1}, \tag{12}$$

and consequently, we find that the derivative of $\hat{\Psi}_0(x)$ obeys

$$\left.\frac{d\hat{\Psi}_0(x)}{dx}\right|_{x=R} \approx \frac{\alpha}{\varphi_0^{(1)}(R)} = -\left\{ \varphi_0^{(1)}(R)\int_0^R \frac{dx}{[\varphi_0^{(1)}(x)]^2}\right\}^{-1}. \tag{13}$$

Similarly, for the derivative of the ground-state wave function in the vicinity of $x=-R$ one finds

$$\left.\frac{d\hat{\Psi}_0(x)}{dx}\right|_{x=-R} \approx -\left\{ \varphi_0^{(1)}(-R)\int_0^{-R} \frac{dx}{[\varphi_0^{(1)}(x)]^2}\right\}^{-1}. \tag{14}$$

Now, combining Eqs. (10) to (14) we have

$$E_0(R,\{\phi\})\int_{-R}^{R} dx\,\varphi_0^{(1)}(x)\hat{\Psi}_0(x) = \left\{ \int_{-R}^{0} \frac{dx}{[\varphi_0^{(1)}(x)]^2}\right\}^{-1} + \left\{ \int_0^R \frac{dx}{[\varphi_0^{(1)}(x)]^2}\right\}^{-1} \tag{15}$$

and, finally, replacing in the integral on the left hand side of Eq. (15) the function $\hat{\Psi}_0(x)$ by $\varphi_0^{(1)}(x)$ (which results in exponentially small with R errors [20]) we find the following explicit expression for the ground-state energy on a finite interval:

$$E_0(R,\{\phi\}) \approx \frac{1}{\int_{-R}^{0}[\varphi_0^{(1)}(x)]^{-2}dx \int_{-R}^{R}[\varphi_0^{(1)}(x')]^2 dx'} + \frac{1}{\int_0^R[\varphi_0^{(1)}(x)]^{-2}dx \int_{-R}^{R}[\varphi_0^{(1)}(x')]^2 dx'}. \tag{16}$$

Equation (16) reproduces the result of [20], derived in terms of a different approach, when the notation $V(x)=\phi^2(x)+d\phi(x)/dx$ is introduced. We note also parenthetically that expressions of quite identical structure were recently obtained for the diffusion constant for the random motion in an external periodic potential [21–24]. Therefore the results which will be obtained in the following also apply to this problem, provided that the potential is defined as in Eqs. (2) and (3).

B. Random potentials $\phi(x)$

We now have to explain why Eq. (16) is still valid in the case of the random potentials $\phi(x)$. We first mention that the analysis of the previous subsection is based on the assumption that the wave functions are normalizable on the whole line. In case of random potentials $\phi(x)$, as defined in Eqs. (2) and (3), the integral $\int_0^R dx\,\phi(x)$ may show an unbounded growth when $R\to\infty$ and thus the wave functions are not normalizable. Therefore, it is not possible to use directly the results of the preceding subsection. One may, however, notice that due to the presence of disorder the zero-energy solutions are localized. This observation allows us to get rid of the problem at infinity. For a given realization of the random potential $\phi(x)$ one chooses R in such a way that

$$\phi(-R)>0 \quad \text{and} \quad \phi(R)<0 \tag{17}$$

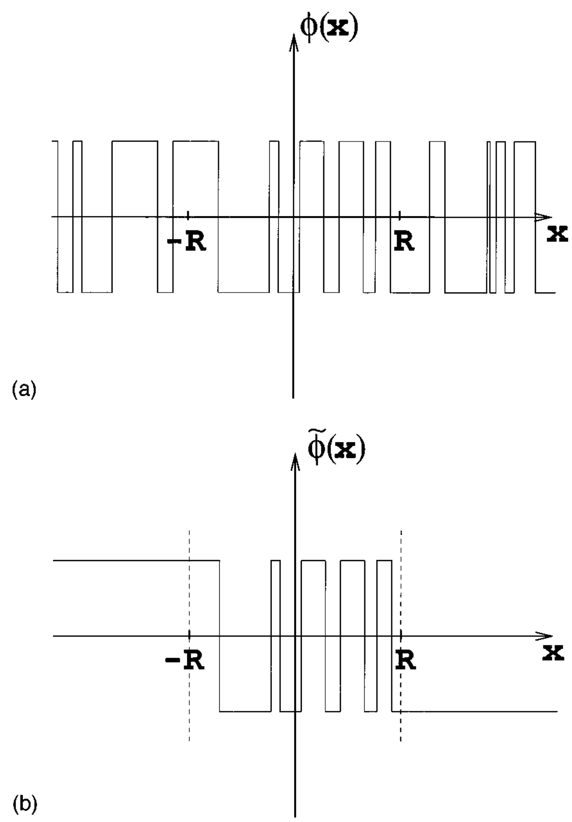

FIG. 1. Schematic picture of a two-level random potential $\phi(x)$. (b) An auxiliary configuration $\tilde{\phi}(x)$.

and then define an auxiliary configuration $\tilde{\phi}(x)$ [see Figs. 1(a) and 1(b)] such that,

$$\tilde{\phi}(x)=\phi(x) \quad \text{for} \quad -R<x<R, \tag{18}$$

$$\tilde{\phi}(x)=\phi(-R) \quad \text{for} \quad x<-R, \tag{19}$$

and

$$\tilde{\phi}(x)=\phi(R) \quad \text{for} \quad x>R. \tag{20}$$

The corresponding wave-function $\tilde{\Psi}_0(x)=\exp[\int_0^x\tilde{\phi}(x')dx']$ is therefore an exact zero mode of the Hamiltonian

$$\tilde{H}=-\frac{d^2}{dx^2}+\tilde{\phi}^2(x)+\frac{d\tilde{\phi}(x)}{dx} \tag{21}$$

on the whole line. When R is sufficiently large, it follows from our previous discussion that there exists a ground-state wave function of the Hamiltonian $\tilde{H}$ on the interval $[-R,R]$, whose ground-state energy is given by Eq. (16). Since this state has its support on $[-R,R]$ the functions $\tilde{\Psi}_0(x)$ and $\varphi_0(x)$ coincide, and thus it is also a quasizero mode of $\hat{H}$ with the ground-state energy defined by Eq. (16).

III. TYPICAL REALIZATIONS OF DISORDER AND THE JENSEN INEQUALITY

We start our analysis of the behavior of the disorder-average ground-state energy in Eq. (16) first considering the typical realizations of the random potential $\phi(x)$. To do this, we rewrite the ground-state energy for a given sample with a particular realization of the random potential $\phi(x)$ Eq. (16) in terms of the following exponential functionals of the potential $\phi(x)$:

$$\tau_{\pm}(z,y)=\int_z^y dx[\varphi_0^{(1)}(x)]^{\pm 2}=\int_z^y dx \exp\left[\pm 2\int_0^x \phi(x')dx'\right], \quad (22)$$

which gives

$$E_0(R,\{\phi\})=\frac{1}{\tau_+(-R,R)}\left(\frac{1}{\tau_-(-R,0)}+\frac{1}{\tau_-(0,R)}\right). \quad (23)$$

We notice now that the function $W(x)=\int_0^x dx\,\phi(x)$, which appears in the definition of $\tau_-(0,R)$ Eq. (22) for the potentials as in Eqs. (2) and (3) is simply a trajectory of a symmetric random walk. Consequently, for the typical realizations of the random potential $\phi(x)$ the value of $W(R)$ should be of the order $(\sigma R)^{1/2}$; hence $\tau_-(0,R)$ should grow typically as $\exp[2(\sigma R)^{1/2}]$ and then Eq. (23) entails $E_0(R) \propto \exp[-4(\sigma R)^{1/2}]$. Therefore for most realizations of $\phi(x)$ one may expect that $E_0(R,\{\phi\})$ vanishes with an increase of R as

$$E_0(R,\{\phi\})_{typ} \sim \exp(-\sigma^{1/2}R^{1/2}). \quad (24)$$

This typical behavior may be used to evaluate correctly the low-energy asymptotic behavior of the density of states of the Hamiltonian in Eq. (1). Equation (24) means that a wave function of low-energy E typically has a spatial extension $2R$ such that

$$R \propto \frac{\ln^2(E)}{\sigma}. \quad (25)$$

Therefore the number of such states per unit length behaves typically as

$$N(E) \propto \frac{1}{2R} \propto \frac{\sigma}{\ln^2(E)}. \quad (26)$$

It is now interesting to compare Eq. (26) with the exact result [9]

$$N(E)=\frac{2\sigma}{\pi^2}\frac{1}{J_0^2(z)+N_0^2(z)}, \quad (27)$$

where $z=\sqrt{E}/\sigma$ and J_0, N_0 are Bessel functions. In the limit $E\to 0$ one has from Eq. (27) that $N(E)\sim 2\sigma \ln^{-2}(E/4\sigma^2)$, i.e., the behavior which is quite consistent with our estimate in Eq. (26). Therefore Eq. (26) shows that anomalous singular behavior of the integrated density of states is supported by the typical realizations of the disorder and thus is quite distinct from the Lifschitz singularity [17], which is most often encountered in the disordered quantum mechanical systems.

We now show that the estimate based on typical realizations of the disorder represents actually a lower bound on the DAGS energy. To show it explicitly we invoke the standard Jensen inequality between the average of the exponential of some function F and the exponent of the averaged value of F,

$$\langle \exp[-F]\rangle \geqslant \exp[-\langle F\rangle]. \quad (28)$$

Now, choosing

$$F=-\ln E_0(R,\{\phi\}) \quad (29)$$

we find, taking into account Eq. (23), that $E_0(R)$ is bounded from below by

$$E_0(R) \geqslant \exp\{\langle \ln[\tau_-(-R,R)]\rangle - \langle \ln[\tau_+(-R,R)]\rangle - \langle \ln[\tau_-(-R,0)]\rangle - \langle \ln[\tau_-(0,R)]\rangle\}. \quad (30)$$

One may readily notice that for any random function $\phi(x)$ of zero mean, not all terms in the exponent on the right-hand side of Eq. (30) are to be calculated independently; obviously,

$$\langle \ln[\tau_-(-R,R)]\rangle = \langle \ln[\tau_+(-R,R)]\rangle \quad (31)$$

and

$$\langle \ln[\tau_-(-R,0)]\rangle = \langle \ln[\tau_-(0,R)]\rangle. \quad (32)$$

Consequently, the first two terms on the rhs of Eq. (30) cancel each other and we have only to perform the averaging of $\ln[\tau_-(0,R)]$.

These functionals $\tau_{\pm}(0,R)$ appear in different physical backgrounds [25,26]. Their discrete-x counterpart, which is the sum of the products of the independent random variables of the form

$$\tau_-(N)=1+z_1+z_1z_2+z_1z_2z_3+\cdots+z_1z_2z_3\ldots z_N,$$

with

$$z_n=\exp(\phi_n),$$

is known as the Kesten variable [27] and plays an important role in the theory of the renewal processes. The distribution function of the continuous-x functional $\tau_-(0,R)$ has been recently examined in [18,28,29,19] within the context of diffusion in the presence of a random quenched force (the Sinai diffusion [30,9]) and also in the literature on mathematical finance [31].

The average logarithm of the functional $\tau_-(0,R)$ can be obtained from the probability distribution of this functional [28,29,19]

$$\langle \ln[\tau_-(0,R)]\rangle = \frac{2}{\pi}\int_0^\infty \frac{dk}{k^2}[1-\exp(-2\sigma R k^2)\pi k \coth(\pi k)] - \Gamma'(1) - \ln(2\sigma) \quad (33a)$$

$$\approx \left(\frac{8\sigma R}{\pi}\right)^{1/2} - \Gamma'(1) - \ln(2\sigma) + O\left(\frac{1}{\sigma R}\right), \quad (33b)$$

where the notation $O(1/R)$ means that the neglected terms multiplied by R will give a constant as $R\to\infty$. Equation (33b) has been recently rederived in [32] which tested predictions of the replica variational approximation [33] for a particular physical system—a classical particle in a one-dimensional box subjected to a random potential which constitutes a Wiener process on the coordinate axis [30,9]. A detailed discussion of the average logarithm of the functional $\tau_-(0,R)$ can be found in [34].

Accordingly, for the DAGS energy we obtain

$$E_0(R) \geqslant \exp\left(-4\left(\frac{2\sigma R}{\pi}\right)^{1/2}\right), \tag{34}$$

which thus shows that $E_0(R)$ vanishes with an increase of the sample-size R not faster than a stretched-exponential function $\exp(-R^z)$ with $z=1/2$. However, this lower bound, which is supported by the typical realizations of the disorder may be improved as we will see in the next section.

IV. LOWER AND UPPER BOUNDS ON $E_0(R)$

In this section we set out to show that, in the limit $R\to\infty$, the dependence of the disorder-averaged ground-state energy $E_0(R)$ on R is quite different from that in Eq. (34). Here we will derive more accurate bounds which show that in the large-R limit, the actual dependence of the disorder-averaged ground-state energy $E_0(R)$ on R is described by a stretched-exponential function $\exp(-R^z)$ but with a smaller exponent, $z=1/3$. This means that the large-R behavior of $E_0(R)$ is supported by the atypical realizations of $\phi(x)$. These realizations will also be specified below.

A. A lower bound

Let us begin with the derivation of a lower bound on $E_0(R)$. We first note that since $\phi(x)$ enters the expression for $E_0(R,\{\phi\})$ only in the form $\int dx\,\phi(x)$, averaging with respect to realization of $\phi(x)$ amounts actually to the averaging over different trajectories $W(x)$ of a symmetric random walk. Therefore we can formally write down the average as a product of two path-integrals

$$E_0(R) = \langle E_0(R,\{\phi\})\rangle = \int_\Omega D\{W(x)\}\int_{\Omega'} D\{W'(x)\} \times P[W(x)]P[W'(x)]E_0(R,\{\phi\}), \tag{35}$$

where the notations used have the following meaning: The symbol Ω denotes the set of *all possible* (unrestricted) trajectories $W(x)$ of a symmetric random walk, which "starts" at $x=0$ at the origin $W(0)=0$, and "time" variable x is defined on the interval $[0,R]$. We describe schematically the set Ω in Fig. 2, where for notational convenience we use the discrete-x picture and depict it using the axis $W(x)$ and x, i.e., using "directed polymers"—like representation. The trajectories (1) and (2) are two examples of possible trajectories which belong to the set Ω. The symbol Ω' denotes, correspondingly, the set of all possible trajectories $W'(x)$ with the "time" variable x defined on the interval $[0,-R]$. The trajectories in Ω and Ω' are statistically uncorrelated. Finally, the symbols $D\{W(x)\}$ and $D\{W'(x)\}$ denote that the integration is performed along the trajectories $W(x)$ and $W'(x)$; $P[W(x)]$ (or $P[W'(x)]$) is the corresponding measure of a given trajectory $W(x)$ [or $W'(x)$], which is the standard Wiener measure.

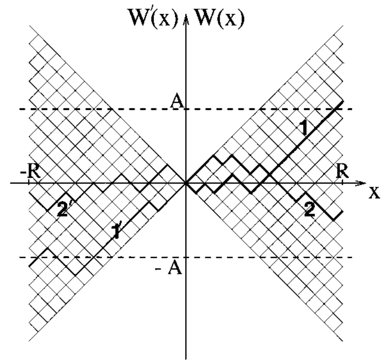

FIG. 2. Schematic representation of the sets Ω, Ω', and the subsets ω, ω'. The set Ω (dashed triangle on the half-plane $x>0$) comprises all possible realizations of an R-step random walk trajectories $W(x)$ with $x\epsilon[0,R]$. The set Ω' (dashed triangle on the half-plane $x<0$) comprises, respectively, all possible trajectories of an R-step random walk with $x\epsilon[0,-R]$. The subsets ω and ω' are the areas cut from the sets Ω and Ω' by the lines $W(x)=A$ and $W(x)=-A$.

The next essential step is as follows. Suppose that from the entire set Ω (and Ω') we select some amount of trajectories having certain prescribed properties and denote this subset of Ω (Ω') as ω (ω'). Then, for any positive definite functional $E_0(R,\{\phi\})$ the following inequality holds:

$$\int_\Omega D\{W(x)\}\int_{\Omega'} D\{W'(x)\}P[W(x)]P[W'(x)]E_0(R,\{\phi\}) \geqslant \int_\omega D\{W(x)\}\int_{\omega'} D\{W(x)\} \times P[W(x)]P[W'(x)]E_0(R,\{\phi\}), \tag{36}$$

where the integrations on the rhs of Eq. (36) extend only over the trajectories which belong to the subsets ω and ω' of the entire sets Ω and Ω'. Employing the inequality in Eq. (36), we get the following bound:

$$E_0(R) \geqslant \int_\omega D\{W(x)\} \times \int_{\omega'} D\{W'(x)\}P[W(x)]P[W'(x)]E_0(R,\{\phi\}). \tag{37}$$

Now we define the subset ω (ω') as follows (Fig. 2): ω (ω') is the set of all trajectories $W(x)$ [$W'(x)$], which, for any x from the interval $[0,R]$ ($[0,-R]$ for ω'), remain inside the strip $[-A,A]$, i.e., such trajectories $W(x)$ [and $W'(x)$] which obey $-A \leqslant W(x), W'(x) \leqslant A$ for any x from the interval $[0,R]$ ($[0,-R]$ for ω'). In Fig. 2 trajectories which form the subsets ω and ω' are exemplified by (2) and (2').

Next, we diminish the rhs of Eq. (37), i.e., enhance the inequality in Eq. (37), by substituting instead of $E_0(R,\{\phi\})$

its minimal value on the subsets ω and ω'. By the definition of ω and ω', which implies that $|W(x)|\leqslant A$ and $|W'(x)|\leqslant A$ we have

$$\int_{-R}^{R} dx \exp[2W(x)]\leqslant 2R\exp(2A), \tag{38}$$

$$\int_{-R}^{0} dx \exp[-2W(x)]\leqslant R\exp(2A), \tag{39a}$$

and

$$\int_{0}^{R} dx \exp[-2W(x)]\leqslant R\exp(2A). \tag{39b}$$

Consequently, for any realization of $W(x)$ or $W'(x)$ which belongs to the subsets ω and ω', the following inequality holds:

$$E_0(R,\{\phi\})\geqslant \min_{\omega,\omega'}\{E_0(R,\{\phi\})\}=\frac{\exp(-4A)}{2R^2}. \tag{40}$$

Substituting Eq. (40) into Eq. (37) we find the following bound:

$$E_0(R)\geqslant \frac{\exp(-4A)}{2R^2}\int_{\omega} D\{W(x)\} \times \int_{\omega'} D\{W'(x)\}P[W(x)]P[W'(x)]. \tag{41}$$

We now notice that the product of the integrals along the two ''restricted'' (statistically independent) paths $W(x)$ and $W'(x)$ on the rhs of Eq. (41) is equal to the probability that the two independent random walkers during ''time'' R will remain within the strip $[-A,A]$, which means that

$$\int_{\omega} D\{W(x)\}\int_{\omega'} D\{W'(x)\}P[W(x)]P[W'(x)]=P^2(A,R), \tag{42}$$

where $P(A,R)$ is the corresponding probability for a single random walker [35]

$$P(A,R)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(-\frac{(2k+1)^2\pi^2\sigma R}{8A^2}\right). \tag{43}$$

Combining Eq. (41) and (43) we thus obtain

$$E_0(R)\geqslant \frac{8\exp(-4A)}{\pi^2R^2}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \times\exp\left(-\frac{(2k+1)^2\pi^2\sigma R}{8A^2}\right)\right]^2. \tag{44}$$

The function on the rhs of Eq. (44) contains a free trial parameter A. The inequality in Eq. (44) holds for any value of this parameter and thus represents a family of lower bounds. Therefore, we will choose such a value of A, which maximizes the rhs of Eq. (44) and thus defines the maximal lower bound. For R sufficiently large the maximal contribution to the probability distribution in Eq. (43) comes from the term with $k=0$, i.e.,

$$P(A,R)\approx \frac{4}{\pi}\exp\left(-\frac{\pi^2\sigma R}{8A^2}\right), \tag{45}$$

and consequently,

$$E_0(R)\geqslant \frac{8}{\pi^2R^2}\exp\left(-4A-\frac{\pi^2\sigma R}{4A^2}\right). \tag{46}$$

Taking the derivative of the rhs of Eq. (46) with respect to the parameter A, we find that

$$A=A^*=\frac{1}{2}(\pi^2\sigma R)^{1/3} \tag{47}$$

provides its maximal value. Substituting Eq. (47) into Eq. (46) we thus arrive at the following ''maximal'' lower bound:

$$E_0(R)\geqslant \frac{8}{\pi^2R^2}\exp[-3(\pi^2\sigma R)^{1/3}], \tag{48}$$

which shows that in the large-R limit the DAGS energy vanishes not faster than the $\exp(-R^{1/3})$, i.e. at a slower rate than ''typical behavior'' in Eq. (34). This improved lower bound in Eq. (48) is supported by the atypical realizations of $W(x)$, such that $W(x)\propto x^{1/3}$, i.e., by trajectories of $W(x)$ which are spatially more confined than the ''typical'' realizations of the random walk trajectories for which $W(x)\propto x^{1/2}$.

B. An upper bound

Let us now discuss the derivation of an upper bound on the DAGS energy. We first note that the rhs of Eq. (16) for any given realization of $W(x)$ can be bounded from above

$$\begin{aligned}
&\frac{1}{\int_{-R}^{0}[\varphi_0^{(1)}(x)]^{-2}dx\int_{-R}^{R}[\varphi_0^{(1)}(x')]^2dx'} + \frac{1}{\int_{0}^{R}[\varphi_0^{(1)}(x)]^{-2}dx\int_{-R}^{R}[\varphi_0^{(1)}(x')]^2dx'}\\
&\leqslant \frac{1}{\int_{-R}^{0}[\varphi_0^{(1)}(x)]^{-2}dx\int_{-R}^{0}[\varphi_0^{(1)}(x')]^2dx'} + \frac{1}{\int_{0}^{R}[\varphi_0^{(1)}(x)]^{-2}dx\int_{0}^{R}[\varphi_0^{(1)}(x')]^2dx'}\\
&= \frac{1}{\int_{-R}^{0}\int_{-R}^{0}dx\,dx'\exp[2W'(x')-2W'(x)]} + \frac{1}{\int_{0}^{R}\int_{0}^{R}dx\,dx'\exp[2W(x')-2W(x)]}.
\end{aligned} \tag{49}$$

As one may readily notice, the inequality in Eq. (49) is obtained by simply diminishing the limits of integration; in the first term we change the limits of integration over the variable x' from $[-R,R]$ to $[-R,0]$, while in the second one the limits are changed from $[-R,R]$ to $[0,R]$. Since $\varphi_0^{(1)}(x)$ is positive definite, the diminishing of limits decreases the value of the integral and consequently, increases the terms on the rhs of Eq. (49).

Now we will try to find an appropriate functional of the extremes of the random function $W(x)$ which will bound the integrals in Eq. (49) from below, and thus in such a way will enhance the bound in Eq. (49).

We note here parenthetically that this problem turns out to be rather nontrivial. In particular, standard integral inequalities (such as, for instance, the Schwarz inequality) are obviously insufficient since they predict an algebraic growth of the integral

$$\int_0^{\pm R}\int_0^{\pm R} dx dx' \exp[2W(x')-2W(x)], \tag{50}$$

while the simple analysis of the ''typical'' behavior shows that Eq. (50) grows at least as a stretched-exponential function of R. In addition, the integrands in Eq. (50) do not possess well-defined derivatives and thus one can not expand the integrands in the vicinity of the extremes of function $W(x)$ and make use of the standard saddle-point-like estimates.

To illustrate the derivation of such a bound we first turn to the more lucid discrete-space picture, assuming that x and y are discrete variables $x,y=0,1,\ldots,R$, and then approximate the integrals in Eq. (50) as products of two sums

$$\int_0^{\pm R} dx \int_0^{\pm R} dx' \exp[2W(x')-2W(x)] \approx \sum_{x=0}^{R}\sum_{y=0}^{R} a_{xy}, \tag{51a}$$

with

$$a_{xy}=\exp[2W(x)-2W(y)]. \tag{51b}$$

The derivation of the corresponding upper bound in the continuous space, which is substantially more lengthy, will be merely outlined in the Appendix.

We notice that the rhs of Eqs. (51) is the sequence of $(R+1)^2$ positive terms, each of which is an exponential of the distance between the positions of a given trajectory $W(x)$ taken at two different moments of ''time'' x (summed over all possible x from the interval $[0,R]$). From this sequence of positive terms $\{a_{xy}\}$ we choose the maximal term, $\max_{x,y\epsilon[0,R]}\{a_{xy}\}$, which is evidently the exponential of the difference of the maximal positive displacement M_+, $(M_+>0)$, of the trajectory $W(x)$ (which is achieved at some moment $x=x^*$) and the maximal negative displacement M_-, $(M_-<0)$, of the same trajectory (achieved at the moment $y=y^*$, both x^* and y^* belonging to the interval $[0,R]$),

$$M_+=\max_{x\epsilon[0,R]}\{W(x)\}=W(x^*), \tag{52}$$

$$M_-=\min_{x\epsilon[0,R]}\{W(x)\}=W(y^*). \tag{53}$$

Since all $a_{xy}\geqslant 0$, the sum on the rhs of Eq. (51) is evidently larger than the maximal term of this sequence, i.e.,

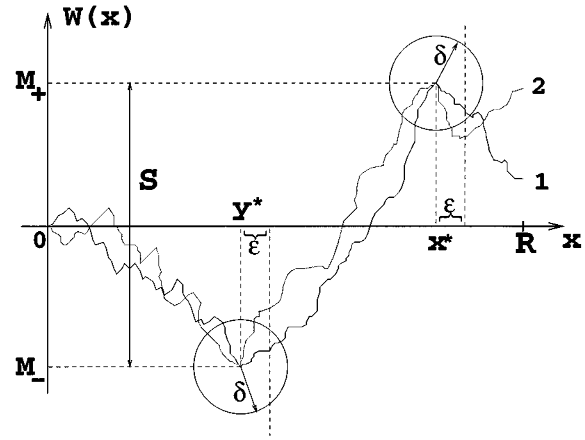

FIG. 3. Maximal positive, maximal negative displacements, and the span of the trajectory $W(x)$ with x defined on the interval $[0,R]$.

$$\sum_{x=0}^{R}\sum_{y=0}^{R} a_{xy} \geqslant \max_{x,y\epsilon[0,R]}\{a_{xy}\}=\exp[2(M_+-M_-)]. \tag{54}$$

Equation (54) represents the (discrete-space) formulation of the desired bound on the integrals in Eq. (50).

Let us now see how this bound can be employed for the derivation of the upper bound on the DAGS. Making use of Eqs. (49) and (54) we have that, at a given realization of $W(x)$, the ground-state energy can be bounded from above as

$$E_0(R,\{\phi\}) \leqslant \exp(-2S')+\exp(-2S), \tag{55}$$

where we denote by

$$S'=\max_{x\epsilon[0,-R]}\{W'(x)\}-\min_{x\epsilon[0,-R]}\{W'(x)\}=M'_+-M'_-$$

and

$$S=\max_{x\epsilon[0,R]}\{W(x)\}-\min_{x\epsilon[0,R]}\{W(x)\}=M_+-M_-.$$

Random variables as S' (or S) are known in the statistics of random walks as a span of the random walk (see Fig. 3), which can be visualized (in the d-dimensional space) as the dimensions of the smallest box with sides parallel to the coordinate axes that entirely contain the trajectory of a random walk [35]. The exact probability distribution $P(S,R)$ of the random variable S is well known [35]; in the case of large R, a convenient representation reads

$$P(S,R)=\frac{8\sigma R}{S^3}\sum_{k=0}^{\infty}\left[\frac{\pi^2(2k+1)^2\sigma R}{S^2}-1\right] \times \exp\left(-\frac{\pi^2(2k+1)^2\sigma R}{2S^2}\right). \tag{56}$$

Therefore the calculation of the upper bound on the DAGS energy reduces to the calculation of the integral

$$E_0(R) \leqslant 2\int_0^{\infty} dS \exp(-2S) P(S,R), \tag{57}$$

in which, noticing that S and S' have identical distribution functions [although for given realizations of $W(x)$ and $W'(x)$ they may have different values] the contribution of S' may be simply accounted for by introducing a multiplier 2.

Let us now consider the behavior of the integral in Eq. (57) in the limit of large R. We first note that in this limit in Eq. (56) only the term with $k=0$ is relevant. Second, noticing that the integrand is a bell-shaped function, we perform the integral using the saddle point approximation. Maximizing the terms in the exponent we get that the saddle-point depends on R as

$$S=S^*=\left(\frac{\pi^2\sigma R}{2}\right)^{1/3} \tag{58}$$

and consequently, the bound in Eq. (57) attains the form

$$E_0(R)\leq 32\left(\frac{2\sigma R}{3\pi}\right)^{1/2}\exp\left[-\frac{3}{2^{1/3}}(\pi^2\sigma R)^{1/3}\right]. \tag{59}$$

Therefore the upper bound on the DAGS energy shows a stretched-exponential dependence on R with the characteristic exponent $z=1/3$, i.e., aside from the numerical factor $2^{-1/3}$ in the exponent and pre-exponential multipliers (which are not reliable in view of the approximation involved), essentially the same behavior as the lower bound in Eq. (48). Since both lower and upper bounds have the same dependence on R and also, on physical grounds, $E_0(R)$ is a monotonically decreasing function of R, we may infer that the stretched-exponential dependence with $z=1/3$ is the asymptotically exact result for $E_0(R)$. It is also important to note that both the lower and the upper bounds turn out to be supported by the same ''class'' of trajectories $W(x)$, such that $W(x)\propto x^{1/3}$.

To close this subsection we remark that the coincidence in the R dependence of the lower and the upper bounds is, in essence, due to the fact that the measure of the restricted trajectories, used in the derivation of the lower bound, and the probability distribution of the maximal displacement (or of the span S) are intrinsically related to each other [35]. Actually, the probability $P(A,R)$ that a random walker, starting at the moment $R=0$ at the origin, remains within an interval $[-A,A]$ in an R-step walk is just the probability that the maximal displacement of this random walker is less than A. It is clear that the probability of having the maximal displacement exactly equal to A is given by [35]

$$V(A,R)\approx\frac{\partial P(A,R)}{\partial A}, \tag{60}$$

and consequently, the probability of having the span of an R-step walk equal to S will follow:

$$P(S,R)\approx\left.\frac{\partial P(A,R)}{\partial A}\right|_{A=S/2}. \tag{61}$$

Further on, evaluating the lower bound we have searched for such an A which maximizes the product $\exp(-4A)P(A,R)$. On the other hand, the upper bound was eventually reduced to the integral in Eq. (57), which, using Eq. (61) can be rewritten as

$$\int_0^\infty dS\exp(-2S)P(S,R)\approx\int_0^\infty dS\exp(-2S)\left.\frac{\partial P(A,R)}{\partial A}\right|_{A=S/2}. \tag{62}$$

Integrating Eq. (62) by parts we arrive at performing an integral with the integrand $\exp(-4A)P(A,R)$. Since the integrand is a bell-shaped function of A and thus the saddle-point approximation can be used, calculation of this integral also reduces to maximizing the integrand.

C. Random Gaussian potentials $\phi(x)$ with correlations

In this subsection we will briefly discuss the behavior of the DAGS in the case where fluctuations of $\phi(x)$ are correlated. Now, as we have already mentioned the relevant property is the integral $W(x)=\int_0^x dx'\phi(x')$, rather than $\phi(x)$ itself. It is therefore convenient to define the correlations in the random potential $\phi(x)$ in terms of the integral $W(x)$. We consider here the case where $W(x)$ is zero on average, as in Eq. (2), and define the second moment as follows:

$$\langle W(x)W(x')\rangle\sim|x-x'|^{1+\lambda},\quad -1\leq\lambda\leq 1. \tag{63}$$

The parameter λ in Eq. (63) determines the nature of the correlations in the random potential $\phi(x)$. The borderline case $\lambda=0$ corresponds to the δ correlated fluctuations of $\phi(x)$, when $W(x)$ is a trajectory of the conventional Brownian motion. This case has been examined in detail in previous sections. The case of positive λ, $(\lambda>0)$, describes the situations in which fluctuations of $\phi(x)$ in two neighboring points x and x' tend to be of the same sign. Here the trajectories $W(x)$ have strong persistency; thinking in terms of the random walk one may say that here the random walker most likely continues the motion in the direction of the previous step than changes the direction of motion. Consequently, its trajectories are more ''swollen'' and spatially more extended compared to the case $\lambda=0$. Finally, the case $\lambda<0$ describes disorder with negative correlations when the values of the potential $\phi(x)$ in two neighboring points x and x' tend to have different signs. Here the random walker has a tendency of changing the direction of its motion at each step and its trajectories $W(x)$ are essentially more compact, compared to the case of the conventional random walk.

Using Eq. (63) one can readily estimate the typical behavior of the DAGS. Since for the typical realizations of $W(x)$ one expects that $W(x)\sim x^{(1+\lambda)/2}$, we will have $\langle\ln\tau_+(0,R)\rangle\sim R^{(1+\lambda)/2}$, and consequently,

$$E_0(R)_{\text{typ}}\sim\exp(-R^{(1+\lambda)/2}) \tag{64}$$

Consider now the behavior of the DAGS stemming from the atypical realizations and generalize the formalism employed for the derivation of the lower bound. Anticipating the reasonings which underly the inequality in Eq. (36) and Eqs. (41), we have that the DAGS can be estimated as

$$E_0(R) \geqslant \exp(-4A) P_\lambda^2(A,R), \tag{65}$$

where $P_\lambda(A,R)$ denotes the probability that a random walker, which is at the origin at $R=0$ and whose trajectories obey Eq. (74) will remain inside the strip $[-A,A]$ during the time interval $[0,R]$. Such a probability can be estimated as [35,36]

$$P_\lambda(A,R) \sim \exp(-R/A^{d_\omega}), \tag{66}$$

where $d_\omega = 2/(1+\lambda)$ is the ''fractal'' dimension [36] of the random walk defined by Eq. (63). Plugging Eq. (66) into the Eq. (65) and maximizing the product with respect to A we obtain the following estimate:

$$E_0(R) \sim \exp(-R^{1/(1+d_\omega)}), \quad R \gg 1 \tag{67}$$

or, in terms of the parameter λ,

$$E_0(R) \sim \exp(-R^{(1+\lambda)/(3+\lambda)}), \quad R \gg 1. \tag{68}$$

Behavior as in Eqs. (67) and (68) is thus supported by such atypical trajectories $W(x)$ which grow with x as $x^{(1+\lambda)/(3+\lambda)}$. It is important to note that again the estimate in Eqs. (67) and (68) shows a slower dependence on R as compared to the typical behavior in Eq. (64).

To close this section we will explain what we have in mind when saying that realizations of disorder which support the anomalous stretched-exponential behavior of the DAGS share common features with the realizations of trajectories which support the anomalous long-time decay of the survival probability of a particle diffusing in the presence of randomly placed traps or Lifschitz tails in the low-energy density of states of an electron in the presence of randomly dispersed scatterers [15,38].

Let us remember, on the example of the trapping problem, some basic formulations and results. Suppose a one-dimensional, infinite in both directions, line with immobile traps B which are placed completely at random at a mean concentration n_B. At $t=0$ we introduce on the line some concentration of particles of another type, say A, and let them diffuse independently of each other. As soon as an A particle approaches a B trap, the A particle gets annihilated, while the trap is unaffected. The question of interest is to define the time evolution of the concentration of A particles (or the survival probability), averaged with respect to the spatial arrangement of traps.

Let $C(x,t)$ denote the local concentration of A particles at the point x at time t. It obeys the diffusion equation

$$\dot{C}(x,t) = D \frac{\partial^2}{\partial x^2} C(x,t), \tag{69}$$

where D is the diffusion coefficient of A particles. Equation (69) is to be solved subject to the adsorbing boundary conditions imposed at the points occupied by traps; that is,

$$C(x=X_i,t)=0, \tag{70}$$

for any X_i from $\{X_i\}$, where X_i defines the position of the ith trap, $-\infty \leqslant i \leqslant \infty$, and $\{X_i\}$ denotes the set of traps' positions.

A nice feature of the one-dimensional geometry is that this problem can be solved exactly [15,37], by simply noticing that the evolution of $C(x,t)$ on some interval $[X_i, X_{i+1}]$ is independent of other intervals. Consequently, one has to find the solution of the diffusion equation on a finite interval of fixed length W, subjected to the adsorbing boundary conditions at the ends of the interval, and then perform averaging with respect to the distribution of the interval's length. Such a solution is given by Eq. (43), which in the limit of sufficiently large times reads

$$P(W,t) \approx \exp\left(-\pi^2 \frac{Dt}{W^2}\right). \tag{71}$$

Now, the disorder-average concentration of A particles at time t will be defined as

$$\langle C(x,t)\rangle \approx \int_0^\infty dW P(W,t) P(W), \tag{72}$$

where $P(W)$ is the probability of having a trap free interval of length W. For Poisson distribution of traps $P(W)$ behaves as

$$P(W) \propto \exp(-n_B W). \tag{73}$$

Substituting Eqs. (73), (71) into the Eq. (72) we thus arrive at an integral of essentially the same structure as that in Eq. (57), which yields [15,16,35–37]

$$\langle C(x,t)\rangle \approx \exp\left[-3\left(\frac{\pi^2}{4} n_B^2 D t\right)^{1/3}\right]. \tag{74}$$

The behavior as in Eq. (74) shows that the long-time decay of the disorder-average concentration is supported by such bounded realizations $W(t)$ of A particles' random walks which obey $|W(t)| \leqslant A \propto t^{1/3}$, i.e., the same class of trajectories which support the large-R behavior of the DAGS in the problem studied in the present paper.

V. CONCLUSIONS

To conclude, we have studied a new aspect of a one-dimensional localization problem associated with the supersymmetric Hamiltonian in Eq. (1) in which the potential $\phi(x)$ is a Gaussian random function of the spatial variable x. We have derived an explicit expression for the ground-state energy of the Hamiltonian (1) defined on a finite interval of the x axis for a given realization of disorder and analyzed the dependence of the disorder-average ground-state energy on the length R of the interval. We have shown that it is described by a stretched-exponential function of the form $\exp(-R^z)$, in which the characteristic exponent z is dependent merely on the nature of correlations in a random potential. In the case when fluctuations in random potential are δ correlated we found $z=1/3$. In the case when fluctuations are defined by Eq. (74) we have deduced that $z=(1+\lambda)/(3+\lambda)$. We have shown that such a behavior is quite different from the one expected when only the typical realization of the disorder are considered and thus is supported by atypical realizations of the random potential, which behave as

$$\int^{x} dx' \phi(x') \propto x^{(1+\lambda)/(3+\lambda)}. \tag{75}$$

We have also shown that such realizations belong to the class of trajectories which support an anomalous long-time behavior of the survival probability of a random walk in the presence of randomly placed traps.

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APPENDIX

In this appendix, we outline the derivation of the upper bound for the DAGS in the continuous-space limit. Consider the integral in Eq. (50) (for simplicity we suppose that limits of the integration are from 0 to $+R$) and, as was done before, assume that a given trajectory $W(x)$ reaches its maximal value at the point $x=x^*$ and its minimal value at the point $x=y^*$. Let us choose some positive constant ε, such that $0<\varepsilon\ll R$ and $x^*+\varepsilon, y^*+\varepsilon\leq R$. Since the integrand in Eq. (50) is positive definite, the following inequality holds:

$$\int_0^R dx \int_0^R dy \exp[2W(x)-2W(y)].$$

$$\geq \int_{x^*}^{x^*+\varepsilon} dx \int_{y^*}^{y^*+\varepsilon} dy \exp[2W(x)-2W(y)]. \tag{A1}$$

Now, taking advantage of the inequality in Eq. (49) we have for the DAGS

$$\begin{aligned}E_0(R) &\leq \left\langle \frac{1}{\int_0^R dx \int_0^R dy \exp[2W(x)-2W(y)]} \right\rangle \\ &\quad + \left\langle \frac{1}{\int_{-R}^0 dx \int_{-R}^0 dy \exp[2W(x)-2W(y)]} \right\rangle \\ &= 2\left\langle \frac{1}{\int_0^R dx \int_0^R dy \exp[2W(x)-2W(y)]} \right\rangle \\ &= 2\int_0^\infty dS P(S,R) \int \int dM_+ dM_- \delta(S-M_+ + M_-) \\ &\quad \times \left\langle \frac{1}{\int_0^R dx \int_0^R dy \exp[2W(x)-2W(y)]} \right\rangle \\ &\quad \times |[M_+ = W(x^*); M_- = W(y^*)],\end{aligned} \tag{A2}$$

where the brackets with the subscript $[M_+=W(x^*);$ $M_-=W(y^*)]$ mean that the average is taken with respect to the trajectories $W(x)$ whose maximal positive displacement is equal to M_+ and the maximal negative displacement is equal to M_-.

Further on, the inequality in Eq. (60) enables us to enhance the bound in Eq. (61) and write

$$\begin{aligned}E_0(R) &\leq 2\int_0^\infty dS P(S,R) \int \int dM_+ dM_- \delta(S-M_+ + M_-) \\ &\quad \times \left\langle \frac{1}{\int_{x^*}^{x^*+\varepsilon} dx \int_{y^*}^{y^*+\varepsilon} dy \exp[2W(x)-2W(y)]} \right\rangle \\ &\quad \times |[M_+ = W(x^*); M_- = W(y^*)].\end{aligned} \tag{A3}$$

Let us now estimate the value of the following functional:

$$\begin{aligned}E &= \int_0^\infty dS P(S,R) \int \int dM_+ dM_- \delta(S-M_+ + M_-) \\ &\quad \times \left\langle \frac{1}{\int_{x^*}^{x^*+\varepsilon} dx \int_{y^*}^{y^*+\varepsilon} dy \exp[2W(x)-2W(y)]} \right\rangle \\ &\quad \times |[M_+ = W(x^*); M_- = W(y^*)].\end{aligned} \tag{A4}$$

To do this we enclose the points $W(x^*)$ and $W(y^*)$ by circles of the radius δ (see Fig. 3), where $\delta=\delta(R)$ is a slowly growing function. The choice of the dependence $\delta(R)$ will be made later. Further on, we divide the set of all possible trajectories Ω into two different subsets. The first subset $\{A\}$ comprises all such trajectories $W(x)$ of random walk (with its maxima at M_+ and minima at M_-) which, on the interval $x\epsilon[x^*,x^*+\varepsilon]$ do not cross the circle around the point $W(x^*)$ and on the interval $x\epsilon[y^*,y^*+\varepsilon]$ do not cross the circle around $W(y^*)$ (e.g. the trajectory 1 in Fig. 3). The subset $\{B\}$ comprises the rest of the trajectories (for instance, the trajectory 2 in Fig. 3). We write now

$$E = A + B, \tag{A5}$$

where A stands for the average of the integrand in Eq. (62) with the trajectories forming the subset $\{A\}$, while B denotes the contribution to E coming from the average of the integrand over of the trajectories forming the subset B.

Consider first the contribution coming from the trajectories in the subset $\{A\}$. By definition of $\{A\}$, we have that on the interval $[x^*,x^*+\varepsilon]$ the trajectory $W(x)$ obeys the inequality $M_+-\delta<W(x)\leq M_+$; and on the interval $[y^*,y^*+\varepsilon]$ the trajectory $W(y)$ obeys $M_-\leq W(y)<M_-+\delta$. Consequently, the integrands in Eq. (A4) are bounded from below by

$$\exp[2W(x)] \geq \exp(2M_+ - 2\delta), \tag{A6}$$

$$\exp[-2W(y)] \geq \exp(-2M_- - 2\delta), \tag{A7}$$

and thus A is majorized by

$$A \leq \frac{\exp(4\delta)}{\varepsilon^2} \int_0^\infty dS P(S,R) \exp(-2S). \tag{A8}$$

Next we estimate the contribution from the trajectories of the subset $\{B\}$. Here, for $x\epsilon[x^*,x^*+\varepsilon]$ and $y\epsilon[y^*,y^*+\varepsilon]$ the function $\exp[W(x)-W(y)]$ is always greater than 1 and consequently

$$\int_{x^*}^{x^*+\varepsilon} dx \int_{y^*}^{y^*+\varepsilon} dy \exp[2W(x)-2W(y)] \geqslant \varepsilon^2. \quad \text{(A9)}$$

Accordingly, the contribution coming from the trajectories of the subset $\{B\}$ can be majorized by

$$B \leqslant \frac{1}{\varepsilon^2} P(\{B\}), \quad \text{(A10)}$$

where $P(\{B\})$ denotes the measure of trajectories forming the subset B. When δ is chosen such that $\delta^2 \gg 2\sigma\varepsilon$, this measure vanishes as

$$\ln P(\{B\}) \sim -\frac{\delta^2}{2\sigma\varepsilon}. \quad \text{(A11)}$$

Now, gathering Eqs. (A8) and (A10) we have that E is bounded from above by

$$E \leqslant \frac{\exp(4\delta)}{\varepsilon^2} \int_0^\infty dS P(S,R) \exp(-2S) + \frac{1}{\varepsilon^2} \exp\left(-\frac{\delta^2}{2\sigma\varepsilon}\right). \quad \text{(A12)}$$

Our previous analysis shows that the integral over the span variable S in the first term on the rhs of Eq. (A12) vanishes with R as a stretched-exponential function of the form $\exp(-R^{1/3})$. Thus the rhs of Eq. (A12) behaves as

$$\approx \frac{\exp(4\delta)}{\varepsilon^2} \exp(-R^{1/3}) + \frac{1}{\varepsilon^2} \exp\left(-\frac{\delta^2}{2\sigma\varepsilon}\right). \quad \text{(A13)}$$

Now we have to make the choice of ε and $\delta(R)$. One readily notices what the proper choice will be if we suppose that $\varepsilon = \text{const}$ and $\delta(R) \sim R^\gamma$, where γ is an arbitrary number from the interval $]1/6,1/3[$. If $\gamma > 1/6$ the second term on the rhs of Eq. (A13) vanishes with R faster than the first term and thus the leading large-R behavior will be given by the first term on the rhs of Eq. (A13). On the other hand the requirement $\gamma < 1/3$ insures that the leading large-R behavior follows the $\exp(-R^{1/3})$ dependence, since $R^{-1/3}\delta(R) \to 0$ when $R \to \infty$.

Therefore we have shown that also in the continuous-space limit the upper bound on the DAGS vanishes with R as a stretched-exponential function with the characteristic exponent $z = 1/3$. The bound derived here [although it suffices to prove the asymptotically exact dependence $\exp(-R^{1/3})$] turns out, however, to be worse than the one found in the discrete-space case; it differs from the bound in Eq. (59) by an additional multiplier which grows with R as $\exp(R^\gamma)$. Besides, this bound is not optimal; there are no well-defined values of ε and δ which minimize the upper bound. Apparently, an optimal upper bound in the continuous space can also be devised, but this is beyond the aims of the present paper.

[1] W.P. Su, J.R. Schrieffer and A.J. Heeger, Phys. Rev. Lett. **42**(1), 698 (1979).
[2] R. Jackiw and G. Semenoff, Phys. Rev. Lett. **50**, 439 (1983).
[3] A. Comtet, A. Georges, and P. Le Doussal, Phys. Lett. B **208**, 487 (1988).
[4] E. Witten, Nucl. Phys. B **188**, 513 (1981).
[5] R. Dutt, A. Khare, and U.P. Sukhatme, Am. J. Phys. **59**, 723 (1991).
[6] A. Comtet, A.D. Bandrauk, and D.K. Campbell, Phys. Lett. B **150**, 159 (1985).
[7] A. Inomata and G. Junker, in *Proceedings of International Symposium on Advanced Topics in Quantum Physics*, edited by J.Q. Liang, M.L. Wang, S.N. Qiao, and D.C. Su (Science Press, Beijing, 1993). p.61.
[8] J.P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, Europhys. Lett. **3**, 653 (1987).
[9] J.P. Bouchaud, A. Comtet, A. Georges, and P. Le Doussal, Ann. Phys. **201**, 285 (1990).
[10] A. Comtet, J. Desbois, and C. Monthus, Ann. Phys. **239**, 312 (1995).
[11] C. Monthus, Phys. Rev. E **52**, 2569 (1995).
[12] P.W. Anderson, Phys. Rev. **109**, 1492 (1958).
[13] P. Lloyd, J. Phys. C **2**, 1717 (1969); H.L. Frisch and P. Lloyd, Phys. Rev. **120**, 1175 (1960).
[14] D.J. Thouless, J. Phys. C **5**, 77 (1972).
[15] B.Ya. Balagurov and V.T. Vaks, Zh. Éksp. Teor. Fiz. **65**, 1939 (1973) [Sov. Phys. JETP **38**, 968 (1974)].
[16] M.D. Donsker and S.R.S. Varadhan, Commun. Pure Appl. Math. **28**, 525 (1975); **29**, 389 (1976).
[17] I.M. Lifschitz, Usp. Fiz. Nauk. **83**, 617 (1964). [Sov. Phys. Usp. **7**, 549 (1965)].
[18] S.F. Burlatsky, G. Oshanin, A. Mogutov, and M. Moreau, Phys. Rev. A **45**, 6955 (1992).
[19] G. Oshanin, S.F. Burlatsky, M. Moreau, and B. Gaveau, Chem. Phys. **177**, 803 (1993).
[20] G. Barton, A.J. Bray and A.J. McKane, Am. J. Phys. **58**, 751 (1990).
[21] R. Festa and E.G. d'Agliano, Physica A **90**, 229 (1978).
[22] P.A. Ferrari, S. Goldstein and J.L. Lebowitz, in *Statistical Physics and Dynamical Systems*, edited by J. Fritz, A. Jaffe, and D. Szasz (Birkhauser, Boston, 1985).
[23] K. Golden, S. Goldstein, and J.L. Lebowitz, Phys. Rev. Lett. **55**, 2629 (1985).
[24] R. Tsekov and E. Ruckenstein, J. Chem. Phys. **101**, 7844 (1994).
[25] C. De Calan, J.M. Luck, Th. Nieuwenhuizen, and D. Petritis, J. Phys. A **18**, 501 (1985).
[26] B. Derrida and H.J. Hilhorst, J. Phys. A **16**, 2641 (1983).
[27] H. Kesten, Acta Math. **131**, 208 (1973).
[28] G. Oshanin, A. Mogutov, and M. Moreau, J. Stat. Phys. **73**, 379 (1993).
[29] C. Monthus and A. Comtet, J. Phys. (France) I **4**, 635 (1994).
[30] Ya.G. Sinai, in Lecture Notes in Physics Vol. 53, edited by R. Schrader, R. Seiler, and O. Uhlenbrock (Springer, Berlin, 1981); Theory Probab. Apll. **27**, 247 (1982).
[31] M. Yor, Adv. Appl. Prob. **24**, 509 (1992); J. Appl. Prob. **29**, 202 (1993).

[32] K. Broderix and R. Kree, Europhys. Lett. **32**, 343 (1995).
[33] M. Mézard and G. Parisi, J. Phys. A **23**, L1229 (1990).
[34] A. Comtet, C. Monthus, and M. Yor, J. Appl. Prob. (to be published).
[35] G.H. Weiss and R.J. Rubin, Adv. Chem. Phys. **52**, 363 (1982); see also: P. Erdos and M. Kac, Bull. Am. Soc. **52**, 292 (1946); G.H. Weiss and R.J. Rubin, J. Stat. Phys. **14**, 333 (1976); K. Itô and H.P. McKean, *Diffusion Processes and their Sample Paths* (Springer, Berlin, 1965).
[36] A. Blumen, G. Zumofen, and J. Klafter, in *Optical Spectroscopy of Glasses*, edited by I. Zschokke (Reidel, Dordrecht, 1986).
[37] G.H. Weiss and S. Havlin, J. Stat. Phys. **37**, 17 (1984).
[38] G.V. Ryazanov, Theor. Math. Phys. **10**, 271 (1972).