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Behavior of transport characteristics in several one-dimensional disordered systems

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In this paper we examine the behavior of transport characteristics, such as the mean square displacement, passage times and average flux through a finite segment of the system, in two models of random walks in one-dimensional disordered systems. The first case is the random walk on a regular one-dimensional lattice with asymmetric random hopping probabilities. The second one concerns random motion of particles in a disordered lattice with distance-dependent transfer rates. The main goal of this paper is to answer the question of what are the representative realizations of disorder which support anomalous behavior of transport characteristics. On the basis of such an analysis we recover most of the known results and establish several new ones.

1. Introduction

Transport in disordered media has received much interest due to the considerable fundamental and technological importance of the problem [1–5]. Transport processes underlay a variety of chromatographic separation techniques [4], extraction of oil from porous rocks [4], excitations quenching in amorphous solids and doped crystals [6–9], conductivity of complex media and permeability of disordered membranes [5,10]. Molecular diffusion measurements in disordered media can serve as a tool in characterizing the internal geometry over a range of molecular and macroscopic length scales [4]. In addition, transport processes control the kinetics of chemical reactions in disordered media [6–9].

Most analyses of transport in random media make the assumption that the medium is infinite and focus on the behavior of such dynamical characteristics like the mean-square displacement as a function of time and the distribution function of the displacement. However, for several important applications the medium is bounded in space and the key interest is to establish the dependence of the characteristics on the system size. For example, these applications concern the permeability of disordered membranes or the conductivity of disordered finite samples [5,11]. For these systems one is interested in the behavior of the static characteristics – times required to diffuse over the sample (passage times) and the disorder average flux of particles through the disordered sample as a function of sample length.

It was believed that if one understands transport in an infinite medium, the results can be used to deduce the dependence of the static characteristics in finite media on the underlying parameters. However, this is not always the case. In recent works [12,13] on flux behavior in finite samples with Sinai-type disorder it was established that the disorder average flux follows a dependence which cannot be understood and reproduced from the analysis of the dynamical characteristics. The origin of this distinction stems from the fact that substantially different realizations of disorder underly the behaviors of dynamical and static characteristics.

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In this paper we examine the behavior of transport characteristics in two different types of one-dimensional disordered systems. On the basis of two well-known models, for which several rigorous results are available, we attempt to provide a clear physical understanding of what precisely the realizations of disorder are which support the anomalous disorder-induced behavior of the mean-square displacement, passage times and steady-state disorder average flux. We consider this explanation as one of the most important aims of the present paper since this type of analysis is often lacking in the literature on the subject. On the basis of this analysis we recover most of the known results and establish several new ones.

The models examined in this paper consider random walk in a one-dimensional lattice with random hopping probabilities and are defined by the following (continuous-time) discrete-space master equation,

$$\frac{\partial P_j(t)}{\partial t} = p_{j-1}P_{j-1}(t) + q_{j+1}P_{j+1}(t) - (p_j + q_j)P_j(t)\,, \qquad (1)$$

where $P_j(t)$ stands for the probability to find a particle at site j at time t; q_j and p_j are the probabilities for a particle to move from site j to site $j-1$ or to site $j+1$, respectively. The set $\{p_j\}$ consists of random variables and defines the random environment in which the random walk takes place. The average with respect to the realizations of disorder will be denoted below as $\langle\ \rangle$.

We consider two different types of random environments $\{p_j\}$:

(A) *An asymmetric hopping model.* In section 2 we study the behavior of disorder average characteristics in an asymmetric hopping model. In this model a random walk takes place on a regular lattice in which the pairs of transition probabilities on each site satisfy the conservation law $p_j + q_j = 1$, while the pairs of transition probabilities p_j and q_{j+1} on each link are independently distributed random variables with distribution function

$$P(p_j) = C\delta(p_j - p) + (1-C)\delta(p_j - 1 + p)\,, \qquad (2a)$$

where the value of p is bounded away from 0 and 1. It is also convenient to express the transition probabilities as

$$p_j = \frac{1+E\xi_j}{2}\,, \quad q_j = \frac{1-E\xi_j}{2}\,,$$

where the value E is bounded as $0<E<1$, and $\{\xi_j\}$ has the distribution function

$$P(\xi_j) = C\delta(\xi_j - 1) + (1-C)\delta(\xi_j + 1)\,, \qquad (2b)$$

i.e., ξ_j with probability C equals 1 and is equal to -1 with probability $1-C$. If $C>\frac{1}{2}$ the particle is subject to a constant average bias (perturbed by fluctuations) in the positive direction, i.e. its average mean displacement is greater than zero and grows with time. If $C=\frac{1}{2}$ no global bias is present in the system and all the odd moments of the displacement are zero in average.

The discrete-time analogue of this model, as well as the continuous space–time one, have been extensively studied [2,3,14–18] and a number of rigorous results on the behavior of the dynamical characteristics has been obtained. The critical marginal case $C=\frac{1}{2}$ represents the well-known Sinai model [19]. We will discuss here the behavior of different characteristics for arbitrary values of C and also consider the case of correlated environments. Sections 2.1–2.5 will be devoted to the Sinai case and several ramifications of this model. Section 2.6 concerns the behavior of characteristics in the model with average bias, $C>\frac{1}{2}$.

Let us note that several physical phenomena can be described in terms of this asymmetric random walk model. For example, it is relevant to the domain-wall dynamics in the one-dimensional random field Ising model [2,3], dislocation dynamics in doped crystals [20–22], the motion of polymers in entangled media [23] and the dynamics of the domain wall between the helix and coil phase in a random heteropolymer [15]. Apparently, the last case represents the first application of the mathematical asymmetric hopping model to physical systems. Besides, eq. (1) with such type of hopping probabilities describes the motion of a charged particle constrained on a one-dimensional random structure, e.g., a polymer, in the presence of an external electric field [11,24].

(B) *Hopping model on a disordered lattice.* Section 3 is devoted to the analysis of characteristics of the random walk taking place in a disordered lattice, i.e. a lattice with fluctuating intersite distance, and transfer rates dependent on the distance between neighboring sites. We consider the particular case when the intersite distance has a Poisson distribution and the transfer rate depends exponentially on the intersite distance. This model describes, for example, the transport of triplet excitations in disordered arrays of immobile donor centers [8,9]. More precisely, this model states the following: Consider a one-dimensional lattice with randomly placed donor centers. The distance L_j between the two neighboring donors, the jth and the $(j+1)$th, is a random variable with distribution function

$$P(L_j) = n\exp(-nL_j)\,, \tag{3}$$

where n is the mean concentration of donor centers. An excitation, initially located at the origin, jumps from site j to site $j+1$ $(j-1)$ with rate $R(j)$ $R(j-1)$ which is related to the intersite distance as

$$R(j) = r\exp(-L_j/a)\,,$$

where r is the amplitude of the transfer at the distance of closest approach and a is the characteristic length of the wavefunction.

This model differs substantially from the asymmetric hopping model. The rate of jump from site j to site $j+1$ is equal to the rate of jump in the backward direction – from site $j+1$ to site j. Besides, the "diagonal" multiplier before $P_j(t)$ in eq. (1) is random in contrast to the constant unit value in model A. It leads to the appearance of random waiting times for jumps from site j. In terms of the normalized transition probabilities this type of random motion is described by eq. (1) with

$$p_j = \frac{R(j)}{R(j-1)+R(j)}\,,$$
$$q_j = \frac{R(j-1)}{R(j-1)+R(j-2)}\,. \tag{5}$$

For this model we will also consider the behavior of dynamical and static characteristics and analyze the particular fluctuations in donor placement which support the behavior of the mean-square displacement, passage times and the average steady state flux.

2. An asymmetric hopping model

First we will consider the asymmetric hopping model in eq. (1) with hopping probabilities $\{p_j\}$ (or $\{\xi_j\}$) being independent (δ-correlated) random variables with the distribution function of eqs. (2). We start with the marginal case $C=\frac{1}{2}$, i.e. the case when no global average bias is present in the system.

This model of a random walk in a random environment was first examined by Sinai [19] who has proved rigorously that the mean-square displacement (MSD) of a particle,

$$\langle x^2(t)\rangle = \sum_{-\infty}^{\infty} j^2 P_j(t)\,, \quad x(0)=0\,,$$

for time t sufficiently large, grows in proportion to $\log^4(t)$ in striking contrast to the linear with time growth of the MSD in regular diffusive systems (all $p_j=\frac{1}{2}$ or $E=0$).

This remarkable rigorous results for a random walk in random environment, which is known now as the "Sinai diffusion" or a "random random walk", has attracted a great deal of attention within recent years. Different properties of this model have been extensively analyzed. In particular, the limiting form of the probability distribution [25], behaviors of the diffusion front and sample-to-sample fluctuations [26], typical values of passage times and disorder average passage times [15,27], have been discussed in great detail.

The physical origin of this disorder-induced slowing down to the diffusion process has been also clearly understood [19,28]. In one dimension it is always possible to define a potential energy function $W(j)$ so that the transition probabilities satisfy the detailed balance condition,

$$\frac{1-p_j}{p_j} = \exp[W(j) - W(j-1)]\,,$$

which entails the following form of the potential energy function,

$$W(j) = \sum_i^j \log\left(\frac{1-p_i}{p_i}\right) = \log\left(\frac{1+E}{1-E}\right)\sum_i^j \xi_i\,. \tag{6}$$

For the distribution function in eqs. (2) with $C=\frac{1}{2}$ each term in the right-hand side of eq. (6) takes with equal probabilities the values $\log[(1+E)/(1-E)]$ and $-\log[(1+E)/(1-E)]$. Therefore, the potential energy function $W(j)$ in eq. (6) is a realization of a symmetric j-step random walk trajectory, where the space variable j plays the role of "time". The mean-square $\langle W^2(j)\rangle$ grows with j as

$$\langle W^2(j)\rangle = \log^2\left(\frac{1+E}{1-E}\right) j = 2D_w j ,$$

$$D_w = \frac{1}{2}\log^2\left(\frac{1+E}{1-E}\right),$$

and the fluctuations in $W(j)$ grow as

$$\sqrt{\langle W^2(j)\rangle} \propto \log\left(\frac{1+E}{1-E}\right) j^{1/2} .$$

Thus, the Sinai model defines a random walk in the presence of a potential which itself is a random walk trajectory (in space).

Next we attempt to recover the behavior of the MSD and other characteristics by means of a simple analysis of the underlying disorder realizations, i.e. different realizations of the potential energy function $W(j)$.

2.1. Mean-square displacement and passage times

Consider a particle diffusing in the presence of some random potential, which creates barriers and wells. Assume that on a scale of size X the potential creates a barrier (or well) of height (depth) H. Then, in order to explore the distance X the particle has to overcome this barrier solely due to the thermal activation. Therefore, the time necessary for a particle to diffuse over a distance X (passage time) is the time needed to cross the potential barrier of height H. This, for a given realization of disorder, is given by the Arrhenius law, $t(X) \propto \exp(H)$.

The potential landscape in the model under consideration represents a very familiar picture of the random walk trajectory (fig. 1). It is characterized by the succession of random maxima and minima. The typical value of the barrier height and wells' depth on scale X, i.e., the value which is supported by the majority of $W(j)$ realizations, is $H \propto \sqrt{X}$. Therefore, the typical value of the passage time can be estimated as

$$t_{typ}(X) \propto \exp(\sqrt{X}) .$$

This suggests that the displacement $x(t)$ of a particle in time t should grow like $\log^2(t)$ for almost all realizations of disorder, i.e. for typical realizations. Thus,

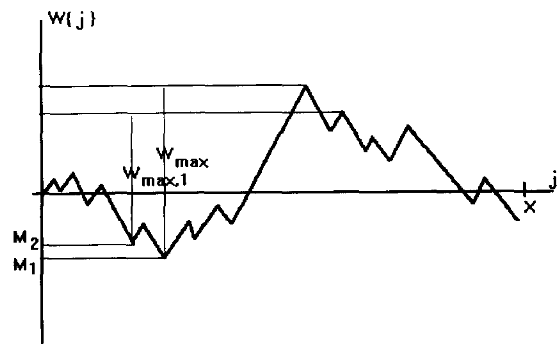

Fig. 1. Realization of potential landscape in Sinai model. M_1 is the maximal negative displacement on the interval $0 \leqslant j \leqslant X$, M_2 is the next negative maximal displacement; W_{max} is the largest potential valley.

one can expect $\langle x^2(t)\rangle \propto \log^4(t)$ for almost all samples.

These simple estimates of the "typical behavior" of the barriers and mean-square displacement can be supported by more striking and more rigorous arguments. To do this, let us consider again a finite segment of length X and discuss more precisely the structural features of the random potential on this segment. For a given realization of the potential $W(j)$ let us select among all maxima and minima the deepest potential well and the highest maximum [#1]. The spatial region which separates these extrema of the potential gives us the deepest potential valley (of the hight W_{max} (fig. 1)) among all the valleys on the segment. Next, we select the valleys of smaller sizes, $W_{max,1}$ (fig. 1), $W_{max,2}$, etc. Sinai [19] and Golosov [26] have proved rigorously that a particle exhibits a stronger form of localization: all the typical time needed to explore the segment of size X the particle spends in the deepest potential valley. This statement can be illustrated by means of a simple observation. Let us estimate the times needed to escape from valleys of heights W_{max} and $W_{max,1}$. These times are

$$t_{typ} \propto \exp(W_{max}) , \quad t_{typ,1} \propto \exp(W_{max,1}) .$$

Both W_{max} and $W_{max,1}$ behave in the same fashion, i.e. grow with X in proportion to $\sqrt{X}$. Therefore, one might be tempted to say that t_{typ} and $t_{typ,1}$ are of the

[#1] Let us emphasize that we consider the typical maxima and minima of order $W \propto \sqrt{X}$, but not the anomalously stretched realizations with low statistical weight.

same order and within time $t_{typ}(X)$ the particle is delocalized within the segment, i.e., it can occur with equal probability at any point within the segment. However, it is incorrect due to the following reason. Consider the ratio of t_{typ} and $t_{typ,1}$,

$$\frac{t_{typ}}{t_{typ,1}} = \exp(W_{max} - W_{max,1}) .$$

It is easy to see that, typically, the difference $W_{max} - W_{max,1}$ also grows in proportion to $\sqrt{X}$. Correspondingly, the time spent in the deepest valley is typically a factor $\exp(\sqrt{X})$ more than the time spent in the smaller valley.

2.2. Disorder average passage time

Let us consider next the behavior of the realization average passage time. It turns out that the disorder average value of the passage time, $\langle t(X)\rangle$, is supported by completely different type of the random potential realizations [15]. For the Sinai chain this value is controlled by the deepest well or the highest barrier, $\langle t(X)\rangle \propto \langle \exp[W_{max}(X)]\rangle$. The distribution function of $W_{max}(X)$ is determined by $\ln P(W_{max}(X)) \propto -W^2_{max}(X)/X$ and hence the value of the disorder average passage time can be estimated as

$$\langle t(X)\rangle \propto \int_0^\infty \exp\left(-\frac{W^2_{max}}{X} + W_{max}\right) \mathrm{d}W_{max} . \qquad (7)$$

The large-X behavior of the integral in eq. (7) is dominated by the saddle point value, which shows that large-X behavior of the average passage time is supported by anomalously stretched realizations of random potential. Namely, such that $W_{max} \propto X$ in contrast to the typical behavior $W(X) \propto X^{1/2}$. This yields a pure exponential growth [15,27] of the disorder average passage time with segment's length

$$\langle t(X)\rangle \propto \exp(X) , \qquad (8)$$

in contrast to a slower stretched-exponential growth of typical passage time $t_{typ}(X)$.

2.3. Disorder average steady state flux

Let us consider some finite segment of size X. The concentrations of particles at the endpoints of this segment, $P_{i=0}(t) = P_0$ and $P_{i=X}(t) = P_X$, are kept fixed for all times. If the difference $\Delta P = P_0 - P_X \neq 0$ there exists a flux of particles through the segment, which, for a fixed environment $\{p_i\}$, approaches at infinite time some constant sample dependent value $J(X)$. This value can be straightforwardly computed from eq. (1) and is equal to

$$J(X) = \frac{P_0}{2\tau^+(X)} - \frac{P_X}{2\tau^-(X)} , \qquad (9)$$

with

$$\tau^+(X) = 1 + \frac{p_1}{q_1} + \frac{p_1 p_2}{q_1 q_2} + \ldots + \frac{p_1 p_2 \cdots p_{X-1}}{q_1 q_2 \cdots q_{X-1}} , \qquad (10)$$

and

$$\tau^-(X) = 1 + \frac{q_{X-1}}{p_{X-1}} + \frac{q_{X-1} q_{X-2}}{p_{X-1} p_{X-2}} + \ldots + \frac{q_{X-1} q_{X-2} \cdots q_1}{p_{X-1} p_{X-2} \cdots p_1} . \qquad (11)$$

For the model under consideration where no global average bias is present the random functions $\tau^+(X)$ and $\tau^-(X)$ have the same distribution functions and, by symmetry [#2], are equal to each other in average,

$$\left\langle \frac{1}{\tau^+(X)} \right\rangle = \left\langle \frac{1}{\tau^-(X)} \right\rangle . \qquad (12)$$

Therefore, in the case of no global average bias the average flux can be written as

$$\langle J(X)\rangle = \frac{\Delta P}{2} \left\langle \frac{1}{\tau^+(X)} \right\rangle , \qquad (13)$$

where the random function $\tau^+(X)$, stated in terms of the potential energy function $W(X)$, reads

$$\tau^+(X) = \sum_{j=0}^{X-1} \exp[-W(j)] . \qquad (14)$$

Next we consider the behavior of the average steady state flux in the Sinai chain. Let us begin with some simple-minded estimates.

Suppose first that we have an infinite one-dimensional chain with the transition probabilities which satisfy the condition $p_i = p_{i+nX}$, where n is an arbitrary positive integer, i.e., we extrapolate periodically a given finite sample of the Sinai chain. Large

[#2] Replacement of p_i by q_i and reflection $i \to X - i$.

scale motion of a particle in such a system will be diffusive, $\langle X^2(t)\rangle = \langle D_{\rm eff}(X)\rangle t$, with the diffusion constant being some decreasing function of the parameter X. One can assume that the behavior of the effective diffusion constant is supported by the typical realizations of the random potential, likewise the behavior of the mean-square displacement. Then,

$$\langle D_{\rm eff}(X)\rangle \propto \exp(-\sqrt{X}) .$$

Conversely, if the value of $\langle D_{\rm eff}(X)\rangle$ is controlled by atypical realizations which support the anomalous behavior of the average passage time, one will expect a dependence of the form

$$\langle D_{\rm eff}(X)\rangle \propto \exp(-X) .$$

To estimate the behavior of the disorder average flux one can employ the Fickian law,

$$J(X) \propto D\,\Delta P/X , \qquad (15)$$

which describes the flux of particles in regular diffusive systems, i.e., systems in which all $p_i=\frac{1}{2}$. Substituting the effective diffusion constant into the Fickian law one can expect that the average flux decreases exponentially fast with the segment length,

$$\langle J(X)\rangle \propto \exp(-X^{\alpha}) ,$$

with α equal to either $\frac{1}{2}$ or 1, depending on the type of the trajectories which support the typical values of passage times or disorder average passage times.

However, it was realized [12] that the disorder average flux through the Sinai chain can be bounded from both sides by algebraically decreasing functions of the segment size,

$$A\frac{\Delta P}{\sqrt{X}\log^2(X)} \leqslant \langle J(X)\rangle \leqslant B\frac{\Delta P}{\sqrt{X}}, \qquad (16)$$

where A and B are X-independent constants.

Therefore, in the system under consideration with exponentially large passage times and logarithmically slow diffusion the flux decreases with X even at a slower rate than the flux in regular diffusive systems, eq. (15). Such a surprising behavior comes out of a particular class of representative realizations of random potential. Below we will discuss the properties of these realizations. This analysis will be based on the bounds on the average steady state flux.

We begin with an upper bound on the average steady state flux. For this propose we first slightly modify the definition of $\tau^+(X)$ in eq. (14). Let us choose among all the values of the potential energy function $W(j)$ on the interval $0\leqslant j\leqslant X$ the maximal negative value, $-\max_{0\leqslant j\leqslant X}[-W(j)]$ (fig. 1). Suppose that it is achieved at the "moment" $j=j_1$ and term it as $M_1=W(j_1)$. Next we choose the next maximal negative displacement (fig. 1); it is achieved at the "moment" $j=j_2$, $M_2=W(j_2)$. Following this line (also for the positive displacements) we will obtain an ordered sequence $M_1\geqslant M_2\geqslant\ldots\geqslant M_{X-1}\geqslant M_X$. Then, $\tau^+(X)$ can be represented as a sum of decreasing terms,

$$\tau^+(X)=\exp(-M_1) +\exp(-M_2)+\ldots+\exp(-M_X) . \qquad (17)$$

Since all the terms in the right-hand side of eq. (17) are positive $\tau^+(X)$ for any given sample is obviously greater than the maximal term

$$\tau^+(X)\geqslant\exp(-M_1) ,$$

and, in turn, the flux through a given sample is less than

$$J(X)\leqslant\exp(M_1) .$$

The value M_1 represents the maximal negative displacement of a symmetric random walk within "time" interval $0\leqslant j\leqslant X$, i.e., the span of 1D symmetric random walk. An average over the realizations of disorder means now the averaging over values of the maximal displacement M_1. Correspondingly, for the disorder average flux we have

$$\langle J(X)\rangle \leqslant \Delta P\int_{-\infty}^{0} dM_1\, P(M_1)\exp(M_1) , \qquad (18)$$

where $P(M_1)$ is the distribution function of the random walk shape. The latter is given by

$$P(M_1)=\left(\frac{2}{\pi D_{\rm w}X}\right)^{1/2}\exp\left(-\frac{M_1^2}{4D_{\rm w}X}\right).$$

Consequently, within the limit of large X one gets for the upper bound

$$\langle J(X)\rangle \leqslant \Delta P\left(\frac{1}{\pi D_{\rm w}X}\right)^{1/2} . \qquad (19)$$

The integral in eq. (18) is supported by $M_1\geqslant -1$. Therefore, the realizations of random potential which support the upper bound typically do not intersect the constant potential level $M_1=-1$. Let us also mention that corrections to the bound in eq. (19) can be obtained accounting for the rest terms in eq. (17) and averaging over the distribution of the next maximum, etc.

Next we consider the lower bound on the disorder average flux. For this purpose we will again employ the fact that averaging over the realizations of disorder is the averaging of random walk trajectory realizations. Therefore, the disorder average flux can be written down as the path integral

$$\langle J(X)\rangle = \Delta P \int_\Omega \dots \int \mathrm{D}\{W(j)\}\, P\{W(j)\}\, [\tau^+(X)]^{-1}\,, \tag{20}$$

where the subscript Ω denotes the set of all possible X-step random walk realizations, $W(j)$ terms the particular realization and $P\{W(j)\}$ denotes its statistical weight.

Next, let us invoke an obvious inequality. Suppose that we want to perform integration not over the entire set Ω but only over some subset ω of random walk realizations having some prescribed properties. Then, for any positively defined functional $F\{W(j)\}$ of the trajectory and for any subset ω, $\omega\subset\Omega$,

$$\int_\Omega \dots \int F\{W(j)\}\, P\{W(j)\}\, \mathrm{D}\{W(j)\} \geqslant \int_\omega \dots \int F\{W(j)\}\, P\{W(j)\}\, \mathrm{D}\{W(j)\}\,.$$

Therefore, if among all the trajectories of the random walk we will choose some subset of realizations with prescribed properties and extend the integration in eq. (20) only over this subset we will arrive at the lower bound on the disorder average flux

$$\langle J(X)\rangle \geqslant \Delta P \int_\omega \dots \int \frac{P\{W(j)\}\, \mathrm{D}\{W(j)\}}{\tau^+(X)}\,. \tag{21}$$

We define the subset ω as follows: ω contains only such trajectories of $W(j)$ which, starting at the origin, within X-steps never cross some growing deterministic function of the "time" j, $\epsilon(j)$. Otherwise stated, this subset can be defined in the following fashion. Suppose that one has a line of integers (space variable is k), an absorbing wall (trap) at point $k=\epsilon(0)<0$ and a particle initially located at position $k=0$. The lattice spacing is equal to $\log[(1+E)/(1-E)]$. The particle moves one step each unit of "time" j. Being at any site k it moves with equal probabilities either to site $k+1$ or to site $k-1$. The absorbing wall moves determenistically in the positive direction with the displacement growing as $\epsilon(j)$, i.e. takes the consecutive positions $\epsilon(j=0)$, $\epsilon(j=1)$, ..., $\epsilon(j=X)$; $\epsilon(0)\leqslant\epsilon(1)\leqslant\epsilon(2)\leqslant\dots\leqslant\epsilon(X)$; as "time" j evolves. Then, the subset ω contains only such trajectories $W(j)$ of the particle, which within X steps escape the moving absorbing wall, i.e. $W(j)-\epsilon(j)>0$ for any value of j within the interval $0\leqslant j\leqslant X$. In these terms, the statistical weight $P\{W(j)\}$ is the probability that diffusive particle survives until "time" X.

Next we enhance the bound in eq. (21) by choosing the maximal value of $\tau^+(X)$ on the subset ω. Evidently, for any realization of $W(j)$ which belongs to the subset ω,

$$\tau^+_\omega(X) \leqslant \tau_{\max}(X) = \sum_{j=0}^{X-1} \exp[-\epsilon(j)]\,, \tag{22}$$

and, consequently,

$$\langle J(X)\rangle \geqslant \frac{\Delta P}{\tau_{\max}(X)} \int_\omega \dots \int P\{W(j)\}\, \mathrm{D}\{W(j)\}\,. \tag{23}$$

Therefore, we have obtained a lower bound on the average steady state flux, which is a functional of the trial dependence $\epsilon(j)$.

The choice of the precise form of $\epsilon(j)$ is constrained by two conditions. First of all, $\epsilon(j)$ have to grow with j sufficiently fast in order to provide the convergence of the sum in the right-hand side of eq. (22) to the X-independent constant,

$$\frac{1}{\sum_{j=0}^{X-1} \exp[-\epsilon(j)]} \to \text{const.}\,,$$

when X tends to infinity. On the other hand, the dependence of $\epsilon(j)$ on j must be slow enough to provide sufficiently large measure to ω. For example, if we choose $\epsilon(j)\propto j^\gamma$ with $\gamma\geqslant\frac{1}{2}$, the sum in the right-hand side of eq. (22) will converge to a constant value. However, the subset ω, which corresponds to such a class of functions $\epsilon(j)$, will have an exponentially decreasing measure – the "diffusive particle" will not escape the trap and its survival probability will drop off exponentially with X, thus providing an exponentially decreasing flux. For our lower bound we choose

$$\epsilon(j) = A\log(\epsilon + j)\ ,$$

with $A>1$ (convergence of the sum in eq. (22)) and $\epsilon<1$ (to make $\epsilon(j=0)$ less than zero). Such a form satisfies both of the constraints. It is easy to see that flux on realizations ω tends to the constant X-independent value $J(A, \epsilon)$. Besides, the average survival probability

$$S(X) = \int_{\omega} \cdots \int P\{W(j)\}\, \mathrm{D}\{W(j)\}\ ,$$

weakly decreases with X. The form of the survival probability of a particle diffusing in the presence of an absorbing wall with logarithmic displacement was discussed in ref. [29]. It was shown that the average survival probability decreases with X as

$$S(X) = \frac{Y_0}{\sqrt{D_{\mathrm{w}} X}}\ ,$$

where Y_0 is given implicitly by the relation

$$Y_0 = \epsilon^2(j^*) = D_{\mathrm{w}} j^*\ .$$

Correspondingly, we arrive at the following lower bound on the flux,

$$\langle J(X)\rangle \geqslant \frac{\Delta P f(A, \epsilon)}{\sqrt{D_{\mathrm{w}} X}}\ , \qquad (24)$$

where f is an X-independent function given by [#3]

$$f(A, \epsilon) = Y_0(A, \epsilon)(A-1)\epsilon^{A-1}\ .$$

Therefore, the set of the potential realizations which support upper and lower bounds, and, correspondingly, the disorder average value of the flux through a finite sample, is as follows. All the points of the trajectories which support the upper bound are located above the fixed level $W(j) = -1$. The trajectories which support the lower bound are located above the curve $\epsilon(j) = A\log(\epsilon + j)$. The difference between these bounds is much smaller than the typical displacement of the realization of random walk $W(X)$ for large X. Note, that the typical depth of the potential minimum is again of the same order as for the typical potential paths. However, it does not alter the steady state flux. The time required for the particle to escape from the potential minimum is still exponentially large, $t \propto \exp(W_{\max})$. At the same time, due to the detailed equilibrium principle the steady state concentration of particles trapped in the local minimum is exponentially high, $C \propto C(0)\exp(W_{\max})$. Therefore, the steady flux is not affected by the existence of the local minima.

Thus, we have proved that disorder average flux in finite system with Sinai-type disorder decreases as an inverse square root of the length of the sample, i.e., at a much slower rate than Fickian flux in regular diffusive systems. Such a behavior is supported by atypical realizations of random potential. It is important to mention that the measure of these atypical realizations decreases with X as the inverse square root of X, which is much slower than the usual exponentially small measure of large deviations. It means that our result on flux behavior does not need extremely large statistics to be observed in numerical simulations [12].

2.4. Continuous-space model

To complete this chapter we would like to mention several results obtained in terms of the continuous space–time analogue of the asymmetric hopping model [2,3]. This is given by the Langevin equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{1}{\gamma} F\{x\} + \eta(t)\ ,$$

where $\eta(t)$ is a thermal noise,

$$\overline{\eta(t)} = 0,\ \overline{\eta(t)\eta(t')} = \frac{2kT}{\gamma}\delta(t-t')\ ,$$

where the overbar denotes thermal averages, kT is the temperature, γ is a friction coefficient; and $F\{x\}$ is a stationary random force. It is a Gaussian white noise with the statistical properties

$$\langle F\{x\}\rangle = F_0\ ,$$

[#3] The bound in eq. (24) represents the family of lower bounds on the average flux for any values of parameters A and ϵ such that $A>1$ and $\epsilon<1$. To find the best lower bound we have to find such values of the variational parameters A and ϵ which provide the maximal value to the function f. The precise form of $f(A, \epsilon)$ and the calculation of its maximal value are presented in ref. [29].

$$\langle F\{x\}F\{x'\}\rangle - F_0^2 = 2D_w\delta(x-x') .$$

The Sinai model corresponds to the case of zero average force, i.e. $F_0=0$. The continuous-space analogue of $P_j(t)$, the probability density $P(x, t)$ of the position x of the particle at time t satisfies, for a fixed environment $F\{x\}$, the usual Fokker–Planck (or Smoluchowski) equation

$$\frac{\partial P(x,t)}{\partial t} = D_0 \frac{\partial^2 P(x,t)}{\partial x^2} - \frac{1}{\gamma}\frac{\partial}{\partial x}[F\{x\}P(x,t)] , \qquad (25)$$

where D_0 is the bare diffusion constant equal to kT/γ. Further on we will set γ equal to 1. The dependence of the characteristics on γ can be easily restored by the correspondent renormalization of scales.

The continuous-space description in terms of eq. (25) has also attracted a great deal of attention (see, e.g., refs. [2,3]). An advantage of this description is that a usual transformation casts eq. (25) into the Schrödinger equation with random potential and various well elaborated canonical techniques can be applied [2,30]. This also established the relation between the model under consideration and certain quantum mechanical systems [2].

Another nice feature of the continuous-space description is that the static characteristics – the disorder average flux and mean passage times, can be calculated exactly for arbitrary values of the segment length X [13]. Besides, it is also possible to obtain more powerful results on the average flux behavior – to calculate explicitly large-X asymptotic forms of the flux moments of arbitrary order and the distribution function [29]. Curiously, it turns out that independently of the order m, all the moments of the flux have the same dependence on the segment size and differ only due to prefactors,

$$\langle J^m(X)\rangle \propto \frac{\Gamma(m)(D_w/D_0)^{m-1/2}}{\sqrt{\pi}} X^{-1/2} , \quad X\to\infty ,$$

where $\Gamma(m)$ is the gamma function.

The distribution function of the flux through the sample was calculated in ref. [29]. For small values of J it shows a log-normal behavior,

$$P(J) \propto \frac{\exp[-\log^2(J)/X]}{J^2\sqrt{X}} ,$$

while in the limit of large J it approaches a negative exponential form,

$$P(J) \propto \frac{\exp(-D_wJ/D_0)}{J^{3/2}\sqrt{X}} .$$

Let us stress that the exponent in the latter equation is independent of X. This clarifies the unusual behavior of the flux moments – their large-X behavior is supported by the large-J tail of the distribution function.

2.5. Correlated environments $\{p_j\}$

The effects of correlations in the random environment $\{p_j\}$ on the behavior of the mean-square displacement and typical times were extensively discussed in recent papers [11,31–33]. It was shown that the presence of correlations in $\{p_j\}$ leads to a modified Sinai law for the MSD, $\langle x^2(t)\rangle \propto \log^{2\nu}(t)$, where the exponent ν is dependent on the nature of correlations in the environments. Therefore, the MSD still exhibits a logarithmic behavior. The typical value of time required to diffuse a distance X becomes in this case $t_{typ} \propto \exp(X^{1/\nu})$, i.e. it is still exponentially large with X. However, correlations can dramatically alter the behavior of higher moments of displacement – even for short range correlations in $\{p_j\}$ there exists a critical value m_c for the order of the moment, such that all moments of order $m<m_c$ depend logarithmically on time, whereas for $m>m_c$ the moments of displacement are a power law of time [11,32]. The details of these results can be found in the original papers [32,33].

Here we will examine the effect of correlations in environments on the behavior of disorder average steady state flux. For this purpose let us first return to our equations (9) and (12), (13), where we have excluded from the consideration random function $\tau^-(X)$ using symmetry arguments. We stress that these arguments are quite correct and the equality in (12) can be proved rigorously. However, it turns out that the analysis of $\tau^-(X)$ gives us a key to the understanding of the average flux behavior in any types of random (and non-random) environments $\{p_j\}$. This behavior can be established most easily in terms of the continuous-space description in eq. (25). Solving eq. (25) in the steady state subject to the boundary conditions

$$P(x=0, t=\infty)=P_0, \quad P(x=X, t=\infty)=P_X,$$

we arrive at the following formula for the steady state flux through a given disordered sample

$$J(X)=D_0\left(\frac{P_0}{\tau^+(X)}-\frac{P_X}{\tau^-(X)}\right),$$

where the continuous-space definition of $\tau^+(X)$ is

$$\tau^+(X)=\int_0^X \mathrm{d}X \exp\left(-\alpha\int_0^X F(\xi)\,\mathrm{d}\xi\right), \quad \alpha=D_\omega/D_0,$$

and

$$\tau^-(X)=\tau^+(X)\left(\frac{\mathrm{d}\tau^+(X)}{\mathrm{d}X}\right)^{-1}.$$

Combining these equations we find

$$J(X)=D_0\left(\frac{P_0}{\tau^+(X)}-P_X\frac{\mathrm{d}}{\mathrm{d}X}\log[\tau^+(X)]\right).$$

Using symmetry arguments one can show that for systems with no global bias ($C=\frac{1}{2}$ or $F_0=0$),

$$\left\langle\frac{1}{\tau^+(X)}\right\rangle=\frac{\mathrm{d}}{\mathrm{d}X}\langle\log[\tau^+(X)]\rangle, \quad (26)$$

and the average flux can be expressed as

$$\langle J(X)\rangle=D_0\,\Delta P\frac{\mathrm{d}}{\mathrm{d}X}\langle\log[\tau^+(X)]\rangle. \quad (27)$$

Therefore, we have evaluated the equation which relates the average flux behavior to the average logarithm of random function $\tau(X)$, i.e. the equation which relates $\langle J(X)\rangle$ to the typical fluctuations of $\tau(X)$.

The typical behavior of $\tau(X)$ can be estimated as follows. For the Sinai model the value of the potential energy function $W(X)=\int^X \mathrm{d}\xi F(\xi)$ is simply a Brownian particle trajectory (see also eq. (14)) with a typical behavior $W(X)\propto X^{1/2}$. It can be straightforwardly computed that $\tau^+_{\mathrm{typ}}(X)\propto\exp(X^{1/2})$ and, thus, one can recover the results of eqs. (19) and (24) substituting $\tau^+_{\mathrm{typ}}(X)$ into eq. (27). An exact calculation of $\langle\log[\tau^+(X)]\rangle$ performed in ref. [29] confirms this simple estimate. Let us discuss now the behavior of $\tau^+_{\mathrm{typ}}(X)$ for correlated environments. The presence of sufficiently strong correlations in $\{p_j\}$ (or in the distribution of $F\{x\}$ for the continuous-space model) results in the anomalous behavior of the potential energy function $W(X)$. It no longer exhibits the familar "random-walk" behavior $W(X)\propto X^{1/2}$ but is characterized by the dependence $W(X)\propto X^\gamma$, where γ, $\gamma\leqslant 1$, is the anomalous exponent dependent on the particular nature of correlations. Correspondingly, for the average logarithm of $\tau^+(N)$ one gets

$$\langle\log[\tau^+(X)]\rangle\propto\log[\tau^+_{\mathrm{typ}}(X)]\propto X^\gamma.$$

Eq. (27) yields the following result for the behavior of the average steady state flux,

$$\langle J(X)\rangle\propto\frac{\Delta P}{X^{1-\gamma}}.$$

Next, let us relate the exponent γ to the exponents characterizing the correlation spectrum of the random environment. Consider the behavior of the potential energy function

$$W(X)=\log\left(\frac{1+E}{1-E}\right)\sum_i^X \xi_i,$$

in the case when the $\{\xi_i\}$ are not δ-correlated. Following refs. [32,33] we assume that our large ($X\gg 1$) but finite system is a part of a periodic system of size L. We further assume that the distribution of ξ_i is symmetric under reversal of all the ξ_i and that the Fourier transforms of the $\{\xi_i\}$, given by

$$\xi_q=\frac{1}{\sqrt{L}}\sum_{l=1}^{L}\xi_l\,\mathrm{e}^{-iql},$$

are correlated through the power-law relation

$$\langle\xi_q\xi_{-q}\rangle\propto\frac{1}{q^\lambda},$$

for small values of q. Correspondingly, if $\lambda=0$ one has the original Sinai model with uncorrelated disorder; if $\lambda>0$ then two neighboring ξ_i tend to be of the same sign; if, conversely, $\lambda<0$ then two neighboring ξ_i tend to be of different sign. It can be shown that for such correlations in the environment the MSD of the potential energy function grows as

$$\langle W^2(X)\rangle\propto X^{1+\lambda},$$

and, therefore, the typical behavior of $W(X)$ can be estimated as

$$W_{\mathrm{typ}}(X)\propto X^{(1+\lambda)/2}.$$

Thus, the exponent γ equals $(1+\lambda)/2$ and the formula for the average flux reads

$$\langle J(X)\rangle \propto \frac{\Delta P}{X^{(1-\lambda)/2}} . \tag{28}$$

We would like to emphasize another important feature of the model under consideration. Let us consider the behavior of the MSD, $\langle x^2(t)\rangle \propto \log^{4/(1+\lambda)}(t)$, the typical value of passage time $t_{\rm typ}(X) \propto \exp(X^{(1+\lambda)/2})$, and, $\langle J(X)\rangle$ in the cases $\lambda<0$ and $\lambda>0$. In the case $\lambda<0$, when neighboring ξ_i tend to be of different signs, the potential landscape is more smoothed and the appearance of a deep valley is less probable than in the Sinai case $\lambda=0$. Consequently, it is quite natural that the MSD grows with time at a faster rate and $t_{\rm typ}$ is smaller than in the case of Sinai diffusion. However, eq. (28) predicts that the average flux decreases at a faster rate, i.e. is smaller than the average flux in the Sinai system. In the case $\lambda>0$ the neighboring ξ's "prefer" to be of the same sign and one thus has larger valleys in the potential landscape than that in the Sinai case. Therefore, the MSD grows slower than $t_{\rm typ}$ is greater than that in the Sinai model. In contrast, the average flux turns out to be greater than that in eq. (24). Such a behavior unambiguously shows that the value of average flux in the disordered system is supported not by the statistics of potential maxima and minima, but only by the probability distribution of the representative trajectories, indicated in the previous chapter. In the case $\lambda<0$ this probability is less than that in the Sinai case, while for the "persistent walk" with $\lambda>0$ this probability is higher.

To complete this chapter we will make a prediction on the behavior of the disorder average flux in another Sinai-type system [28] with correlated random environment $\{p_j\}$. In this model correlations were introduced not directly, like in refs. [31–33], but were generated by the particular algorithm of the construction of the environment. This environment was developed from the Thue–Morse substitutional sequence using random rarefactions of it. The resulting set $\{p_j\}$ is random, in contrast to the Thue–Morse sequence, and exhibits essential fluctuations. For example, the fluctuations in the potential energy function grow with segment's size like X^β with $\beta=\log 3/\log 4$ (the value of β is dependent on the choice of the rarefaction rule). For this model it was found [28] that the typical time grows as $\exp(X^\beta)$ and the MSD grows like $\log^{2/\beta}(t)$. For such a system we predict the following behavior of the average flux

$$\langle J(X)\rangle \propto \frac{1}{X^{1-\log 3/\log 4}} .$$

2.6. Displacement of a particle in the presence of an average global bias. Continuous-space case

Both the discrete- and the continuous-space models in eqs. (1) and (25) in the presence of an average bias have attracted a considerable amount of attention [2,3,14–18]. First results on the anomalous displacement of a particle in such a system have been obtained in refs. [16,17] for the discrete-space model and then have been recovered in terms of the continuous-space description in eq. (25). Here we will consider only the continuous-space model. We consider the case when the disorder average value F_0 of the stationary random force is positive, i.e., the particle is favored to move in the positive direction.

When $F_0\neq 0$ the situation becomes different from the case considered above – on each scale of size X there appears a convective term $-F_0X$ in the potential in addition to the typical fluctuations, which are of order $\sqrt{X}$. The average bias eventually dominates the fluctuations in $W(X)$ after some transient regime, and one should expect the mean displacement of particle to grow linearly in time for times t sufficiently large. However, it turns out that for certain relation between parameters the mean displacement exhibits an anomalous long-time behavior indicating that there exists some type of fluctuations in $W(X)$ which affect the particle's motion. Several conjectures on the origin and properties of these fluctuations will be discussed below.

Let us begin with introducing some characteristic scales and parameters. The typical potential energy barrier encountered by the particle on a scale of size X is of order

$$\Delta W(X) \propto -F_0X \pm \sqrt{2D_{\rm w}X} . \tag{29}$$

This equation defines the first characteristic parameter $X_0=2D_{\rm w}/F_0^2$. When $X\ll X_0$ the second term in eq. (29) dominates. Therefore, on scales of order $X\ll X_0$ the motion is the Sinai diffusion controlled by the typical potential barrier growing in proportion to $\sqrt{X}$. Within the opposite limit the average bias becomes dominant. In order to reach the scale X_0 particle has to overcome a typical barrier of height

$$\Delta W(X_0) \propto \frac{2D_w}{F_0}.$$

Thus one can construct the second parameter which compares the degree of disorder to the strength of thermal activation,

$$\mu = \frac{2kT}{\Delta W(X_0)} = \frac{D_0 F_0}{D_w}.$$

This parameter naturally appears as the control parameter of the model. It was shown in ref. [2] that depending on its value different regimes can be observed. For $\mu < 1$ the mean displacement of the particle grows sublinearly with time, $\langle x(t) \rangle \propto t^{\mu}$. For $1 < \mu < 2$ the displacement grows linearly with time but, however, has an anomalous dispersion, $\langle x(t) \rangle \propto F_0 t \pm t^{1/\mu}$. Eventually, for $\mu > 2$ the normal diffusion with drift is established.

The origin of the slowing down to the diffusion process in the domain $\mu < 1$ has also been discussed in ref. [2]. It was argued that such a behavior arises due to the anomalously large fluctuations of the potential. These barriers act as trapping regions in which particle spends most of the time. These atypical energy barriers (anomalously stretched trajectories of $W(X)$) have exponentially low statistical weight, but the time required to overcome them grows exponentially with their height. As a result, it was guessed that these times, trapping times, have a broad distribution with infinite moments. Thus it was assumed that the physics of the original model in eq. (25) can be well captured by much simplier model of the directed random walk in the presence of temporal traps with a broad distribution of release times.

Below we will present another explanation of the behavior $\langle x(t) \rangle \propto t^{\mu}$. We will show that the origin of the slowing-down of the diffusion process comes out of typical fluctuations in the potential $W(X)$.

Random potential in eq. (25), $W(X) \propto \int^X d\xi F\{\xi\}$, can be represented as the sum

$$W(X) = F_0 X + W_0(X),$$

where the first term gives the regular drift while the second one, $W_0(X)$, is the potential in the Sinai model – a "symmetric random walk" trajectory superposed on the regular drift. The potential $W(X)$ fluctuations around the slope $F_0 X$ on the scale X are distributed as

$$P\{W_0(X)\} = (4\pi D_w X)^{-1/2} \exp\left(-\frac{W_0^2(X)}{4D_w X}\right).$$

On scales $X \ll X_0$ the typical fluctuations in $W(X)$ dominate over the regular bias. However, the time required to overcome the fluctuation-induced barriers is of order

$$t_{typ}(X) \propto \exp\left(\frac{W(X)}{kT}\right), \tag{30}$$

and, therefore, has distribution function with finite moments [#4]. It means that typical fluctuations can only alter the prefactor in the dependence of the mean displacement on time.

Let us consider some other class of typical fluctuations. Suppose that our segment X is a part of a much larger, of size $L \gg X$ (fig. 2), system, so that the entire system is a sequence of L/X segments of size X. Suppose next that we have a *typical* realization of random potential, i.e., such that fluctuations in every segment grow with segments size as $A_i\sqrt{X}$. For any typical realization of potential the amplitudes A_i, however, are not the same for each segment of the sequence. They fluctuate and one can select among them the maximal amplitude $A_{i^*} = \max\{A_i\}$, $1 \leq i \leq L/X$. The fluctuation of the potential on the i^*th segment represents the maximal typical fluctuation of the potential on scale L. For systems with large fluctuations $\mu \leq 1$ the typical passage time is controlled

[#4] The value of X is less than X_0 and $t_{typ}(X)$ is less than $t_{typ}(X_0)$.

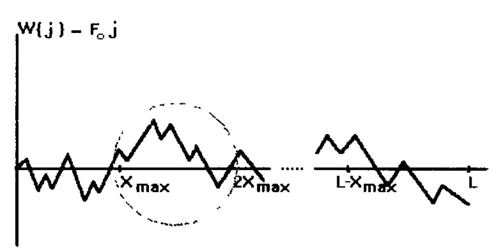

Fig. 2. Potential fluctuations around the slope $-F_0 j$ in the asymmetric hopping model with average bias. In circle: maximal typical potential fluctuation. X_{max} is the scale of maximal typical fluctuation.

by the segment with the maximal typical potential fluctuation the distribution of which can be estimated as

$$P\{W_0(X)\}\frac{L}{X}=1 .$$

For $L/X \gg 1$ it leads to

$$W_{0,\max}(X)=2\sqrt{D_w X\log(L)} , \qquad (31)$$

for the maximal typical fluctuation on the scale X. Introducing (31) into (30) we obtain the following estimate for the typical value of passage time

$$t_{\text{typ}}(X)\propto\exp\left(\frac{-F_0X+2\sqrt{D_wX\log(L)}}{kT}\right). \qquad (32)$$

In order to determine the representative value of potential fluctuation one has to find the size of the maximal typical fluctuation. For this purpose we maximize the right-hand side of eq. (32) with respect to X. The value of X which provides maximum to eq. (32) is equal to

$$X_{\max}=\frac{D_w\log(L)}{F_0^2} .$$

Inserting $X_{\max}$ into eq. (32) we find that typical passage time grows with system size as

$$t_{\text{typ}}(L)\propto L^{D_w/D_0F_0} ,$$

and, eventually, recover the desired result

$$x(t)\propto t^{\mu} ,$$

which shows that the anomalous behavior of the mean displacement in systems with $\mu<1$ stems out of typical potential fluctuations. For systems with small potential fluctuations, $\mu>1$, the latter equation predicts that mean displacement grows faster than displacement of particle in regular diffusive systems with constant force, $x(t)\propto Vt$. It means that for $\mu>1$ the scale of fluctuation barriers $X_{\max}$ associated with maximal typical fluctuation of the potential becomes comparable with L (or greater) and eqs. (31), (32) are no longer valid.

3. Transport in a one-dimensional disordered lattice

In this section we will examine the behavior of transport characteristics in a one-dimensional lattice with fluctuating intersite distance and distance-dependent transfer rates – model B in our previous notations. We consider a particular case when the intersite distances $\{L_j\}$ are mutually independent random variables with Poisson distribution in eq. (3) and the transfer rate from site j to site $j+1$ drops off exponentially with the distance between these sites L_j. We set out to show that all transport characteristics, the MSD, typical times and the average flux, exhibit anomalous behavior. We analyze the structure of fluctuations which support the anomalous behavior of transport characteristics.

Let us also mention that the model B is related to different models of random walk in disordered media. In particular, it describes also the random walk on comb-like structures [1] or ballistic motion of particles in presence of multistate random potential [31]. Therefore, the analysis of the representative disorder realizations performed for the model B can also be applied for several other models.

First of all we would like to emphasize the important difference between the model under consideration and the model A. We have already mentioned that eq. (1), which describes the transport properties in both cases, has different structure of the transition rates for models A and B. In the case of the Sinai model a particle makes one jump per each time unit so that a particle located at site j at time t leaves this site with unit probability at time $t+\Delta t$, Δt being infinitesimally small. All the randomness in the model A arises due to the fact that the pairs of transition probabilities on each link are random. In contrast, for the model B the probability to leave the site is random. The case in the model A is often quoted as the case of "off-diagonal" disorder, while the case in the model B represents the system with "diagonal" disorder. The "diagonal" disorder results in the appearance of waiting times between the successive hops. The difference between these models can be also seen from the analysis of the behavior of the potential energy function $W(X)$ introduced in the previous section. It is easily found that all R_j which enter eq. (6) cancel and $W(X)$ on a finite segment of size X depends only on the values of the first and the last transition rates,

$$W(X)=\log(R_1/R_X) .$$

In contrast to the Sinai model where the potential energy function on a segment of size X is the sum of X independent identically distributed random variables and typically exhibit an abounded growth with the growth of the segment size, in the model B the potential $W(X)$ is the difference of two independent random variables. It means that $W(X)$ does not grow with X and, therefore, there are no large potential barriers or deep wells which can act like temporal traps and slow down excitation transport. However, in the systems with small values of the control parameter $\nu = an$ there are typical fluctuations which make the transport anomalous. These fluctuations are again the maximal typical fluctuations of the distance between sites.

3.1. Mean square displacement and the typical value of passage time

Consider some finite segment of a random lattice, which contains $N+1$ sites (N is a large positive integer, $N \gg 1$, which determines the scale $X = N/n$). Since all the intersite distances are independent random variables, the probability that the maximal intersite distance on a segment with $N+1$ donor centers is exactly equal to $L_{\max}$ can be written as

$$\Pi(L_{\max}) = F^{N-1}(L_{\max}) P^N(L = L_{\max}) \, ,$$

where $P(L = L_{\max})$ is the distribution function in eq. (3) and $F(L_{\max})$ is

$$F(L_{\max}) = \int_0^{L_{\max}} \mathrm{d}X P(X) \, .$$

Since $L_{\max}$ is supposed to be much greater than the typical value of the intersite distance $\bar{L} \propto 1/n$, for large N one has

$$\Pi(L_{\max}) \propto \exp(-[1 - F(L_{\max})]N) P(L = L_{\max}) \, .$$

The maximum of $\Pi(L_{\max})$ is approached when

$$\Pi(L_{\max}) N = 1 \, ,$$

In other words, $L_{\max}$ is the typical largest value of the intersite distances for which the expectation of the number of intervals is of order unity. The typical maximal intersite distance $L_{\max}$ can be estimated from the latter relation and one finds that $L_{\max}$ grows with the number of sites N as

$$L_{\max} \propto \frac{\log(N)}{n} \, .$$

The typical time needed to jump over the interval $L_{\max}$ is given by

$$t_{\mathrm{typ}} = (L_{\max}) \propto \frac{N^{1/\nu}}{rn} \, . \tag{33}$$

Since we have fixed $L_{\max}$ by its typical maximum value all other distances are less than this typical value and, therefore, all other jump times have bounded distributions with finite moments of arbitrary order. In particular, the mean jump rate is equal to

$$\langle t(L, L < L_{\max}) \rangle = \frac{\int_0^{L_{\max}} \exp[(1-\nu)\xi/a] \, \mathrm{d}\xi}{r \int_0^{L_{\max}} \exp[-n\xi] \, \mathrm{d}\xi}$$

$$= \frac{\nu}{1-\nu} N^{1+1/\nu} \, .$$

The sum of N independent jump times for these typical intervals which are smaller than the typical maximal interval is equal to

$$\frac{\nu}{1-\nu} N^{1/\nu} \, ,$$

i.e. is smaller than the right-hand side of eq. (33) due to the smaller value of the prefactor.

Therefore, one can formulate the following statement. For small values of the parameter $\nu < \frac{1}{2}$, the excitation, confined between two typical maximum intervals $L_{\max,1}$ and $L_{\max,2}$, such that $L_{\max,1} > L_{\max,2}$ and both are greater than all the intervals in between, has the smaller probability to cross the largest interval starting from its boundary then the probability of jumping through all the in between intervals until the next maximum.

Correspondingly, the most probable trajectory is organized as follows. An excitation being at site i jumps through the smallest of the two neighboring intervals. It moves in the same direction until it meats the interval which is greater than the first one. Then it changes the direction and jumps until it encounters the interval which is greater than all explored before. Then it again changes the direction and the process repeats. Let us also illustrate this statement for a particular realization of donor placement. Suppose a segment of the lattice, containing seven donors. The intersite distance between the first and the second is

L_1, the distance between the second and the third is L_2, etc. Suppose next that for some particular realization of donor placement (fig. 1) these intervals satisfy the inequality

$$L_6 > L_1 > L_5 > L_3 > L_4 > L_2 ,$$

and the donor is initially located at position $j=4$ having neighboring intervals L_3 and L_4. Then the most probable path for the first 13 steps is as follows (fig. 3). Since $L_3 > L_4$ an excitation jumps through the smallest of these two intervals, $4 \to 5$. Being at site $i=5$ it changes its direction since $L_5 > L_3$ and jumps $5 \to 4 \to 3 \to 2$. Further on, since $L_1 > L_5$ it jumps $2 \to 3 \to 4 \to 5 \to 6$. Next, since $L_6 > L_1$ it again changes the direction and jumps $6 \to 5 \to ... \to 1$.

Since the excitation jumps through the largest interval only from the neighboring donor center and the probability to find the excitation on each donor center between two maximum intervals is of the same order, the time which excitation spends on the chain segment containing N donor centers is proportional to the product $N t_{\text{typ}}(L_{\max})$. This yields the following scaling relation between the typical displacement and time,

$$t \propto N^{1+1/\nu} \propto [\sqrt{\langle x^2(t) \rangle}]^{1+1/\nu} ,$$

and, thus, one obtains for the mean square displacement

$$\langle x^2(t) \rangle \propto t^{2\nu/(1+\nu)} .$$

This result was obtained by means of a mean-field theory approach in ref. [1].

3.2. Disorder average flux

The dependence of the disorder average flux on the segment length can be found using analogous estimates. Taking advantage of eqs. (9)–(11) of the previous section, one can deduce that flux through a given sample of size N is defined by eq. (9) with $\tau^+(N)$ given by

$$\tau^+(N) = \sum_{i=1}^{N} R_i^{-1} . \tag{34}$$

Assuming that disorder average flux is controlled by the largest interval between donors we obtain the following estimate

$$\langle J(N) \rangle \propto \Delta P \int_N^\infty P(L_{\max}) R(L_{\max}) \, dL_{\max} .$$

Performing the integration we evaluate the following dependence of the average flux on the number of donor centers N (the dependence on the sample size is obtained trivially by substitution $N = X/n$),

$$\langle J(N) \rangle \propto \frac{\Delta P}{N^{1/\nu}} . \tag{35}$$

However, the disorder average flux can be calculated exactly. Let us examine the behavior of $\tau^+(N)$ in eq. (34). Using eqs. (3), (4) one can find the distribution function of variable $1/R_i$. It reads

$$P(\tau_i = 1/R_i) = \nu r^{-\nu} \tau_i^{-1-\nu} , \quad 1/r < \tau_i < \infty , \tag{36}$$

Therefore, $\tau^+(N)$ is the sum of N independent identically distributed random variables having a broad distribution with infinite moments. A generalized form of the central limit theorem insures that the suitable rescaled variable, $T(N) = \tau^+(N)/N^{1/\nu}$, possesses a limit distribution $P(T(N))$ when N goes to infinity, $P(T(N)) = L_\nu(T)$, where L_ν is a non-Gaussian stable (Levy) law. The explicit expressions of the stable laws are known only in two particular cases. For $\nu = \frac{1}{2}$, when it describes the limit distribution of first return times of a one-dimensional Brownian motion, and for $\nu = \frac{1}{3}$. However, the large-T and small-T asymptotic forms of $L_\nu(T)$ are established (see, e.g., the Appendix in ref. [2]) for any value of the parameter ν. Besides, the negative moments of $L_\nu(T)$ can be calculated exactly,

$$\int_0^\infty T^{-m} \, dT \, L_\nu(T) = Z^{-m/\nu} \frac{\Gamma(m/\nu)}{\nu \Gamma(m)} ,$$

where

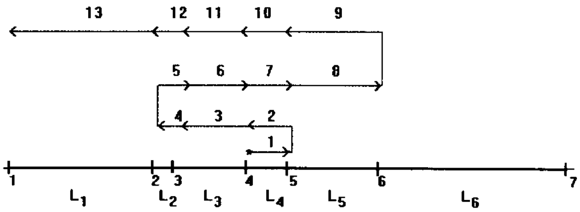

Fig. 3. Most probable trajectory (first 13 steps) of an excitation for fixed realization of donor positions with $L_6 > L_1 > L_5 > L_3 > L_4 > L_2$.

$$Z=\frac{\pi}{\sin(\pi\nu)\Gamma(\nu)r^{\nu}}\,.$$

Let us note that the desired average flux and its moments are essentially the negative moments of the distribution $L_\nu(T)$. After some transformations we obtain the exact expression for the flux moments of an arbitrary order m,

$$\langle J^m(N)\rangle=\frac{\Gamma(m/\nu)(\Delta P)^m}{\nu\Gamma(m)(ZN)^{m/\nu}}\,. \quad (37)$$

This exact formula for the flux confirms our estimate in eq. (35) based on the analysis of disorder realizations. Let us note that in contrast to the Sinai model, the moments of flux exhibit a strong dependence on N. Besides, the average flux decreases with N at a faster rate than the flux in the Sinai model, and even faster than flux in regular diffusive systems (eq. (15)). This is caused by the nature of fluctuations in the system – the potential energy function does not fluctuate and, thus, there are no trajectories which accelerate particle motion, while the fluctuations of the intersite distance can only slow down motion of particles.

Let us next consider the behavior of characteristics as functions of the parameter ν. With the growth of ν the MSD becomes greater and, eventually, when ν is equal to unity, the MSD approaches a normal diffusion law. The dependence of $\langle J(N)\rangle$ on N becomes slower with the growth of the parameter ν and, eventually, when $\nu=1$ the normal Fickian law in eq. (15) is restored.

The knowledge of the moments in eq. (37) or the distribution function of $\tau^+(N)$ suffices to evaluate the distribution function of the flux. The normalized distribution function of the flux reads

$$P(J)=(J^2N^{1/\nu})^{-1}L_\nu(1/JN^{1/\nu})\,.$$

One can evaluate explicitly its asymptotic forms. For $JN^{1/\nu}\gg 1$ the normalized distribution function has a stretched-exponential form,

$$P(J)\propto\left[2\pi(1-\nu)\left(\frac{J^{2-3\nu}}{\nu ZN}\right)^{1/(1-\nu)}\right]^{-1/2}$$
$$\times\exp\left(-\frac{1-\nu}{\nu}(\nu ZNJ^{\nu})^{1/(1-\nu)}\right).$$

Within the opposite limit $N^{1/\nu}J\ll 1$ the distribution function diverges as

$$P(J)\propto\frac{ZN\Gamma(1+\nu)\sin(\pi\nu)}{\pi}J^{\nu-1}\,.$$

4. Conclusions

To conclude, we have examined the origin of the anomalous behavior of transport characteristics in two different models of transport in one-dimensional disordered systems – the Sinai model and disordered lattice with fluctuating intersite distance and distance-dependent transfer rates. We have clarified the underlying realizations of disorder which support the behavior of the mean square displacement, typical passage times and average fluxes through a finite segment of the system. We have shown that the anomalous behavior of the MSD is connected with the typical realizations of disorder. For the Sinai model, where the source of disorder is a random stationary force, these are the typical realizations of random potential. For the model with average bias (Kesten–Kozlov–Spitzer or Derrida–Pomeau model) these are the maximal typical fluctuations of the potential on the scale under consideration. For the transport in the disordered lattice these are the maximal typical fluctuations of the intersite distance. In all the cases considered here these fluctuations induce the sublinear growth of the MSD with time.

The behavior of the average flux through a finite segment of the system is more complicated and is connected with the type of disorder. For the Sinai model the average flux is supported by the realizations of the random potential which favor motion through the sample. The precise form of the flux dependence on the sample size is defined entirely by the measure of these realizations. The average flux in such a system turns out to be substantially greater than the flux in regular diffusive systems. In contrast, in systems where disorder is caused by the fluctuations of the intersite distance, the potential is not fluctuating and, correspondingly, there are no particur realizations of the potential which can enhance the flux. Fluctuations of the intersite distance essentially diminish the average flux through the sample as compared to the flux in the Sinai system and, even in regular diffusive systems. The Fickian regime is established only in the trivial case when the wave functions of donors essentially overlap ($an \to 1$) and transport becomes diffusive.

We would like to stress that several of the presented analytical results are well known and some of them are confirmed by rigorous calculations. Our aim in this paper was to provide a qualitative explanation to these results. We have performed here an analysis of the particular realizations of disorder which induce anomalous behavior of transport characteristics. We consider this analysis as the major outcome of our work. Besides, we have established several new results on the behavior of disorder average flux in the Sinai system with correlated fields and in disordered lattices.

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