Problem: Landau theory of Fermi liquids

We start from a system of non-interacting electrons (a Fermi gas) at zero temperature, with Hamiltonian

\[ H = \sum_{p,\sigma} \epsilon_0^p c_{p,\sigma}^\dagger c_{p,\sigma} \]

with \( \epsilon_0^p = p^2/2m \) and \( \sigma = \pm 1/2 \). We suppose the system is isotropic and set \( \hbar = 1 \). In the Fermi gas, infinitely long lived particle-hole excitations are constructed from the one-particle spectrum \( \epsilon_0^p \).

The ground state is characterized by the Fermi-dirac step function \( n_{0,\sigma}^p = \theta(p_F - p) \), and excited states by \( \delta n_{p,\sigma}^p = \delta_{p,p'}\delta_{\sigma,\sigma'} \), while a hole excitation of momentum \( p' \) (\( p' < p_F \)) and spin \( \sigma' \) is described by \( \delta n_{p,\sigma}^p = -\delta_{p,p'}\delta_{\sigma,\sigma'} \).

1. For a periodic system of \( N \) noninteracting electrons of mass \( m \) in a large volume \( V \), what is the Fermi surface? Compute the Fermi momentum \( p_F \) and Fermi energy \( \epsilon_0^F \). What is the density of state \( \rho_0(\epsilon) \)? What is the energy of a generic excited state \( \delta n_{p,\sigma}^p \)? Show that the chemical potential is equal to the energy of a particle on the Fermi surface.

We now work in the continuum, and work with a coarse grained distribution \( \bar{n}_{p,\sigma}(0 \leq \bar{n}_{p,\sigma} \leq 1) \). Consider a cluster of \( N \) neighbouring values of \( n_{p,\sigma} \), with \( \bar{n}_{p,\sigma} \) their average distribution. Show that the log of the number of microscopic configurations of the cluster leading to a \( \bar{n}_{p,\sigma} = x \) is (for large \( N \) \( \gg \) 1)

\[ -N \left[ x \log x + (1 - x) \log(1 - x) \right] \]

3. We set \( k_B = 1 \). Show that the entropy of the Fermi gas in the state \( \bar{n}_{p,\sigma} \) is

\[ S[\bar{n}] = -V \int \frac{d^3p}{(2\pi)^3} \sum \sigma \left[ \bar{n}_{p,\sigma} \ln \bar{n}_{p,\sigma} + (1 - \bar{n}_{p,\sigma}) \ln(1 - \bar{n}_{p,\sigma}) \right] \]

(1)

From now on we drop the bar in \( \bar{n}_{p,\sigma} \) and simply write \( n_{p,\sigma} \), keeping in mind that we are dealing with the coarse grained distribution.

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1For instance we can consider cubic clusters centered around \( P \in M^22\pi \mathbb{Z}^3 \) and of size \( (M^22\pi)^3 \). Each such cluster contain \( N = M^3 \) momenta, and the coarse grained distribution \( \bar{n}_{p,\sigma} \) is simply the average of \( n_{p,\sigma} \) over those \( N \) values. At the microscopic level we have \( n_{p,\sigma} = 0 \) or 1, therefore \( \bar{n}_{p,\sigma} \in \{n/N, n = 0, 1, \cdots N\} \). For large values of \( N \), we end up with a continuous distribution \( \bar{n}_{p,\sigma} \), which can take any value between 0 and 1.
4. The thermodynamic potential is given by

\[ \Omega[n] = E[n] - \mu N[n] - TS[n] \]  

(2)

where \( E[n] = \sum_{p,\sigma}^0 \epsilon_p n_{p,\sigma} \) and \( N[n] = \sum_{p,\sigma} n_{p,\sigma} \). The equilibrium distribution \( n^{eq} \) is obtained as \( \delta \Omega[n]/\delta n_{p,\sigma} = 0 \). Recover the Fermi-Dirac distribution.

1 The quasiparticle concept

Upon adding interactions, the property of the electronic liquid can change drastically \((i.e. \text{ lead to a phase transition, for instance to a superconducting state})\), but in some situations the electronic liquid retains many properties of the non-interacting gas: such a Fermi liquid is then said to be normal.

Landau Fermi liquid theory is a effective theory describing the low-energy degrees of freedom of normal interacting gas of electrons. The main idea underlying this construction is adiabaticity. Starting from the non-interacting fermion system, and slowly switching on the interactions, Landau argued that the Fermi gas ground state adiabatically transforms into the ground state of the interacting system. During this adiabatic process, conserved quantities such as spin, charge or momentum remain unchanged, while dynamical properties such as mass, magnetic moment, etc are renormalized to new values.

By slowly switching on the interaction, the non-interacting excited state \( \delta n_{p,\sigma} \) get dressed and becomes an eigenstate of the interacting system, which we still label by the bare distribution \( \delta n_{p,\sigma} \). The underlying hypothesis is that this adiabatic process leads to a smooth deformation of the non-interacting states, without hitting any singular behavior. Basically we assume that the interactions do not lead to a phase transition.

This adiabatic mapping provides a one-to-one correspondence between the elementary excitations of the Fermi gas and the Fermi liquid, but is is important to keep in mind that the excitations of the Fermi liquid are fully interacting dressed states, and are not simply obtained by adding particles and holes to the ground state. Another crucial difference with the Fermi gas is that these elementary excitations interact with each other. As a result they acquire a finite life time, and for this reason these excitations are called quasiparticles \((\text{and quasiholes})\). Fortunately these quasiparticles are better and better defined as one approaches the Fermi surface\(^2\), and as long as we consider only low energy excitations \((i.e. \text{ close to the FS})\), the quasiparticle damping can be neglected.

In the interacting system, \( n_p \) describes the distribution of quasiparticles, and is measured by the departure from the ground state distribution \( \delta n_{p,\sigma} = n_{p,\sigma} - n_{p,\sigma}^0 \). We will only consider low energy excitations for which \( \delta n_p \) is small, and non zero only for \( p \) close to the FS. In this

\(^2\)At zero temperature, this lifetime varies as the inverse square of the energy separation to the fermi surface \((cf \text{ calculation of the self-energy})\).
regime the energy can be developed in $\delta n_p$:

$$E[n] = E_0 + \sum_{p,\sigma} \epsilon_p \delta n_{p,\sigma} + \frac{1}{2V} \sum_{p,\sigma',\sigma''} f^{\sigma\sigma'}_{pp'} \delta n_{p,\sigma} \delta n_{p',\sigma'} + O(\delta n^3)$$  \hspace{1cm} (3)

The quasiparticle dispersion relation $\epsilon_p$ can be expanded around the Fermi surface as

$$\epsilon_p \sim \epsilon_F + v^*_F (|p| - p_F), \quad v^*_F = \frac{p_F}{m^*}$$ \hspace{1cm} (4)

which defines the (renormalized) Fermi velocity $v^*_F$ and effective mass $m^*$. By analogy with the non-interacting case, we define the quasiparticle density at the Fermi surface as

$$\rho^*(\epsilon_F) = \frac{V m^* p_F}{\pi^2}.$$

The phenomenological theory of Fermi liquids as proposed by Landau takes into account interaction between quasiparticles through an extra quadratic term where $f^{\sigma\sigma'}_{pp'}$ are the Landau parameters (which are symmetric $f^{\sigma\sigma'}_{pp'} = f^{\sigma'\sigma}_{pp'}$).

5. If $V$ is the total volume, what is the order of $f^{\sigma\sigma'}_{pp'}$ in $V$? Give a physical justification.

6. Suppose that $\delta n_{p,\sigma}$ is only significant for $|p - p_F| < \delta$. Show that both the linear term and the quadratic term are of the same order in $\delta$.

7. What is the energy $\tilde{\epsilon}_p$ of an additional quasiparticle with momentum $p$?

For a time-reversal invariant system, we can split the Landau parameters into symmetric and antisymmetric coefficients:

$$f^{\sigma\sigma'}_{pp'} = f^s_{pp'} + f^a_{pp'}$$

$$f^{\sigma\sigma'}_{pp'} = f^s_{pp'} - f^a_{pp'}.$$

As only wave vectors near the Fermi surface are considered in our isotropic system, we can set $|p| = |p'| = p_F$, and only the relative angle $\theta$ between $p = p_F \hat{p}$ and $p' = p_F \hat{p}'$ is important. We can expand the coefficients in terms of Legendre polynomials:

$$f^{s(a)}_{pp'} = \sum_{l=0}^{\infty} f^{s(a)}_{l} P_l(\cos \theta), \quad f^{s(a)}_{l} = (2l + 1) \int \frac{d\hat{p}'}{4\pi} f^{s(a)}_{pp'} P_l(\cos \theta)$$ \hspace{1cm} (5)

We recall the orthogonality relation: $\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta mn$ and give $P_0(x) = 1$ and $P_1(x) = x$. It is convenient to introduce the dimensionless parameters

$$F^{s(a)}_l = \rho^*(\epsilon_F) f^{s(a)}_l = \frac{V m^* p_F}{\pi^2} f^{s(a)}_l.$$

Now, we will determine the properties of Fermi liquids and compare them with those of the Fermi gas. First, we relate the effective mass with the Landau parameters. Then, we calculate the specific heat, the spin and charge susceptibility and the first sound velocity.
2 Effective mass

We consider the system from a moving frame at an infinitesimal velocity $v$ with respect to the laboratory frame.

8. Let $|\Psi\rangle$ be an eigenstate of $H$ with energy $E$ and total momentum $P$. What is the energy $E'$ in the moving frame as a function of $v$, of the initial energy $E$, of the total mass $M$ and of the total momentum $P$? * Figure out how to implement Galilean transformations using a (time-dependent) unitary transformation.

9. We take as our reference (excited) state the ground state of $H$ viewed from the moving frame. Show that

$$
\delta n_p = -\frac{m}{m^*} \mathbf{p} \cdot \mathbf{v} \delta (\epsilon_p - \epsilon_F).
$$

10. We add a quasiparticle of momentum $p$ (in the moving frame) and of spin $\sigma$ to the system. Calculate its energy $\epsilon'_p \sigma$ in the moving frame as a function of $\epsilon_p \sigma$, $\mathbf{v} \cdot \mathbf{p}$, $m$ and $m^*$. Now, calculate it from the energy of Q.7.

$$
\frac{m^*}{m} = 1 + \frac{1}{3} F_{1}^{\alpha} \tag{6}
$$

3 Magnetic susceptibility

We now determine the spin susceptibility $\chi$ of a Fermi liquid. $\chi = \frac{1}{V} \frac{dM}{dB} |_{B \to 0}$, where $B$ is the external applied magnetic field in the $z$ direction. The Zeeman coupling causes a change of energy of $-\gamma \sigma B$, where $\gamma$ is the gyromagnetic ratio. Our reference state is the equilibrium state of the Fermi liquid under the Hamiltonian $H$.

11. Is the chemical potential $\mu$ affected by $B$ to first order in $B$ (for a constant number of particles)? Why? Relate $\delta n_{p,\sigma}$ to $\delta n_{p,-\sigma}$.

12. What is the energy $\tau_{p\sigma}$ of a quasiparticle near the Fermi surface? Express it as a function of $\Delta n_{\sigma} = \sum_{p} \delta n_{p\sigma}$.

13. Calculate $\Delta n_{\sigma}$ and deduce that

$$
\chi = \frac{\gamma^2 N^0}{4V(1 + F_{0}^{\alpha})} \tag{7}
$$

4 Compressibility

14. The compressibility is given by $\kappa = \frac{1}{\rho^2} \frac{\partial \rho}{\partial \mu}$ where $\rho$ is the particle density. Show that

$$
\kappa = \frac{p_F m^*}{\rho^2 \pi^2 h^3 (1 + F_{0}'')} \tag{8}
$$

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