ICFP Master Program
Condensed Matter Theory (B. Douçot, B. Estienne, L. Messio)
Monday December 17th 2018, 14:00 to 18:00.

The goal of the problem is to couple a 1D Luttinger liquid to a 1D electric field. We recall that the Fourier modes of density fluctuations of right and left moving particles are given respectively by the operators $\rho_{\mathrm{R}}(q)$ and $\rho_{\mathrm{L}}(q)$. A traditional formulation of the Luttinger model, which we have not given in the lectures, involves a pair of canonically conjugate real scalar fields $\Phi(x)$, $\Pi(x)$, which are given explicitely by:

$$\begin{aligned}
\Phi(x) &= \sum_{q\neq 0} -\frac{i\pi}{qL}(\rho_{\mathrm{R}}(q)+\rho_{\mathrm{L}}(q))e^{-iqx} - \frac{\pi}{L}(\hat{N}_{\mathrm{R}}+\hat{N}_{\mathrm{L}})x + \bar{\phi} \\
\Pi(x) &= \sum_{q\neq 0} \frac{1}{L}(\rho_{\mathrm{R}}(q)-\rho_{\mathrm{L}}(q))e^{-iqx} + \frac{1}{L}(\hat{N}_{\mathrm{R}}-\hat{N}_{\mathrm{L}})
\end{aligned}$$

The $\bar{\phi}$ operator is characterized by the fact that it commutes with density modes $\rho_{\mathrm{R}}(q)$, $\rho_{\mathrm{L}}(q)$ and that it satisfies: $[\bar{\phi},\hat{N}_{\mathrm{R}}-\hat{N}_{\mathrm{L}}]=i$. To simplify notations, we shall use the convention $\hbar=1$ in this problem. L is the linear size of the periodic 1D domain.

1 Luttinger model in terms of scalar bosonic fields

1) Check that the field operators $\Phi(x)$ and $\Pi(x)$ are self adjoint. This is what is meant by the expression *scalar fields*.

2) Check that these fields obey canonical commutation rules, i.e. $[\Phi(x),\Phi(x')]=0$, $[\Pi(x),\Pi(x')]=0$, and $[\Phi(x),\Pi(x')]=i\delta(x-x')$.

3) Check that we can relate the Φ field to the long wave-length modulations of the electronic density by $\rho_{\mathrm{R}}(x)+\rho_{\mathrm{L}}(x)=-\frac{1}{\pi}\frac{\partial\Phi}{\partial x}$. What kind of physical interpretation of the phase field Φ does this suggest?

2 Coupling to an external electric field

We are now taking the 1D electronic system to be infinite, so $L\to\infty$. One can easily show that the kinetic energy Hamiltonian H_0 for the Luttinger model with linear dispersion relations on the right and left moving branches can be recast into the quadratic bosonic Hamiltonian:

$$H_0=\frac{v_{\mathrm{F}}}{2}\int_{-\infty}^{\infty}dx\,\left(\pi\Pi^2+\frac{1}{\pi}\left(\frac{\partial\Phi}{\partial x}\right)^2\right)$$

We couple this 1D system to an external electromagnetic field, described in terms of the scalar potential $V(x,t)$ and the component $A(x,t)$ of the vector potential taken along the wire. The component of the electric field which is tangent to the wire is then given by: $E(x,t)=-\frac{\partial V}{\partial x}-\frac{\partial A}{\partial t}$. We recall that this electric field is invariant under gauge transformations defined by: $A\to A+\frac{\partial f}{\partial x}$ and $V\to V-\frac{\partial f}{\partial t}$, where f is an arbitrary function of x and t.

The effect of this electric field on the electronic system is assumed to be described by the additional term:

$$H_{\mathrm{el}}(t)=-e\int_{-\infty}^{\infty}dx\,\left(\frac{V(x,t)}{\pi}\frac{\partial\Phi}{\partial x}(x)+v_{\mathrm{F}}A(x,t)\Pi(x)\right)$$

where $H_{\rm el}(t)$ has the status of a time-dependent operator in Schrödinger's picture. The total Hamiltonian acting on the 1D system is then $H(t) = H_0 + H_{\rm el}(t)$.

In the remaining questions, we shall adopt the following notations for Fourier transforms, adapted to the case of an infinite 1D system:

$$
\begin{aligned}
f(x,t) &= \int_{-\infty}^{\infty} \frac{dq}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i(qx-\omega t)} \tilde{f}(q,\omega) \\
\tilde{f}(q,\omega) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt\, e^{-i(qx-\omega t)} f(x,t)
\end{aligned}
$$

4) Show that the time evolution of Heisenberg operators $\Phi(x,t)$ and $\Pi(x,t)$ is determined by:

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t}(x,t) &= \pi v_{\rm F}\Pi(x,t) - e v_{\rm F} A(x,t) \\
\frac{\partial \Pi}{\partial t}(x,t) &= \frac{v_{\rm F}}{\pi}\frac{\partial^2 \Phi}{\partial x^2}(x,t) - \frac{e}{\pi}\frac{\partial V}{\partial x}(x,t)
\end{aligned}
$$

5) In the above system of equations, eliminate Π to obtain an equation connecting Φ to (V, A). Check that it is gauge invariant, provided $\Phi(x,t)$ behaves in a well-chosen way under gauge transformations. Can you justify this choice on a physical ground?

6) Explain to which extent we can consider that the local electric charge density $\rho(x,t)$ and the local electric current density $j(x,t)$ are given by:

$$
\begin{aligned}
\rho(x,t) &= -\frac{e}{\pi}\frac{\partial \Phi}{\partial x}(x,t) \\
j(x,t) &= \frac{e}{\pi}\frac{\partial \Phi}{\partial t}(x,t)
\end{aligned}
$$

Motivate then the choice of $H_{\rm el}$.

7) Using previous questions, show that there exists a *linear* relation between the current $j(x,t)$ and the field $E(x,t)$. Setting $\tilde{j}(q,\omega) = \tilde{\sigma}_0(q,\omega)\tilde{E}(q,\omega)$, compute $\tilde{\sigma}_0(q,\omega)$. How can you interpret physically such result?

8) We consider the special case $E(x,t) = E\theta(t)$ where $\theta(t) = 1$ if $t > 0$ and $\theta(t) = 0$ if $t < 0$. The initial condition is $j(x,t) = 0$ for all $t < 0$. Evaluate then $j(x,t)$ for $t \geq 0$. Does this fit with your previous physical interpretation of $\tilde{\sigma}_0(q,\omega)$?

9) We consider now the case of a static external potential $V(x)$. Show that the induced charge $\rho(x)$ is *linear* in V. Setting $\tilde{\rho}(q) = \tilde{\chi}^0(q)\tilde{V}(q)$, compute $\tilde{\chi}^0(q)$. What do you think of this result? Under which conditions does this sound realistic? Which microscopic processes are not captured by this simple model?

3 A toy model of 1D electrodynamics

Besides the matter-field coupling associated to $H_{\rm el}$, it is necessary to attribute its own dynamics to the field, in the presence of source terms (ρ, j). We make the following choice: $\frac{\partial E}{\partial x} = \rho$ and $-\frac{\partial E}{\partial t} = j$.

10) Give a physical motivation for this choice. What is the potential $V(x)$ which is created by a point charge located at $x = 0$, so $\rho(x) = \delta(x)$? Are you surprised by the result?

11) Give the complete evolution equation for the E field, after eliminating the source terms (ρ, j) using results of section 2. What kind of dispersion relation do you get? How do you interpret physically this result?

12) We wish to evaluate the response of this system to an externally imposed field $E_{\text{ext}}(x,t)$. For this, we proceed in the following way. The electronic system responds to the total field $E = E_{\text{ext}} + E_{\text{ind}}$, where E_{ind} is the induced field, solution of $\frac{\partial E_{\text{ind}}}{\partial x} = \rho$ and $-\frac{\partial E_{\text{ind}}}{\partial t} = j$.

Consider first the case of a static external potential $\tilde{V}_{\text{ext}}(x)$. Show that the induced charge $\delta\rho(x)$ in the system is linear in $V_{\text{ext}}(x)$, so that we may write $\delta\tilde{\rho} = \tilde{\chi}(q)V_{\text{ext}}(q)$. Compute $\tilde{\chi}(q)$. In the particular case where $V_{\text{ext}}(x)$ is created by a point like external charge, (so $\frac{\partial^2 V_{\text{ext}}}{\partial x^2} + \delta(x) = 0$), compute $V(x)$ and compare it to $V_{\text{ext}}(x)$. What do you think of the result?

13) We now assume an arbitrary profile for $E_{\text{ext}}(x,t)$. Show that the induced current $j(x,t)$ is still linear in E_{ext}, so that we have: $\tilde{j}(q,\omega) = \tilde{\sigma}(q,\omega)\tilde{E}_{\text{ext}}(q,\omega)$. Express $\tilde{\sigma}(q,\omega)$ as a function of $\tilde{\sigma}_0(q,\omega)$. What is the limit of $\tilde{\sigma}(q,\omega)$ when $(q,\omega) \to (0,0)$? How would you qualify such system?

14) Suppose that we replace the interaction between electrons in this 1D model by the usual 3D Coulomb potential e^2/r. You may use the fact that the Fourier transform of $1/r$ in 1D is approximately $2\log(1/|q|a)$ at small q, where a is a short distance cut-off. How would this affect the previous results?