ICFP Master Program
Condensed Matter Theory (B. Douçot, B. Estienne, L. Messio)
Final exam, 14h to 18h, December 18th, 2017.

The two parts of this exam are completely independent.

1 Quantum fluctuations of the number of fermions in a finite subsystem

Let us consider a non-interacting spinless fermion system in 3D at zero temperature. We shall denote by $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$ the annihilation and creation operators of a single fermion at position $\mathbf{r}$. These operators are assumed to obey the standard fermionic anticommutation rules, in particular $\{\Psi(\mathbf{r}), \Psi^\dagger(\mathbf{r}')\} = \delta^{(3)}(\mathbf{r}-\mathbf{r}')$. It will be convenient to assume that the system is infinite, and therefore, we shall use here the following definition for Fourier modes: $\Psi(\mathbf{r}) = \int \frac{d^{(3)}\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}} c(\mathbf{k})$ and $c^\dagger(\mathbf{k}) = \int d^{(3)}\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} \Psi^\dagger(\mathbf{r})$. It follows that $\{c(\mathbf{k}), c^\dagger(\mathbf{k}')\} = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$. We will study quantum fluctuations of the total particle number $\hat{N}_B$ in a box B whose linear size is finite but large compared to the Fermi wave-length. Explicitely, $\hat{N}_B = \int_B d^{(3)}\mathbf{r}\ \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})$, where the integral is taken over the volume of the box. As usual, the Fermi momentum will be denoted by k_F.

1) Give the expression of $\langle \hat{N}_B \rangle$ in the non-interacting Fermi sea.

2) Show that the variance of the particle number can be expressed as:

$$\langle (\Delta \hat{N}_B)^2 \rangle = \langle \hat{N}_B^2 \rangle - \langle \hat{N}_B \rangle^2 = \int \frac{d^{(3)}\mathbf{q}}{(2\pi)^6}\, F_{3B}(\mathbf{q}) \mathcal{V}_3(k_F, \mathbf{q})$$

where

$$F_{3B}(\mathbf{q}) = \left| \int_B d^{(3)}\mathbf{r}\ e^{i\mathbf{q}\cdot\mathbf{r}} \right|^2$$

and

$$\mathcal{V}_3(k_F, \mathbf{q}) = \int d^{(3)}\mathbf{k}\ \theta(k_F - |\mathbf{k}|)\theta(|\mathbf{k}+\mathbf{q}| - k_F)$$

3) Show that when B is a sphere of radius R, then: $F_{3B}(\mathbf{q}) = 16\pi^2 \frac{R^2}{q^4}\left(\cos(qR) - \frac{\sin(qR)}{qR}\right)^2$. Here q stands for $|\mathbf{q}|$. Draw a qualitative plot of the variations of $F_{3B}(\mathbf{q})$ with q. What happens in the limit of small q, and of large q?

4) Try to show (without wasting too much time on this) that $\mathcal{V}_3(k_F, \mathbf{q}) = \pi q(k_F^2 - \frac{q^2}{12})$ for $q \leq 2k_F$. What happens for $q \geq 2k_F$? Draw a qualitative plot of the variations of $\mathcal{V}_3(k_F, \mathbf{q})$ with q.

5) Show that the integral giving $\langle (\Delta \hat{N}_B)^2 \rangle$ converges for large values of q. Give an estimate of the contribution of large wave-vectors ($q \geq 2k_F$) to the integral, when the box is large in the sense that $k_F R \gg 1$. Do not hesitate to make all the approximations which you find reasonable, here and for the next two questions.

6) Still for a large box, estimate now the contribution of small wave-vectors ($qR \ll 1$). Show that it has the same order of magnitude as the previous one.

7) Finally, estimate the contribution from the intermediate range ($R^{-1} < q < 2k_F$). Show that it dominates over the two previous contributions. What do you think of the result? Which physical ingredients are responsible for it?

8) To put these results in a broader perspective, we now consider a 1D system of infinite length, in which the box B corresponds to the interval $[-R, R]$. Show that $\langle (\Delta \hat{N}_B)^2 \rangle$ is given by a similar integral as in 3D, namely:

$$\langle (\Delta \hat{N}_B)^2 \rangle = \int \frac{dq}{(2\pi)^2} \, F_{1B}(q) \mathcal{V}_1(k_F, q)$$

9) Give the expressions of $F_{1B}(q)$ and $\mathcal{V}_1(k_F, q)$.

10) Show that the situation is qualitatively similar as in 3D, namely the expression for $\langle (\Delta \hat{N}_B)^2 \rangle$ is dominated by contributions from wave-vectors q such that $R^{-1} < q < 2k_F$. What do you think of the result?

11) In one dimension, we can use the mapping to harmonic density modes in order to access the probability distribution of $\hat{N}_B$. The goal is to show that, to a reasonable approximation, the fluctuations of $\hat{N}_B$ are gaussian. For this, we remind that for a gaussian random variable X with zero mean value, and for an arbitrary real number λ, we have:

$$\langle e^{\lambda X} \rangle = e^{\frac{\lambda^2}{2} \langle X^2 \rangle}$$

The natural strategy is then to evaluate the expectation value $\langle e^{\lambda \hat{N}_B} \rangle$ in the non-interacting ground-state. Keeping in mind the results of the previous question, show that it is reasonable to approximate the local fermion density $\Psi^\dagger(x)\Psi(x)$ by $\Psi_R^\dagger(x)\Psi_R(x) + \Psi_L^\dagger(x)\Psi_L(x)$ in the expression of $\hat{N}_B$.

12) From this, express $\Delta \hat{N}_B = \hat{N}_B - \langle \hat{N}_B \rangle$ in terms of the density modes $\rho_R(q)$ and $\rho_L(q)$. In this part, you can assume that the system is a ring of total length L, quantized with periodic boundary conditions, in order to use the same notations as in the lectures. Of course, we impose that $2R < L$. Express $\Delta \hat{N}_B$ as a sum of two operators of the form $\mathcal{O}^\dagger + \mathcal{O}$, where both $\mathcal{O}^\dagger$ and $\mathcal{O}$ are linear in density mode operators, and such that $\mathcal{O}^\dagger$ creates particle-hole excitations and $\mathcal{O}$ annihilates them.

13) Evaluate the commutator $[\mathcal{O}, \mathcal{O}^\dagger]$, and use it to compute $\langle e^{\lambda \Delta \hat{N}_B} \rangle$. What do you infer from this? Is this consistent with what was obtained by the simpler treatment in question (10)? What is missing in this approach?

2 Dissipation for a particle moving in a Fermi liquid

In this section, we consider an additional particle moving along a classical trajectory $\mathbf{R}(t)$ in a non-interacting Fermi liquid. The coupling between the particle and the Fermi liquid is taken to be of the form:

$$H_c(t) = \int d^{(3)}\mathbf{r} \; V(\mathbf{R}(t) - \mathbf{r}) \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r})$$

Therefore, the force exerted by the Fermi liquid on the particle at time t is: $\mathbf{F}(t) = -\boldsymbol{\nabla}_{\mathbf{R}} H_c(t)$. Here, $V(\mathbf{R} - \mathbf{r})$ is typically a short range potential, whose Fourier transform will be denoted by $\tilde{V}(\mathbf{q})$. The volume of the system will be denoted by Ω. Here, we return to the finite volume normalizations used in the lectures, namely: $\Psi(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{r}} c(\mathbf{k})$, and $c^\dagger(\mathbf{k}) = \frac{1}{\sqrt{\Omega}} \int d^{(3)}\mathbf{r}\ e^{i\mathbf{k}.\mathbf{r}} \Psi^\dagger(\mathbf{r})$.

14) Express $H_c(t)$ and $\mathbf{F}(t)$ in terms of the particle position $\mathbf{R}(t)$ and of the density modes $\rho(\mathbf{q})$.

15) To evaluate the friction force exerted by the Fermi liquid on the particle, we assume that the former is initially in its ground-state, so that $\langle \mathbf{F}(t) \rangle \to 0$ as $t \to -\infty$. Show that, within linear response, we can write:

$$\langle \mathbf{F}(t) \rangle = -\frac{i}{\Omega} \sum_{\mathbf{q}} \mathbf{q}\, |\tilde{V}(\mathbf{q})|^2 \int_{-\infty}^{\infty} dt'\ e^{i\mathbf{q}.(\mathbf{R}(t)-\mathbf{R}(t'))} \chi_c(\mathbf{q}, t-t')$$

where $\chi_c(\mathbf{q}, t-t')$ is the charge susceptibility of the fluid.

16) Specialize this expression to the case when the particle moves with a constant velocity $\mathbf{v}$, so that $\mathbf{R}(t) - \mathbf{R}(t') = \mathbf{v}(t-t')$.

17) Using a spectral decomposition, show that $\chi_c(\mathbf{q}, \omega)^* = \chi_c(-\mathbf{q}, -\omega)$.

18) The Fermi liquid is subjected to the total Hamiltonian $H(t) = H_c(t) + H_0$, where H_0 is the kinetic energy part. We are interested in the energy power dissipated into the fluid, i.e. the time derivative of the expectation value of the total energy $\langle H(t) \rangle$. Assuming that the Fermi liquid wave-function obeys the Schrödinger equation with the time-dependent Hamiltonian $H(t)$, recall why we have:

$$\frac{d}{dt}\langle H(t) \rangle = \langle \frac{\partial}{\partial t}(H(t)) \rangle$$

and show that the dissipated power per unit time $P(t)$ is related to the particle velocity and the instantaneous friction force $\mathbf{F}(t)$ in the expected manner.

19) We now assume that the particle moves with the constant velocity $\mathbf{v}$. Using the fact that the potential V is real and the symmetry relation in question (17), show that the dissipated power P is equal to:

$$P = -\frac{1}{\Omega} \sum_{\mathbf{q}} |\tilde{V}(\mathbf{q})|^2 \mathbf{q}.\mathbf{v} \Im \chi_c(\mathbf{q}, \mathbf{q}.\mathbf{v})$$

Check that this dissipated power is always *positive*. Could you give a more physical interpretation of this formula?

20) Suppose now that $v = |\mathbf{v}|$ is much smaller than the Fermi velocity v_F. Show that the sum in the previous formula for P is dominated by the contribution of momentum transfers $q \leq 2k_F$. Using the low energy approximation for the particle-hole density of states given in the lectures, and taking the thermodynamic limit $\Omega \to \infty$, give an explicit expression for P. Check that it has the correct dimension, and comment the result.

21) Reconsider now the problem for a 1D system of non-interacting fermions. Show that, by contrast to the 3D case, the dissipated power is now due to momentum transfers in a narrow window around $2k_F$.

22) Give then an explicit formula for P and comment the result.

23) We wish now to study friction for a particle moving in a 1D Luttinger liquid. Show, using the previous discussion, that a reasonable description for the particle-liquid coupling is:

$$H_c(t) = \int dx\, V(X(t) - x) \left(\Psi_R^\dagger(x)\Psi_L(x) + \Psi_L^\dagger(x)\Psi_R(x) \right)$$

where $X(t)$ is the particle position at time t. Note that this implies that the instantaneous friction force is given by the operator:

$$\mathbf{F}(t) = -\int dx\, V'(X(t) - x) \left(\Psi_R^\dagger(x)\Psi_L(x) + \Psi_L^\dagger(x)\Psi_R(x) \right)$$

V' being the spatial derivative of V.

24) Using linear response, show that:

$$\langle \mathbf{F}(t) \rangle = \frac{i}{\hbar} \int dx \int dx' \int dt'\, V'(X(t)-x) V(X(t')-x') \left(\mathcal{C}(x - x', t - t') - \mathcal{C}(x' - x, t' - t) \right) \theta(t-t')$$

Here, we have introduced the notation $\mathcal{C}(x-x', t-t') = \langle [\mathcal{O}(x,t), \mathcal{O}^\dagger(x',t')] \rangle$, the expectation value being taken in the Luttinger liquid ground-state. The operator $\mathcal{O}(x,t)$ is itself defined as a bilinear of Heisenberg picture operators: $\mathcal{O}(x,t) = \Psi_R^\dagger(x,t)\Psi_L(x,t)$. Explain why several other terms which might have appeared in this formula turn out to be equal to zero.

25) Neglecting zero-mode contributions, and using standard Lunttinger liquid notations and techniques, try to establish (without spending too much time on it) that we can write the desired expectation value as:

$$\begin{aligned}
\langle \mathcal{O}(x,t)\mathcal{O}^\dagger(x',t') \rangle &= e^{-2ik_F(x-x')} \frac{\epsilon^{2(K-1)}}{L^2} F(x - x' - \tilde{v}_F(t-t')) F(-(x - x' + \tilde{v}_F(t-t'))) \\
\langle \mathcal{O}^\dagger(x',t')\mathcal{O}(x,t) \rangle &= e^{-2ik_F(x-x')} \frac{\epsilon^{2(K-1)}}{L^2} F(-(x - x' - \tilde{v}_F(t-t'))) F(x - x' + \tilde{v}_F(t-t'))
\end{aligned}$$

Here, K is the usual dimensionless interaction parameter of the Luttinger liquid, $\tilde{v}_F$ is the renormalized Fermi velocity, and the function $F(x)$ is defined as:

$$F(x) = \left(1 - e^{-\epsilon} e^{i\frac{2\pi}{L}x} \right)^{-K}$$

where ϵ is a small dimensionless convergence parameter.

26) Expanding $F(x)$ as $F(x) = \sum_{m=1}^{\infty} c_m e^{-m\epsilon} e^{i\frac{2\pi}{L}mx}$, show that we can write the dissipative force as:

$$\langle \mathbf{F} \rangle = -\frac{2\pi\epsilon^{2(K-1)}}{\hbar^2 L^2} \sum_{m,n} c_m c_n (2k_F + q(m,n)) |\tilde{V}(2k_F + q(m,n))|^2 \delta(\omega(m,n) - v(2k_F + q(m,n)))$$

Here, $\omega(m,n) = \frac{2\pi}{L}\tilde{v}_F(m+n)$ and $q(m,n) = \frac{2\pi}{L}(m-n)$. Comment on the possible similarities and differences compared to the formula for non-interacting systems given in question (19).

27) Let us assume that v is much smaller than $\tilde{v}_F$. Taking the thermodynamic limit in the previous formula, show that the average friction force is no longer linear in v, but that it obeys a power law behavior, whose exponent can be obtained by a simple scaling argument. We recall that at large m, c_m is proportional to m^{K-1}. What do you think of this result? What happens at very large repulsion? Does this sound reasonable to you? What would you suggest to improve the treatment?