ICFP Master Program
Condensed Matter Theory (B. Douçot, B. Estienne, L. Messio)
Monday January 9th 2017, 14:00 to 18:00.

You are allowed to use your lecture notes. There are many questions, but most of them involve only mimimal calculations, so don't be scared! The conventions for Fourier transforms are the same as in the lecture notes, namely:

$$\Psi_\eta(x) = \frac{1}{\sqrt{L}}\sum_k c_\eta(k)e^{ikx}, \qquad c_\eta(k) = \frac{1}{\sqrt{L}}\int_0^L dx\, e^{-ikx}\Psi_\eta(x)$$

and the fermionic fields $\Psi_\eta(x)$ are periodic and normalized as $\{\Psi^\dagger_\eta(x), \Psi_{\eta'}(x')\} = \delta_{\eta\eta'}\delta(x-x')$.

1 The model and its symmetries

The subject of the problem is the famous Gross-Neveu model. It has been studied a lot in the 1970's in the high energy physics community, as a theoretical laboratory to investigate non-perturbative phenomena in strongly interacting quantum field theories. From a solid-state physics standpoint, it appears as a rather natural generalization of the Luttinger model with spin, where the global symmetry group $SU(2)$ is replaced by $SU(N)$, N arbitrary positive integer. A lot of attention has been dedicated to the large N limit, for which a rather appealing physical picture has been proposed by E. Witten in 1978, using an approach based on path-integrals. The following problem aims at presenting this physics from the viewpoint and methods used in the course. With our favorite notations, the Gross-Neveu Hamiltonian reads:

$$H_{\mathrm{GN}} = \sum_k\sum_{\sigma=1}^{N} k(:c^\dagger_{R\sigma}(k)c_{R\sigma}(k): - :c^\dagger_{L\sigma}(k)c_{L\sigma}(k):) + Ng_1\int_0^L dx\, :\mathcal{O}(x)\mathcal{O}^\dagger(x): + \frac{g_2}{N}\int_0^L dx\, :n_R(x)n_L(x):$$

We use units in which $\hbar = 1$, $v_{\mathrm{F}=1}$. We consider periodic boundary conditions and we have set $k_{\mathrm{F}} = 0$ here. This is mostly to simplify the notation, without altering the physics. Indeed, we still have two Fermi points, both at $k = 0$, one for each sign of the group velocity. As usual, :: stands for normal-ordering, R for right-moving fermions and L for left-moving fermions. The generalized spin index σ runs now from 1 to N. The operator $\mathcal{O}(x)$ is defined by:

$$\mathcal{O}(x) = \frac{1}{N}\sum_{\sigma=1}^{N}\Psi^\dagger_{R\sigma}(x)\Psi_{L\sigma}(x)$$

The local charge densities are $n_R(x) = \sum_{\sigma=1}^N \Psi^\dagger_{R\sigma}(x)\Psi_{R\sigma}(x)$ and $n_L(x) = \sum_{\sigma=1}^N \Psi^\dagger_{L\sigma}(x)\Psi_{L\sigma}(x)$. The total charges are $N_R = \int_0^L dx\ :n_R(x):$, $N_L = \int_0^L dx\ :n_L(x):$. In the sequel, it will be often made use of $G = N_R - N_L$.

1) What kind of more physical model could motivate the Hamiltonian H_{GN} ?

2) What are the commutators $[G, \mathcal{O}(x)]$ and $[G, \mathcal{O}^\dagger(x)]$?

3) Check that G commutes with H_{GN}. The associated symmetry is often called *chiral symmetry* in the high energy physics literature.

4) It is interesting to see how this symmetry acts on basic operators. Let us introduce $U(\phi) = e^{i\frac{\phi}{2}G}$. Evaluate then $U(\phi)\Psi^\dagger_{R\sigma}(x)U(\phi)^{-1}$, $U(\phi)\Psi^\dagger_{L\sigma}(x)U(\phi)^{-1}$, $U(\phi)n_R(x)U(\phi)^{-1}$, $U(\phi)n_L(x)U(\phi)^{-1}$, $U(\phi)\mathcal{O}(x)U(\phi)^{-1}$, and $U(\phi)\mathcal{O}^\dagger(x)U(\phi)^{-1}$.

5) Another symmetry of $H_{\rm GN}$ is particle-hole symmetry. It is implemented by the linear operator $\mathcal{C}$ defined by:

$$\mathcal{C}^2 = 1,\ \ \mathcal{C}\Psi^\dagger_{R\sigma}(x)\mathcal{C} = \Psi_{R\sigma}(x),\ \ \mathcal{C}\Psi^\dagger_{L\sigma}(x)\mathcal{C} = -\Psi_{L\sigma}(x)$$

What are the operators $\mathcal{C}\mathcal{O}(x)\mathcal{C}$ and $\mathcal{C}\mathcal{O}^\dagger(x)\mathcal{C}$?

6) Finally, we will also need reflection symmetry $\mathcal{P}$. It is defined by:

$$\mathcal{P}^2 = 1,\ \ \mathcal{P}\Psi^\dagger_{R\sigma}(x)\mathcal{P} = \Psi^\dagger_{L\sigma}(-x),\ \ \mathcal{P}\Psi^\dagger_{L\sigma}(x)\mathcal{P} = \Psi^\dagger_{R\sigma}(-x)$$

What are the operators $\mathcal{P}\mathcal{O}(x)\mathcal{P}$ and $\mathcal{P}\mathcal{O}^\dagger(x)\mathcal{P}$?

2 Renormalization group analysis

It is possible to derive the renormalization group equations for the effective couplings g_1 and g_2. At the one loop approximation, these equations are:

$$\frac{dg_1}{d\log\Lambda} = \frac{g_1^2}{2\pi},\quad \frac{dg_2}{d\log\Lambda} = \frac{g_1^2}{2\pi N}$$

7) Draw the corresponding renormalization trajectories in the g_1, g_2 plane. What are the possible phases suggested by such diagram ?

8) What happens in the $N \to \infty$ limit ?

3 Mean-field approximation

The rest of the problem attempts to understand better the strong coupling phase whose existence is suggested by the previous renormalization group analysis. In this part, we take $g_2 = 0$ to simplify the discussion, and we write g for g_1 The mean-field approximation amounts, as usual, to replace the quartic operator $:\mathcal{O}\mathcal{O}^\dagger:$ by the combination $\left\langle\mathcal{O}^\dagger\right\rangle\mathcal{O} + \langle\mathcal{O}\rangle\,\mathcal{O}^\dagger$.

9) What could be the role of the parameter N in this problem ?

10) Introducing $\Delta = g\langle\mathcal{O}\rangle = |\Delta|e^{i\phi}$, check that the mean-field Hamiltonian H_{MF} is diagonalized in the quasiparticle basis whose associated creation operators are given by:

$$\begin{aligned} d^\dagger_{+\sigma}(k) &= \sin(\frac{\theta_k}{2})e^{-i\phi/2}c^\dagger_{R\sigma}(k) + \cos(\frac{\theta_k}{2})e^{i\phi/2}c^\dagger_{L\sigma}(k) \\ d^\dagger_{-\sigma}(k) &= \cos(\frac{\theta_k}{2})e^{-i\phi/2}c^\dagger_{R\sigma}(k) - \sin(\frac{\theta_k}{2})e^{i\phi/2}c^\dagger_{L\sigma}(k) \end{aligned}$$

Here, we have introduced the angle θ_k defined by:

$$\cos\theta_k = -\frac{k}{\sqrt{k^2+|\Delta|^2}},\quad \sin\theta_k = \frac{|\Delta|}{\sqrt{k^2+|\Delta|^2}}$$

11) What is the energy spectrum of H_{MF} ?

12) Write explicitely the self-consistency equation for the parameter Δ. Does the phase ϕ enter in this equation ? Explain why.

13) Building from the previous question, do you think that these mean-field solutions can be taken literally ? What could be a physical argument against them ?

14) For which values of g is there a non-trivial solution, i.e. with $\Delta \neq 0$? Evaluate $|\Delta|$ as a function of g. For this, it is necessary to introduce an ultra-violet cut-off on allowed momenta, i.e. to impose $|k| \leq \Lambda$. We assume that the coupling is not too large, so that $|\Delta| \ll \Lambda$.

15) To which extent can this simple mean-field picture be connected with the renormalization group flow of the previous section ?

4 Collective modes (I)

In this section, we shall consider the collective excitations of the system in the vicinity of the self-consistent mean-field ground-state with $\phi = 0$. This means that $\langle \mathcal{O} \rangle$ is real. In the sequel, it will be useful to distinguish between *amplitude* and *phase* fluctuations of the *order-parameter* $\langle \mathcal{O} \rangle$. For this, we define two hermitian components:

$$\mathcal{O}_a(x) = \frac{1}{2}(\mathcal{O}(x) + \mathcal{O}^\dagger(x)), \quad \mathcal{O}_b(x) = \frac{i}{2}(\mathcal{O}^\dagger(x) - \mathcal{O}(x))$$

The a-direction in order parameter plane is then associated to amplitude fluctuations and the b-direction to phase fluctuations. Note that the interaction term in H_{GN} now reads:

$$H_{\text{int}} = Ng \int_0^L dx \, \left(:\mathcal{O}_a(x)^2: + :\mathcal{O}_b(x)^2: \right)$$

The mean-field Hamiltonian, with these notations, becomes:

$$H_{\text{MF}} = H_0 + 2Ng \int_0^L dx \, \left(\langle \mathcal{O}_a(x) \rangle \mathcal{O}_a(x) + \langle \mathcal{O}_b(x) \rangle \mathcal{O}_b(x) \right)$$

We will study collective modes in the spirit of the *Random Phase Approximation* (RPA). For this, we impose to the system, initially in a mean-field ground-state of H_{GN}, an external perturbation:

$$\delta H(t) = \int_0^L dx \, \left(h_{\text{ext},a}(x,t)\mathcal{O}_a(x) + h_{\text{ext},b}(x,t)\mathcal{O}_b(x) \right)$$

The perturbation will modify the expectations values of $\mathcal{O}_a$ and $\mathcal{O}_b$. As usual in the RPA, we take interactions into account by a space and time dependent deformation of the self-consistent field acting on the underlying particles. The RPA assumes then that the fermions respond as if they followed the mean-field Hamiltonian H_{MF}, in the presence of local fields $h_{\text{loc},a}, h_{\text{loc},b}$, where:

$$\begin{aligned} h_{\text{loc},a} &= h_{\text{ext},a} + 2Ng\delta \langle \mathcal{O}_a(x) \rangle \\ h_{\text{loc},b} &= h_{\text{ext},b} + 2Ng\delta \langle \mathcal{O}_b(x) \rangle \end{aligned}$$

In order to relate $\delta \langle \mathcal{O}_a(x) \rangle$ and $\delta \langle \mathcal{O}_a(x) \rangle$ to local fields, we use response functions $R_{ij}(q,\omega) \equiv R_{\mathcal{O}_i,\mathcal{O}_j}(q,\omega)$, $(i,j = a \text{ or } b)$, evaluated in the ground-state of the *spinless* version of H_{MF}. This choice allows a simple tracking of the N-dependency of the order-parameter dynamics. In clear, we assume:

$$\delta \langle \mathcal{O}_i(x) \rangle (q,\omega) = \frac{1}{N} \sum_{j=a,b} R_{ij}(q,\omega) h_{\text{loc},j}(q,\omega)$$

16) Explain the $1/N$ factor in the previous equation.

17) Give general expressions for the local fields $h_{\text{loc},i}(q,\omega)$ and the order parameter fluctuations $\delta \langle \mathcal{O}_i(x) \rangle (q,\omega)$ in terms of the external fields $h_{\text{ext},j}(q,\omega)$. In particular, the latter relations will serve to define dressed response functions $R_{ij}^{\text{RPA}}(q,\omega)$ in the RPA.

18) What happens to $R_{ij}^{\text{RPA}}(q,\omega)$ in the $N \to \infty$ limit ? Does this sound reasonable ?

19) Now, a rather remarkable fact happens, explained in section 5 below, that $R_{ab}(q,\omega) = 0$. What are the resulting dressed response functions $R_{ij}^{\text{RPA}}(q,\omega)$?

20) Another very interesting phenomenon appears: there is a pole in $R_{bb}^{\text{RPA}}(q,\omega)$ at $(q,\omega) = (0,0)$. Show that this is not a coincidence and that the existence of this pole can be predicted.

5 Symmetries and response functions

This results of this part will be used in section 6 below. For a system with an unperturbed Hamiltonian H_0 and a time dependent perturbation $\delta H(t) = \lambda(t) B(t)$, the response function $R_{AB}(t-t')$ is defined by:

$$\delta \langle A \rangle (t) = \int dt' \, R_{AB}(t-t') \lambda(t')$$

where $\delta \langle A \rangle (t)$ denotes the change in the expectation value of the observable A at time t induced by the perturbation. We recall that if the system is initially in its ground-state $|\Psi_0\rangle$, we have:

$$R_{AB}(t-t') = -\frac{i}{\hbar} \langle \Psi_0 | [A(t), B(t')] | \Psi_0 \rangle \, \theta(t-t')$$

where $A(t)$ and $B(t')$ are in the interaction picture with respect to H_0. We suppose that there is a symmetry operation, described by the unitary operator U, which leaves both H_0 and $|\Psi_0\rangle$ invariant, i.e. $UH_0 = H_0 U$ and $U|\Psi_0\rangle = |\Psi_0\rangle$. We may also assume that A and B depend on a spatial coordinate, in which case we will write $A(x)$ and $B(x)$. Then, it is possible to expand these operators in Fourier modes:

$$A(x) = \sum_q e^{iqx} A_{-q}, \quad B(x) = \sum_q e^{iqx} B_{-q}$$

Here A_q and B_q carry momentum q. This means that the action of A_q or B_q on any state with momentum k produces a state with momentum $k+q$.

21) If we denote $A' = UAU^{-1}$, and $B' = UBU^{-1}$, then show that $R_{AB}(t-t') = R_{A'B'}(t-t')$.

22) Write down a spectral decomposition of the Fourier transform (with respect to both space and time) of $R_{A(x)A(0)}$.

23) We suppose that the unitary symmetry operator U satisfies $UA_qU^{-1} = \tau A_{-q}$, with $\tau = \pm 1$. Show that $R_{AA}(q,\omega)$ is even in q, that $\Im R_{AA}(q,\omega)$ is odd in ω and that $\Re R_{AA}(q,\omega)$ is even in ω.

6 Collective modes (II)

24) What are the operators $\mathcal{C}\mathcal{O}_a(x)\mathcal{C}$ and $\mathcal{C}\mathcal{O}_b(x)\mathcal{C}$?

25) Show that H_{MF} for $\phi = 0$ commutes with the particle-hole symmetry operator $\mathcal{C}$ and that its ground-state is invariant under $\mathcal{C}$.

26) What can we infer from this for $R_{ab}(q, \omega)$?

27) Show that H_{MF} for $\phi = 0$ commutes with the reflection symmetry operator $\mathcal{P}$ and that its ground-state is invariant under $\mathcal{P}$.

28) Use this to show that $R_{aa}(q, \omega)$ and $R_{bb}(q, \omega)$ are even functions of q.

29) Show also that $\Re R_{aa}(q, \omega)$ and $\Re R_{bb}(q, \omega)$ are even functions of ω.

30) Show that $\Im R_{aa}(q, \omega)$ and $\Im R_{bb}(q, \omega)$ are identically zero for ω not too large.

31) The previous remarks show that the Taylor expansion of $R_{aa}(q, \omega)$ and $R_{bb}(q, \omega)$ near $(q, \omega) = (0, 0)$ has then the form:

$$R_{ii}(q, \omega) = R_{ii}(0, 0) - \lambda_i \omega^2 + \mu_i q^2 + ..., \quad i = a, b$$

The coefficients $R_{ii}(0, 0), \lambda_i, \mu_i$ are real numbers. Show that $\lambda_i > 0$. We will assume that $\mu_i > 0$. Give some argument supporting this assumption.

32) Deduce from these informations the generic form of the dressed responses $R_{aa}^{\mathrm{RPA}}(q, \omega)$ and $R_{bb}^{\mathrm{RPA}}(q, \omega)$ near $(q, \omega) = (0, 0)$. You can use the fact that a complete calculation shows that $R_{bb}(0, 0) < R_{aa}(0, 0) < 0$.

33) We define the symmetrized correlation function $C_{ii}(x, t)_s$ by:

$$C_{ii}(x, t)_s \equiv \frac{1}{2} \left(\langle \mathcal{O}_i(x, t) \mathcal{O}_i(0, 0) \rangle + \langle \mathcal{O}_i(0, 0) \mathcal{O}_i(x, t) \rangle \right).$$

The fluctuation-dissipation relation at zero temperature states that the Fourier transform of this correlation function is related to the corresponding response function by:

$$C_{ii}(q, \omega)_s = -\Im R_{ii}(q, \omega) \mathrm{sign}(\omega).$$

Use it to estimate the qualitative behavior of the correlation functions $C_{ii}(x, t = 0)_s$. How these results precise the physical picture for the low energy dynamics in the large N limit ?

7 A glimpse of Bosonization

34) Show that the Gross-Neveu Hamiltonian can be expressed as a generalized Sine-Gordon model involving N coupled scalar bosonic fields ϕ_σ and their canonically conjugate fields Π_σ, $1 \leq \sigma \leq N$. Show also that the linear combination $\phi_c = \frac{1}{N} \sum_\sigma \phi_\sigma$ and its canonically conjugate $\Pi_c = \sum_\sigma \Pi_\sigma$ decouple and remain gapless.

35) How does this fit with the conclusion drawn from the RPA analysis ?

36) What could be the analogues, in the Sine-Gordon description, of the gapful fermionic quasi-particles present in the mean-field approximation ?

8 Closing remarks

It turns out that the Gross-Neveu model is exactly solvable by Bethe Ansatz! This was shown by Andrei and Lowenstein in 1980. They noticed soon afterwards that a rather simple modification of the model maps it into the Kondo model (when $N = 2$), which was then solved as well!

37) Can you guess the relation between the two models ?