III. QUANTUM GASES AND LIQUIDS: BOSE-EINSTEIN CONDENSATION, MEAN-FIELD THEORY

We will consider the very basic properties of dilute Bose gases in homogeneous and trapped systems. We remind the basic results of scattering theory, and argue that the interaction of quantum gases at low temperatures is well described by the first phase shifts, in many cases, via the s-wave scattering length only. There are various formalisms to account for that, here we use a simple zero-range pseudo-potential. The Hartree ansatz for the wavefunction leads directly to the Gross-Pitaevskii equation for the condensate at zero temperature. We will derive the finite-temperature mean-field equation and the Bose-Einstein transition in mean-field by using minimizing a variational upper bound for the free energy.

We will look at the problem of interacting Bosons, mainly with the aim to describe the experiments of Bose-Einstein condensation of alkaline atoms in trapped systems. Some time will be spend with the two-body problem. Many properties of Bose condensates can be quantitatively explained by a simple non-linear Schrödinger equation, the Gross-Pitaevskii equation. However, in the non-linearity enters the scattering length, and not the bare-potential. For analytical calculations, zero-range pseudo-potentials are quite useful.

A. Interacting Bosons: the two-body problem

In order to understand the N-body problem, we need some basic concepts and results from quantum scattering theory concerning simple two-body collisions. Quite generally, we can assume that the interaction potential between two particles, $V(r_1, r_2) = V(|r|)$, depends only on the relative coordinate, $r = r_1 - r_2$. We further assume, that the external potential also separates in relative, and center-of-mass coordinates, so that the complete Hamiltonian of the two particles writes

$$H_{12} = H_{cm} + \frac{\hbar^2}{2\mu} \Delta_r + V(|r|)$$

where $\mu = m/2$ and $H_{cm}$ depends only on the center-of-mass coordinates $r_{cm} = (r_1 + r_2)/2$, e.g. a homogeneous system, or a system in an external harmonic potential. The wavefunction therefore separates

$$\Psi(r_1, r_2) = \psi_{cm}(r_{cm})\phi(r)$$

and we will concentrate on $\varphi(r)$ in the following, which contains all collisional properties.

Using rotational symmetry, we introduce angular coordinates $(r, \theta, \varphi)$ and we separate the angular part of the wavefunction using the eigenfunctions of angular momentum

$$\phi(r) = \sum_{n,l,m} a_{nml} \frac{u_{nl}(r)}{r} Y_{lm}(\theta, \varphi)$$

where the spherical harmonics $Y_{lm}(\theta, \varphi) \propto P_l(\cos \theta)e^{im\varphi}$ can be expressed using the Legendre polynomials $P_l(x)$. The radial part satisfies a one-dimensional differential equation for each angular momentum $l$

$$-\frac{\hbar^2}{2\mu} \frac{d^2 u_{nl}(r)}{dr^2} + \left[\frac{l(l+1)\hbar^2}{2m\mu} + V(r)\right] u_{nl}(r) = E_{nl} u_{nl}(r)$$

The interatomic potential $V$ has a characteristic length scale $R_c$; for distances $r > R_c$ we can safely neglect the potential, $V(r > R_c) \equiv 0$. For vanishing potential corresponding to free particles, the solution of Eq. (4) are the spherical Bessel functions, $j_l(\kappa r)$ and $n_l(\kappa r)$, with $\kappa = \sqrt{2\mu E_{nl}/\hbar^2}$. The solution of Eq. (4) for $r < R_c$ has to be calculated for any specific potential $V(r)$ and can then be matched at $r = R_c$ to the free solution, such that the
wavefunction and its derivative are continuous functions at \( r = R \). The outside solution is a superposition of \( j_l \) and \( n_l \) which is conveniently written as

\[
\frac{u_l(kr)}{kr} = \frac{u_{nl}(r)}{kr} = 2 \left[ \cos \delta_l j_l(kr) - \sin \delta_l n_l(kr) \right]
\]  

Writing the superposition coefficient in terms of the phase shift \( \delta_l(k) \) is motivated by considering the large distance asymptotic of the Bessel functions

\[
j_l(x) \to \frac{\sin \left[ x - \frac{l \pi}{2} \right]}{x}, \quad x \to \infty
\]

\[
n_l(x) \to -\frac{\cos \left[ x - \frac{l \pi}{2} \right]}{x}, \quad x \to \infty
\]

so that we get

\[
\frac{u_l(kr)}{kr} \to 2 \frac{\sin \left[ kr - \frac{l \pi}{2} + \delta_l \right]}{kr} \approx 2j_l(kr + \delta_l), \quad kr \to \infty
\]  

We have chosen the normalization for the radial wavefunction such that

\[
\int_0^\infty dru_l(kr) u_l(kr) = 2\pi \delta(k - k')
\]  

in the continuum limit.

Let us denote \( u_{nl}^{(0)}(r) \) the free solution of Eq. (4) for vanishing potential

\[
-\frac{\hbar^2}{2\mu} \frac{d^2u_{nl}^{(0)}(r)}{dr^2} + \frac{l(l+1)\hbar^2}{2mr^2} u_{nl}^{(0)}(r) = E_n u_{nl}^{(0)}(r)
\]

\[
\frac{u_{nl}^{(0)}(kr)}{kr} = 2j_l(kr)
\]

We multiply Eq. (4) with \( u_{nl}^{(0)}(r) \) from the left and subtract Eq. (10) multiplied by \( u_{nl}(r) \). Integrating over \( r \) from 0 to \( R \) we obtain

\[
\frac{2\mu}{\hbar^2} \int_0^R dr u_{nl}^{(0)}(kr)V(r)u_{nl}(kr) = -u_{nl}(kR) \frac{du_{nl}(kr)}{dr} \bigg|_{r=R} + u_{nl}^{(0)}(kR) \frac{du_{nl}(kr)}{dr} \bigg|_{r=R}
\]

where we have integrated by parts and used that \( u_l \) and \( u_{nl}^{(0)} \) vanish at \( r = 0 \). For large \( R \) we can use the asymptotic expressions and obtain

\[
u_{nl}(kR) \frac{du_{nl}^{(0)}(kr)}{dr} \bigg|_{r=R} - u_{nl}^{(0)}(kR) \frac{du_{nl}(kr)}{dr} \bigg|_{r=R} = -4k \sin \left( kr - \frac{l \pi}{2} + \delta_l \right) \cos \left( kr - \frac{l \pi}{2} \right) + 4k \sin \left( kr - \frac{l \pi}{2} \right) \cos \left( kr - \frac{l \pi}{2} + \delta_l \right)
\]

\[
= -4k \sin \delta_l
\]

We obtain a quite useful result

\[
\int dr \frac{u_{nl}^{(0)}(rk)}{2kr} V(r) \frac{u_{nl}(rk)}{2kr} = -\frac{4\pi \hbar^2}{m} \frac{\sin \delta_l}{k}
\]

**Hard sphere scattering.** Note that for small arguments, the Bessel functions behave as

\[
j_l(x) \to \frac{x^l}{(2l+1)!!}, \quad x \to 0
\]

\[
n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}}, \quad x \to 0
\]
For hard spheres, the wavefunction has to vanish at \( r = a \) where \( a \) is the diameter of one sphere, or \( u_0(ka) \equiv 0 \). Using the asymptotic expansion (16), valid for low-energy scattering, in the expression (5), we obtain for the phase-shift

\[
\tan \delta_l \simeq -\frac{(ka)^{2l+1}}{(2l+1)!!(2l-1)!!}, \quad ka \to 0
\]  

(17)

In this low energy limit \( ka \to 0 \), the dominant contribution comes from \( l = 0 \), called s-wave scattering. Note that the s-wave phase shift \( \delta_0 = -ka \) is negative for the repulsive potential.

**Scattering length, alcaline atoms.** For low-energy scattering \( (k \to 0) \) one introduces the s-wave scattering length \( a_s \) for a general interaction potential writing

\[
\frac{k}{\tan \delta_0} = -\frac{1}{a_s} + \frac{1}{2} r_{eff} k^2
\]  

(18)

where \( r_{eff} \) is called effective range of the potential. The phase shift, and therefore also the scattering length can be measured. Let us consider alcaline atoms. The short range interactions at the range of the Bohr radius is dominated by the exchange energies of the overlapping electronic wavefunctions, and strongly repulsive. The long-range yail of the interaction can be described by a van-der-Waals interaction \( V(r) = -C_6/r^6 \) with characteristic distance \( R_e \equiv (4mC_6/h^2)^{1/4} \) with \( h/(2\mu R_e^2) = C_6/R_e^6 \). For alcalines \( R_e \) ranges from 20 Å for Li, 80 Å for Rb, up to 100 Å for Cs. For larger distances, the scattering properties are described by the phase shifts, corresponding to a scattering length of 55 Å for Rb, and -600 Å for Cs.

**Universality, pseudo-potentials.** If we consider properties where the distance between partiicles is large compared to the characteristic length scale of the interaction \( R_e \), we expect that we can express all properties by the phase shifts. The properties which depend only on the phase shifts are universal with respect to different interactions. Since we can obtain the same phase-shifts for different microscopic interactions, we are free to choose the one which is best suited for our calculations. These potentials are called pseudo-potentials.

A particular simple potential for analytical calculations is a delta-functional like potential

\[
V_\delta(r) = g\delta(r)
\]  

(19)

so that we can extend the free particle wavefunction to the limit of vanishing distances. However, inserting (19) into the relation Eq. (14), and using the outside potential wavefunctions, Eq. (4) and Eq. (11), we obtain a divergent term, since \( n_0(x \to 0) \to -1/x \). The simple delta-function potential can therefore not be used in general, however, we can eliminate the problem using a regularized delta-function potential

\[
\tilde{V}_\delta(r) = g\delta(r) \left[ \frac{d}{dr} r \right]
\]  

(20)

where the differential operator on the rhs eliminates the term proportional to \( 1/r \) in the outside wavefunction. We can read off the relation of the coupling constant \( g \) with the phaseshift from Eq. (14),

\[
g = -\frac{4\pi \hbar^2 \tan \delta_0}{km} = \frac{4\pi \hbar^2 a_s}{m}
\]  

(21)

where we inserted the scattering length expression, Eq. (18). Note that acting with the regularized pseudopotentials, Eq. (20) on functions which are regular at the origin, reduces to the action of a simple delta-function potential, Eq. (19).

**B. Ground state wavefunction for a Bose gas (\( T = 0 \)): variational calculation**

We are in the following considering the experiments on Bose-Einstein condensation with (metastable) alcaline atoms. Typically \( N \sim 10^7 \) atoms are confined in a harmonic trapping potential with typical peak densities \( n \lesssim 10^{15} \text{cm}^{-3} \). The typical interparticle distance \( n^{-1/3} \lesssim 10^3 \text{ Å} \) is large compared to the characteristic distance \( R_e \) of the van-der Waals interaction, and many low-temperature properties should be well universal in terms of the scattering length \( a_s \). Further, we also have \( na_s^3 \ll 1 \) characteristic for a dilute gas.

Since the system is dilute, on might consider an ideal gas Hartree wavefunction as a first starting point for the ground state wavefunction

\[
\Psi_0(R) = \prod_i \varphi(r_i)
\]  

(22)
and minimize the variational energy $\langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle$ to obtain the best single particle wavefunction $\varphi$. However, the real interparticle interaction between alkali atoms is not smooth at all, it is rather diverging at distances comparable to the Bohr radius. Therefore, the variational energy of an ideal gas wavefunction using the bare interatomic potential is not giving any reasonable result; certainly it is not a good starting point, we have to consider effects of the interactions already in the lowest order.

We can use a Jastrow wavefunction to take into account the correlations of the particles,

$$
\Psi_J = \prod_i \varphi(r_i) \prod_{j<n} f(r_{jn})
$$

(23)

For dilute systems $na_s^3 \ll 1$, we expect that the correlation function is essentially small ranged and given by the solution of the two-body problem, $f(r) \approx u(r)/(kr)$. This can be done, however, the non-orthogonality of the wavefunctions, essentially due to overlapping configurations, necessitates a big overhead for analytical calculations. In order to avoid the overlap, one can constrain $f$ to go rapidly to one within the mean interparticle distance and one can obtain analytical results.

Still, working with the bare interparticle potential might be dangerous. Actually the experiments on alkali gases are not in the ground state (which is a solid), but they are metastable. In general, the variational principle gives only upper bounds for the energy of the true ground state and might lead into serious troubles when applied to high energy metastable states.

In order to avoid bound states already in the two-body scattering problem, one conveniently introduces pseudo-potentials which lead to the same scattering properties but eliminate bound states. We will see that they also facilitate analytical calculations. Therefore we consider from now on the generic many-body Hamiltonian

$$
H = \int d\mathbf{r} \Psi^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2 \nabla^2}{2m} + u(r) - \mu \right] \Psi(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \Psi^\dagger(\mathbf{r}) \Psi^\dagger(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \Psi(\mathbf{r}') \Psi(\mathbf{r})
$$

(24)

using second quantized field operators. The external potential is $u(r)$, $\mu$ is the chemical potential which must be adjusted to obtain the right particle number, and $v$ is the interparticle pseudo-potential

$$
v(r) = g \delta(r) \frac{d}{dr} (r'), \quad g = \frac{4\pi \hbar^2 a}{m}
$$

(25)

and we denote the $s$-wave scattering length $a$.

Due to the introduction of the pseudo-potential, we can now try an uncorrelated Hartree wavefunction

$$
\Psi_0(\mathbf{R}) = \prod_i \varphi(r_i)
$$

(26)

to obtain an upper bound for the total energy

$$
E_0 \leq E[\varphi, \nabla \varphi] = N \int d\mathbf{r} \left[ \frac{\hbar^2 \nabla^2 \varphi}{2m} + |u(r) - \mu| |\varphi(r)|^2 + \frac{(N - 1)g}{2} |\varphi|^4 \right]
$$

(27)

We have used a partial integration, and vary the energy as a functional of $\varphi$ and $\nabla \varphi$ independently using the Euler-Lagrange equations

$$
\frac{\delta E}{\delta \varphi(\mathbf{r})} - \nabla \cdot \frac{\delta E}{\delta (\nabla \varphi)} = 0
$$

(28)

We obtain a non-linear Schrödinger equation

$$
\left( -\frac{\hbar^2 \nabla^2}{2m} + u(r) + Ng|\varphi(r)|^2 \right) \varphi(\mathbf{r}) = \mu \varphi(\mathbf{r})
$$

(29)

Since all particles are in the same single-particle state $\varphi(\mathbf{r})$, this single particle wavefunction describes a Bose-Einstein condensate and the non-linear Schrödinger equation is called Gross-Pitaevskii equation in this context. The condensate wavefunction must be normalized to one, and the chemical potential corresponds to the mean energy per particle in the system at zero temperature. The density is given by

$$
n(\mathbf{r}) = N|\varphi(\mathbf{r})|^2
$$

(30)
Ground state energy of the homogeneous system, healing length. The ground state wavefunction for a homogeneous system \( u(r) \equiv 0 \) must be constant and one immediately gets
\[
\mu = gn
\]  
(31)
The length scale, \( \xi \) associated with the interaction energy,
\[
\frac{\hbar^2}{2m\xi^2} = gn
\]  
(32)
or
\[
\xi = \sqrt{\frac{\hbar^2}{2mg_n}}
\]  
(33)
is called healing length.

**Thomas-Fermi approximation.** Let us consider a very smoothly varying external potential. We can regard the system as locally homogeneous and treat the system in the local density approximation. The kinetic energy contribution corresponding to the laplacian in the Gross-Pitaevskii equation, Eq. (29), is expected to be small compared to the interaction contribution. This allows us to express the chemical potential as a functional of the local density \( n(r) \)
\[
\mu[n(r)] = gn(r) + u(r)
\]  
(34)
However, for thermodynamic stability, the chemical potential must be constant in space, and we can determine the density distribution in an external trapping potential
\[
n(r) = \frac{1}{g} [\mu - u(r)] = n(0) - \frac{1}{g} u(r), \quad \mu - u(r) \geq 0
\]  
(35)
and
\[
n(r) \equiv 0, \quad \mu - u(r) < 0
\]  
(36)
From the normalization we can easily calculate the total number of atoms \( N = \int dr n(r) \) as a function of the chemical potential. Further, we can check, if the kinetic energy can be indeed neglected. Typically, the kinetic energy contribution changes the wavefunction at the edge and smoothes the distribution in order to obtain a density profile with continuous derivatives. The Thomas-Fermi distribution further breaks down for small systems.

**Harmonic trapping potential.** In most of the experiments, the atoms are trapped by a harmonic oscillator potential
\[
u(r) = \frac{1}{2} m\omega^2 r^2
\]  
(37)
with small (or even strong) asymmetric oscillator strengths. The typical length scale of the trapping potential is then given by the harmonic oscillator length, \( a_0 \),
\[
\frac{1}{2} m\omega^2 a_0^2 = \frac{\hbar^2}{2ma_0^2}
\]  
(38)
or
\[
a_0 = \sqrt{\frac{\hbar}{m\omega}}
\]  
(39)
The kinetic energy per particle is roughly \( \sim \hbar^2/(2ma_0^2) \sim \hbar\omega \), whereas the interaction energy is proportional to the density \( gn(r) \). For \( gn(r) \sim \mu \gg \hbar\omega \) or \( \xi \ll a_0 \), the Thomas-Fermi approximation is justified, and the density distribution has a typical parabolic shape. Whenever, the kinetic energy dominates, the wavefunction will be rather similar to the non-interacting gaussian, since the level-spacing of the trap is dominating.

**Collective excitations, vortices.** Stationary solutions of the Gross-Pitaevskii equation (29) with higher energies than the ground state correspond to collective excitations of the condensate. Solutions with non-zero angular momentum are called vortices. The centrifugal potential forces that the condensate wavefunction to vanish around the vortex core which is of size of the healing length \( \xi \).

**Stability, Bogoliubov excitations.** The stability of the condensate with respect to small variations in the wavefunctions will lead to the Bogoliubov excitation spectrum in linear response. These excitations remain occupied even at zero temperature and lead to a depletion of the condensate density of order of \( (na^3)^{1/2} \).
C. Bose-Einstein condensation

1. Ideal Gas $g = 0$

For an ideal Bose gas the total energy of the system is given by a summation of the occupation number $n_i$ of each energy eigenstate of energy $\varepsilon_i$

$$E(\{n_i\}) = \sum_i n_i \varepsilon_i$$  \hspace{1cm} (40)

and the partition function at temperature $T = 1/\beta$ in the canonical ensemble writes

$$Z_N = \sum_{\{n_i\}} e^{-\beta E(\{n_i\})}$$  \hspace{1cm} (41)

where the summation over all occupation numbers is done with the constraint that they sum up to the total number of Bosons $N$,

$$\sum_{\{n_i\}} = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \ldots \delta \sum_{n_i=N}$$  \hspace{1cm} (42)

Going over to the grand canonical ensemble, we can avoid this constraint introducing the chemical potential $\mu$ and we get

$$Z = \sum_N e^{\beta \mu N} Z_N$$  \hspace{1cm} (43)

$$= \sum_{n_1=1}^{\infty} e^{-\beta n_1 (\varepsilon_1 - \mu)} \sum_{n_2=1}^{\infty} e^{-\beta n_2 (\varepsilon_2 - \mu)} \ldots$$  \hspace{1cm} (44)

$$= \prod_i \left[ \frac{1}{1 - e^{-\beta (\varepsilon_i - \mu)}} \right]$$  \hspace{1cm} (45)

We get the mean occupation number of mode $i$ by varying $\varepsilon_i$, e.g.,

$$N_i \equiv \langle n_i \rangle = -\frac{\delta \log Z}{\delta \varepsilon_i}$$  \hspace{1cm} (46)

$$= \frac{1}{e^{\beta (\varepsilon_i - \mu)} - 1}$$  \hspace{1cm} (47)

which is the familiar Bose-Einstein distribution.

**Bose Einstein condensation.** The number of particles $N = \sum_i \langle n_i \rangle$ is an increasing function for increasing (negative) chemical potential. However, at $\mu = \varepsilon_0 = 0$ a singularity appears, where

$$N_0 \simeq \frac{1}{\beta |\mu|}, \quad \text{for } \mu \to 0$$  \hspace{1cm} (48)

**Homogenous system.** For a homogeneous system, where the energy eigenstates are plane waves with energies $\varepsilon_k = \hbar^2 k^2 / 2m$, and the density can be expressed by

$$n = \frac{1}{V} \sum_k N_k = \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta (\hbar^2 k^2 / 2m - \mu)} - 1} = \left( \frac{2m}{\hbar^2 \beta} \right)^{3/2} \frac{1}{8\pi^3} \int d^3x \sum_{n=1}^{\infty} e^{-n(x^2 - \beta \mu)} = \lambda^{-3} \sum_n \frac{e^{n\beta \mu}}{n^{3/2}}$$  \hspace{1cm} (49)

where we have introduced the thermal wavelength $\lambda = \sqrt{2\pi \hbar^2 / (mT)}$. Defining

$$g_z(x) = \sum_n n^{-2} x^n$$  \hspace{1cm} (50)
we have
\[ n\lambda^3 = g_{3/2}(e^{\beta\mu}) \]  
(51)

Einstein condensation occurs at a finite density/temperature in three dimensions, since the density at \( \mu = 0 \) is finite
\[ n_c\lambda^3 \simeq 2.61 \ldots \]  
(52)

This is the critical density of Bose Einstein condensation. Note that we approached \( \mu \to 0 \) such that the ground state occupation density \( n_0 = N_0/V \) remains zero.

Below the critical temperature (or above the critical density),
\[ \beta\mu \sim -\frac{1}{V} \]  
(53)
and the ground state density \( n_0 \sim 1 \). However, whereas the condensate is macroscopically occupied, the excited state occupation remains of order one.

2. Dilute gas: \( g > 0 \), mean-field

At zero temperature, the variational principle could be applied to find the lowest energy state and we obtained the Gross-Pitaevskii equation. At finite temperature we can use a similar variational principle, namely any trial density matrix, \( \rho_0 \), will lead to an upper bound of the free energy, \( F = E - TS = -T \log Z \) of the system.

We will write the normalized trial density matrix using a trial Hamiltonian, \( H_0 \),
\[ \rho_0 = e^{-\beta H_0} Z_0 \]  
(54)
The mean energy using this trial density matrix writes \( \langle H \rangle_0 \) whereas \( -\langle \log \rho_0 \rangle_0 \) is the entropy \( \langle \ldots \rangle_0 \) denotes the quantum mechanical average using \( \rho_0 \). The total free energy is then bounded by
\[ F \leq T\langle \log \rho_0 \rangle_0 + \langle H \rangle_0 - T \log Z_0 + \langle H - H_0 \rangle_0 \]  
(55)
where we can easily identify \( F_0 = -T \log Z_0 \) as the free energy of a system with Hamiltonian \( H_0 \).

**Thermal cloud** \( T > T_c \), homogeneous. Let us consider that \( H_0 \) is essentially given by the non-interacting system
\[ H_0 = \sum \frac{\hbar^2 k^2}{2m} + \xi - \mu \]  
(56)
where \( a_k \) is the annihilation operator of state \( k \), and \( \xi \) is our variational parameter, a global shift of all energies, or equivalently, an effective change of the chemical potential. The corresponding free energy, \( F_0 \), is almost the same as the ideal system calculated above. Since
\[ \langle \Psi^\dagger(r)\Psi^\dagger(r)\Psi(r)\Psi(r) \rangle_0 = 2n(r) = 2n \]  
(57)
we have \( \langle H - H_0 \rangle_0 = \frac{\sqrt{g}}{T} 2[n(\xi)]^2 - \xi V n(\xi) \), with
\[ n(\xi) = g_{3/2}(e^{\beta(\mu - \xi)}) \]  
(58)
so that we have
\[ F \leq F(\xi) \equiv F_0(\xi) + V g[n(\xi)]^2 - \xi V n(\xi) \]  
(59)
Since
\[ \frac{\partial F_0(\xi)}{\partial \xi} = -\frac{\partial F_0(\xi)}{\partial \mu} = V n(\xi) \]  
(60)
we can easily minimize the right hand side of Eq. (59)
\[ \frac{\partial \tilde{F}(\xi)}{\partial \xi} = V n(\xi) + 2V g n(\xi) n'(\xi) - V n(\xi) - \xi V n'(\xi) = [gn(\xi) - \xi]V n'(\xi) \]  
(61)
We see that the (true) minimum is reached for $\xi = 2gn(\xi)$. This leads to the mean-field (Hartree-Fock) equations which have to be solved self-consistently

$$n_{mf} = g_{3/2} \left( e^{\beta(\mu - 2gn_{mf})} \right)$$

(62)

since the rhs depends on $n_{mf}$. In the mean-field approach, Bose-Einstein condensation occurs for $\mu = 2gn_{mf}$. However, since the mean-field just leads to a shift of all energy levels, the critical density/temperature is just the same as for the ideal gas

$$n_{c,mf} \lambda^3 = 2.61...$$

(63)

**Mean-field condensed system.** For a system below the Bose-Einstein transition, we have to split of the finite condensate density, $n_0$. We will use $n_0$ as a parameter in favour of the chemical potential $\mu$. We describe the excited state by a thermal trial density matrix with parameter $\xi$ as above. However, since $n_0$ is treated as a parameter, $H - H_0$ contains all terms of $H$ which involve $n_0$. We first minimize with respect to $\xi$, where we get

$$\xi = 2g[n(\xi) + n_0]$$

(64)

Mimizining the residuals with respect to $n_0$ we can eliminate $\mu$ via

$$\mu = gn_0$$

(65)

This constitutes a simple set of mean-field equations to describe the condensed phase.