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A comparative introduction to the renormalization methods used in statistical mechanics and for dynamical systems

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Introduction

My aim in this talk is to show the very close analogy that exists between:

— *temporal renormalization* used in the study of chaotic dynamical systems and

— *spatial renormalization* previously developed in statistical mechanics, implemented in the framework of field theory.

The comparison leads to a better understanding not only of the fundamental mathematical ideas, but also of the common physical features that lie under all the various applications of renormalization techniques[21].

The key of the analogy is to exchange the spatial extension (and the associated variable $\vec{r}$) of the physical systems described in statistical mechanics and the time length (and the associated time variable t) of the evolution of the dynamical systems.

This points out that the nature of the extensive variable(s) does not matter very much; as a consequence, a direct extension of both "statistical mechanics renormalization" and "dynamical systems renormalization" will be *spatio-temporal renormalization methods*, where space and time have the same status. This leads to a powerful tool of analysis for numerous challenging problems, for example dynamical phase transitions, chaos in extended systems, stochastic partial differential equations or diffusion in disordered media.

Part A - Renormalization in statistical mechanics

Renormalization techniques were first introduced in quantum electrodynamics[2,4] in order to get rid of ultra-violet divergences and to handle singular perturbative expansions. It was only more than a decade later, mainly after Wilson's works[30], that their relevance in statistical mechanics became obvious. Wilson's contribution was to put together the renormalization-group methods of *quantum electrodynamics* and the concept of *scale invariance* pointed out by Kadanoff[20], among others, then to develop from their conjunction a rigorous and nevertheless constructive scheme for the study of the so-called critical phenomena encountered in statistical mechanics (§A.2). He also applied this new operational framework in quantum field theory, and obtained striking advances which led him to the Nobel Prize in 1982.

In this first section, I present the renormalization method designed for the description of phase transitions in the framework of statistical mechanics. All the usual approaches fail close to critical points. I explain why renormalization turns out to be successful, beyond hope since it proves not only the existence but also the universality of scaling behavior, then gives explicit values for the critical exponents. It thus achieves

to classify critical phenomena according to the universal properties they share. Understanding how renormalization works for critical phenomena shows up the fundamental physical and statistical mechanisms which give rise to critical behavior[23,29], which in turn clarifies the conditions for deriving macroscopic deterministic behavior from small scale probabilistic description.

A.1 - Thermodynamic formalism at a fixed temperature

The typical problem to be solved in the statistical mechanics framework[24,26] is to describe the *macroscopic behavior* of a system $\mathcal{S}$ composed of a large number $N \gg 1$ of interacting particles, possibly submitted to an applied constant field B (magnetic field, pressure...). I suppose that $\mathcal{S}$ is *in equilibrium* with a surrounding thermal bath at a given temperature T. Equilibrium means that no change can be perceived when observing $\mathcal{S}$ at *macroscopic* spatial and temporal scales. Knowing composition and dynamics of $\mathcal{S}$ at a microscopic scale, one has to compute the value of all the physical quantities that could be measured in a macroscopic observation, and to describe how they depend upon the control parameters B and T, denoted K in short.

A.1.1 - Ensemble averages

I call *microscopic* the length scale a at which the physical system $\mathcal{S}$ is defined and modelled. It is the smallest scale of the description and it is supposed to be large enough so that a *semi-classical (non-quantum)* description of $\mathcal{S}$ is sufficient. At this given scale, the system $\mathcal{S}$ appears as a population of N interacting particles, N being very large. The name *particle* must be understood in the weak sense of *elementary sub-system of (linear) size a*. By definition of a the finer structure of the particle is not described; its consequences at scales larger than a must be taken into account through *effective terms*.

Each particle is described by a few *individual state variables*, say k, that I shall represent by a k-vector $z \in \mathcal{E}_{el}$; the k components of z are in most cases the position $\vec{x}$ of the particle in the accessible domain of the real physical space $\mathbf{R}^d$, its momentum $\vec{p}$, also having d components, and a local order parameter, for example its spin $\vec{s}$, having n components. The two dimensions d and n will play a major role when classifying critical phenomena[14]. A (Nk)-vector $[z] \equiv (z_\alpha)_{1 \leq \alpha \leq N} \in \mathcal{E}_{el}^{\otimes N}$ is called a *configuration* of $\mathcal{S}$; it thoroughly describes its microscopic state at a given time. Thus the relevant *phase space* of $\mathcal{S}$ at the microscopic scale is the set $\mathcal{E}_N \in \mathcal{E}_{el}^{\otimes N}$ of all the possible configurations when taking into account the macroscopic constraints on $\mathcal{S}$ (geometry, boundary conditions, physical invariants, symmetries...). In the case of particles fixed at the sites of a given lattice, only the spins are relevant (hence $k = n$) since positions are known (used as indices for labelling the particles) and momenta are all equal to 0. The basic ingredient to be specified in modeling $\mathcal{S}$ is its *Hamiltonian*, namely the (classical) state function $H(N, [z])$ giving the energy of any configuration $[z]$ of N particles.

Although one can then write exactly the deterministic Hamiltonian laws of evolution, the study of their solution in unworkable, even for a computer, due to the too large value of the dimension Nk of the phase space $\mathcal{E}_N$ (a typical value for N, the Avogadro number, is about $6.\,10^{23}$). Only a *statistical approach* will be possible. The relevant microscopic information on $\mathcal{S}$ is contained in the statistical weights of the configurations $[z]$ belonging to its phase space $\mathcal{E}_N$. For a classical system $\mathcal{S}$ in thermal equilibrium, these weights are given by the Boltzmann-Gibbs distribution at T. Equilibrium implies *statistical stationarity* so that time no more intervene.

A.1.2 - Partition function and state functions

Denoting $\beta = 1/k_{_B}T$ the inverse temperature[1], the Boltzmann-Gibbs distribution of $\mathcal{S}$ is:

$$p\,(N,\beta,[z]) = \frac{1}{Z(N,\beta)}\;\; e^{-\beta H(N,[z])} \qquad\qquad (\,[z]\in\mathcal{E}_N\,)$$

where the normalization factor:

$$Z\,(N,\beta) = \sum_{[z]\in\mathcal{E}_N} e^{-\beta H(N,[z])}$$

is called the *partition function* of the system $\mathcal{S}$; it also depends on any parameter, for example B, which appears in H. Knowing the probabilities $p\,(N,\beta,[z])$, it is possible to compute the statistical mean of any state function[2]. Most of the averages can be directly deduced from the knowledge of the partition function:

— the *free energy* is given by $F(N,\beta) = -\,\frac{\ln Z(N,\beta)}{\beta}$.

— the *internal energy* is given by $U(N,\beta) = <H> = -\,\frac{\partial \ln Z(N,\beta)}{\partial\beta}$.

— the *entropy* is given by $S(N,\beta) = \frac{U-F}{T} = -\,k_{_B}\,\beta^2\,\frac{\partial}{\partial\beta}\left(\frac{\ln Z(N,\beta)}{\beta}\right)$.

The *thermodynamic limit* [27] is defined as the limiting operation, performed on the previous statistical averages after division by N in order to obtain intensive (rather than extensive) quantities:

$$\begin{cases} N\to\infty \\ V/N = \text{Cte} \end{cases}$$

where V is the accessible volume in the real space $\mathbf{R}^d$. The resulting quantities are expected to give the deterministic values of the observed macroscopic quantities (per particle), as described in classical thermodynamics. When it is the case, statistical mechanics provide microscopic foundations for the macroscopic and deterministic thermodynamics, allowing to relate macroscopic measurements and microscopic parameters.

[1] Here $k_{_B}$ is the well-known Boltzmann constant.

[2] A *state function* is associated to any physical characteristic of $\mathcal{S}$ which is thoroughly determined when knowing the configuration $[z]\in\mathcal{E}_N$ of $\mathcal{S}$; it is thus a real function defined on $\mathcal{E}_N$.

A.1.3 - The conjugate space

The so-called *conjugate space* is a counterpart, obtained through a Fourier transform, of the space $\mathbf{R}^d$ of positions $\bar{x}$ known as the *real space*. The local order parameter $\bar{s}_j$ can be viewed as a n-vector function of the position $\bar{x}_j$ of the j-th particle. But due to the finite minimum scale $a > 0$, the position $\bar{x}$ takes its values in the discrete space $(a\mathbf{Z})^d$, so that $\bar{s}(\bar{x})$ does not define a continuous field. Intending to take advantage of the *field theory framework*[33], one computes the *Fourier transform* of the discrete field $\bar{s}(\bar{x})$:

$$\hat{s}(\bar{q}) = a^d \sum_{\bar{m} \in \mathbf{Z}^d} e^{-ia\bar{m}.\bar{q}}\ \bar{s}(a\bar{m})$$

which is a n-vector complex field on $\mathbf{R}^d$. Inverse transform writes:

$$\bar{s}(a\bar{m}) = \int_0^{\frac{2\pi}{a}} \dots \int_0^{\frac{2\pi}{a}} e^{+ia\bar{m}.\bar{q}}\ \hat{s}(\bar{q})\ \frac{d^d\bar{q}}{(2\pi)^d}$$

The transformed field $\hat{s}(\bar{q})$ defined in the conjugate space has the desired regularity property[3] without any continuous limit $a \to 0$, even if $\bar{s}$ takes only discrete values. Whereas the physical meaning of the contributions in H are better understood in the real space, H is often directly constructed in the conjugate space when the whole subsequent analysis is performed there: it is given as a functional of the field $\hat{s}(\bar{q})$, typically involving an integration over a local expression.

Let us stress that the counterpart of a minimum scale $||\Delta\bar{x}|| = a > 0$ in the real space is a *finite cutoff* $\Lambda = 2\pi/a$ in the conjugate space. This intrinsic bound, imposed with the choice of a theoretical model, prevents from any ultra-violet divergence.

We shall see that renormalization, originally conceived and designed in the real space, is more often implemented in the conjugate space, where the analogy with quantum field theory is obvious (at least for specialists!) and fruitful.

A.2 - Critical points

A.2.1 - Critical behavior

Let $\mathcal{S}$ be a N-particle system observed at a macroscopic scale in the thermodynamic limit $N \to \infty$ and for varying control parameter(s) K. From a physical (experimental) viewpoint, $\mathcal{S}$ is said[4] to have a "*critical behavior*" if it exhibits the following features when K reaches some special value K_c:

— large scale inhomogeneities are observed;

— statistical fluctuations cannot be ignored;

[3] Indeed, the real physical quantity locally described by $\bar{s}$ has a finite spatial extension; even in an "ideal" model of infinite spatial extension (as it is the case in the thermodynamic limit), one may thus impose that $\bar{s}(\bar{x})$ is rapidly decreasing at infinity, so that $\hat{s}(\bar{q})$ is differentiable at any order.

[4] One also speaks of "critical point" or "critical phenomena" in such a situation; K_c is called a "critical value" and $\mathcal{S}(K_c)$ a "critical system".

— localized perturbations generate long distance disturbances during a long time;

— some thermodynamic quantities (or their derivatives) diverge in K_c.

Divergence[5] of the thermodynamic quantity $x(K) = \lim_{N\to\infty} X(N,K)/N$ for some special value K_c of K expresses that there are strong non-additive contributions in $X(N,K)$; their origin lies in correlation effects at all scales. As a consequence, the system cannot be split into uncorrelated cells of finite size. Such a situation is a characteristic feature of a *critical phenomenon*. It indicates that a major critical feature is the *divergence of the correlation length.*

◇ *Divergence of correlation length and its consequences*

The statistical properties of a critical phenomenon cannot be explained in the framework of classical thermodynamics. Hence, a more theoretical definition of criticality is the following:

A system $\mathcal{S}$ will be said to be "critical" if the predictions of classical thermodynamics fail to reproduce its observed properties.

Closely related to the study of critical phenomena, a key issue in statistical mechanics is thus to ask whether it is correct to describe the macroscopic behavior of $\mathcal{S}$ in terms of computable statistical averages taken in the thermodynamic limit $N \to \infty$. The answer is positive as soon as the *strong law of large number* applies to the sequence $(Z_j)_{1\leq j\leq N}$ of elementary random variables globally distributed in $\mathcal{E}_N$ according to the Boltzmann-Gibbs distribution (Z_j describing the random state of the particle j). Indeed, this law allows to *identify almost surely* any observable quantity, which corresponds to a realization of the average $F_N = N^{-1}\sum_{j=1}^{N} f(Z_j)$ over the N particles (f being any local state function), with its statistical average $< F_N >$ in the thermodynamic limit $N \to \infty$. It can be shown that the statistical feature that invalidates the strong law of large numbers (hence characterizes a critical system) is the *divergence of the correlation length*: $\xi(K_c) = \infty$. In that case, the macroscopic quantities are still random variables, so that a low-dimensional deterministic description of $\mathcal{S}$ makes no sense. This assertion is expressed more quantitatively by introducing the relative statistical fluctuation $\delta F_N \equiv F_N - < F_N >$; according to the *central limit theorem*, δF_N is almost surely a correction of order $\sqrt{\xi_K^d/a^d N}$, obeying a *Gaussian statistics* of variance $\xi(K)^d/a^d N$ (d being the dimension of the underlying real space). Hence, the mean amplitude of the fluctuations becomes macroscopic when the correlation length ξ_K diverges. Moreover, $\xi(K)$ is shown[21] to give the *spatial extension of fluctuations*, that is, the typical size of the connected domains where statistical fluctuations exceed a given threshold. The typical size an_K above for which it is possible to identify spatial (that is, over the n_K particles) and statistical averages behaves as $\xi(K)$, hence tends to infinity when K tend to K_c.

It thus becomes obvious that criticality appears in K_c if and only if $\xi(K)$ tends to infinity when K reaches K_c; indeed, if $\xi(K_c) = \infty$, *it is no more possible to ignore*

[5] Let us note that the divergence of $x(K) = \lim_{N\to\infty} X(N,K)/N$ in K_c appears only in the thermodynamic limit $N \to \infty$. For finite N, $X(N,K)/N$ is an everywhere regular function of K.

statistical fluctuations, since both their amplitude (at a fixed position) and their spatial extension (above a given level) diverge with $\xi(K)$. Hence an observable feature of critical systems is the occurrence of giant fluctuations that can be directly perceived at a macroscopic scale (in fact as soon as $\xi(K)$ is greater than the minimum scale of observation). Dynamic study shows that moreover the time life of these fluctuations diverges at a critical point so that they are effectively perceived.

It can also be shown[21] that ξ gives an estimate of the "order scale", that is, of the *spatial extension of collective effects*, for exemple of domains in which all the particles are organized so that they behave in a totally correlated manner. Hence criticality is strongly related to the presence of collective behaviors, inducing *organization, hence structures, at all scales*. Moreover, it can be shown that $\xi = \infty$ jointly occurs with self-similar and scaling properties of the overall behavior, which ends to deduce the whole critical features from the divergence of the correlation length.

◇ *Critical phase transitions*

Typical critical features are illustrated by the so-called *"critical" phase transitions*. Let us first recall that a *phase transition* is an exchange of stability between two states of thermodynamic equilibrium of $\mathcal{S}$, observed at a macroscopic scale when varying some control parameter(s) K, typically the temperature T or the pressure P. These two states, also called *phases*, differ through the expression of the massic thermodynamic quantities describing how their observable properties vary when K is varied. An *order parameter* $M(K)$ is chosen: it can be any thermodynamic quantity which reveals the transition and is greater in the most organized phase (called the *ordered phase*) than in the other one, in which it can be identically zero, for example. A transition occurring in $K = K^*$ is said to be of *first order* if the coexistence of two different phases can be observed for $K = K^*$; this implies that some massic thermodynamic quantities, among which the order parameter, the massic volume and the massic entropy, present a jump discontinuity in $K = K^*$.

On the contrary, the transition is said to be of *second order* if the order parameter and the other observable thermodynamic state functions are continuous in $K = K^*$, but with diverging derivatives. This continuity of the thermodynamic properties implies that it is no more possible to observe the coexistence of two differentiated phases in K^*. The divergence of some derivatives causes singularities in the observable behavior, for example on the heat capacity, which are typical of a critical phenomenon. Moreover, it can be shown that these divergences are the macroscopic observable consequences of statistical critical properties, mainly giant fluctuations and diverging correlation length. In fact, all the critical features listed above can be identified in such "critical phase transitions".

A.2.2 - Failure of conventional methods

Let me point out the main difficulties arising in the analytical study of critical phenomena, which prevent from applying standard approaches used for any non-critical

system.

• *Standard statistical mechanics fail* close to a critical point, since it is impossible to identify macroscopic observations with statistical averages.

• The correlation length ξ gives the order of the variance and of the size of microscopic inhomogeneities so that its divergence forbids to treat the *statistical fluctuations perturbatively*, starting from a macroscopic description using only statistical averages.

• A typical failure in studying critical phenomena is in the *mean field theories*, in which the interactions between the elementary constituents are replaced by an effective homogeneous exterior influence depending only on the statistical properties of the system.

• To be independent, samples of the system must have a size of order ξ: divergence of ξ makes any *local analysis* (on a sample) impossible in the real space.

• Correlations at all scales also forbid any *local analysis in the conjugate space*, as it is impossible to decorrelate ultra-violet and infra-red domains. *All the methods based on scale separation fail* close to a critical point.

• A difficulty arises when handling singular quantities: it is impossible to determine the state functions in the form of *analytic expansions* in $K - K_c$

• The problem of critical divergences becomes crucial in *perturbative methods* with respect to some additional contribution V in H. They involve typically expansions of the form $\sum_j c_j(K)\, V^j$; the successive terms become very large as the critical point K_c is approached, and the convergence of the expansion becomes slower and slower; it is necessary to take more and more terms into account (at K_c, all of them must be taken into account) in order to accurately estimate the quantity described by the expansion.

• A similar difficulty arise when trying to *invert the thermodynamic limit $N \to \infty$ and the "critical" limit $K \to K_c$*. We observe $L_1 = \lim_{K\to K_c} \lim_{N\to\infty}$ but we know how to compute $L_2 = \lim_{N\to\infty} \lim_{K\to K_c}$. Due to critical singularities, L_1 and L_2 are different and $\lim_{N\to\infty}$ cannot be exchanged with $\lim_{K\to K_c}$ nor with differentiation in K.

• As a consequence, implementing at first a perturbative approach in power of $(K-K_c)$ in order to compute macroscopic state functions for finite N and close to a critical point, hence letting N tends to infinity, does not give a correct description of the thermodynamic behavior around K_c.

A.2.3 - Scaling behavior

The main question relative to critical systems is to explain *how short-range (spatial or temporal) physical couplings generate a phenomenon perceptible on a large scale*. The answer lies in the existence of a collective behavior organized hierarchically from the microscopic up through the observation scales. Moreover, the very existence of this organization at all scales implies that the macroscopic observations may be sensitive

to a modification of microscopic features: it is no more the values of the physical quantities which are relevant, but the qualitative features of their dependance with respect to the size N of the system and to the control parameters, denoted K. The analysis should deal not with the microscopic details but with the way in which they cooperate; it should be detached from the specific details of the system, in order to give universal results which should be the same for any system for which some set of generic hypotheses is satisfied. The different scales must be related each to each other in such a way as to make the expected scale invariance explicit; these relations take generically the form of *scaling laws*, for example (here $K = (K_1, K_2)$):

$$X(K_1, K_2) \sim |K_1 - K_{1c}|^{\gamma}\, f(|K_1 - K_{1c}|\, |K_2 - K_{2c}|^{-\alpha}) \qquad \text{with} \quad \alpha > 0$$

where f is a continuous function which behaves like $f(z) \sim A\ |z|^{\rho}$ at infinity. It expresses *in a whol*e the singular behavior of the thermodynamic function X when the parameters K reach the critical point K_c and the associated *scaling invariance* of X:

$$X(K_1, K_2) \sim |K_1 - K_{1c}|^{\gamma}\, f(0) \quad \text{if} \quad K_1 \to K_{1c}, \quad K_2 \text{ being fixed}$$

$$X(K_1, K_2) \sim A\, |K_2 - K_{2c}|^{-\alpha\rho}\, |K_1 - K_{1c}|^{\gamma+\rho} \quad \text{if} \quad K_2 \to K_{2c}, \quad K_1 \text{ being fixed}$$

$$X(K_{1c} + \lambda^{\alpha} k_1, K_{2c} + \lambda k_2) \sim \lambda^{\alpha\gamma}\, X(K_{1c} + k_1, K_{2c} + k_2) \quad \text{for any } \lambda > 0$$

γ, α and ρ are called "*critical exponents*"; they describe *quantitatively* the scaling behavior of the state function X in the vicinity of the critical point.

An important example is given by the statistical pair-correlation function, which is expected to behave according to:

$$\Gamma_{\infty}(K, \bar{x}, \bar{y}) \sim e^{-||\bar{x} - \bar{y}||/\xi(K)} \qquad (K \neq K_c)$$

in the thermodynamic limit $N \to \infty$, except at the critical point K_c where the exponential dependence is replaced by a power law:

$$\Gamma_{\infty}(K_c, \bar{x}, \bar{y}) \sim ||\bar{x} - \bar{y}||^{-\gamma}$$

The characteristic length $\xi(K)$ defines the statistical correlation length and the critical behavior of Γ_{∞} indicates that $\xi(K_c) = \infty$; experimental results suggest that:

$$\xi(K) \sim |K - K_c|^{-\nu}$$

Nevertheless, as the importance of the notion of critical phenomena dawned, the necessity of new tools for describing them correctly became obvious. We shall see that the renormalization methods are the most efficient among these[6].

[6] Before the emergence of renormalization methods, analytical description of critical phenomena was based on *scaling theory*, which amounts to "guess" phenomenological relations describing the scaling properties around a critical point, with experimentally fitted exponents. In the best cases, physical consistency of the theory together with dimensional analysis give rise to new relations which either determine new critical exponents, either establish universal theoretical relations between the experimental exponents (such a property is called *hyperscaling*).

A.3 - Renormalization approaches

The discussion presented in the previous section (§A.2) showed the lack of analytical methods for the investigation of critical systems. Renormalization is designed to make up with the difficulties which arise, due to critical behavior, in the thermodynamic analysis. It has to provide rigorous results about the macroscopic state functions observed in the thermodynamic limit when the system $\mathcal{S}$ is close to a critical point ($N \to \infty$, $K \to K_c$), focusing on their dependence with respect to the parameter(s) K. More specifically, the relevant program of a renormalization method is:

— to prove the existence of scaling laws describing the critical behavior,

— to compute the value of critical exponents,

— to provide expressions for the associated scaling functions,

— to single out the physical parameters that influence these exponents and these functions; it amounts to investigate their universal nature and (simultaneously, in general) to determine the associated universality classes,

— to approximate corrective terms to the leading dependence in $K - K_c$ (for example logarithmic corrections) in order to estimate deviations from the scaling behavior,

— to describe finite-size scaling (for $N < \infty$) where both N and $K - K_c$ arise in the scaling laws, in order to compare theoretical predictions with experimental or numerical results.

I shall now explain how renormalization analysis achieves to fulfill this program.

A.3.1 - Coarse-graining and spin-block renormalization

◇ *Coarse-graining*

The major aim of statistical mechanics is to describe the macroscopic properties of the system at thermal equilibrium in the thermodynamic limit $N \to \infty$. A basic idea is to perform a *coarse-graining*, replacing the short range correlations, structures and mechanisms by *effective ones* at larger scale. The first result is to increase the minimum scale of the model, at which observation must be done to match the parameters of the model with experimental information. In particular, it relates the smallest scale ingredients of the model to their consequences at higher scales. The second result is to reduce the number of degrees of freedom of the model (that is, the dimension of its phase space) while preserving its long range properties, its symmetries and any physical invariance. This procedure amounts to replace the initial statistical model by a simpler[7] one, of lower dimension but with the same macroscopic observable features.

In conjugate space, such a coarse-graining amounts to introduce a cutoff Λ corresponding to the desired minimum length scale $a = 2\pi/\Lambda$; components of wave vector

[7] Of course, information about the small range correlations is lost, hence small scale properties are no more available from the coarse-grained model: coarse-graining is a *non-reversible* (and more often *approximate*) procedure.

$\bar{q}$ for $||\bar{q}|| = q > \Lambda$ are no more described explicitely, and their influence at larger spatial scale is reproduced by modifying the parameter values and by adding effective contributions of wave vectors $q \leq \Lambda$.

Coarse-graining has been used for a long time as a preliminary step intending to produce a workable model. Although including this operation, renormalization is a far more powerful tool since it is devised *to be iterated*. It thus also includes rescalings in order to preserve apparent scales.

◇ *Spin-block renormalization*

The spin-block renormalization implements such iterated coarse-grainings for spin systems, where the spins are fixed on a regular lattice of parameter a. Their Hamiltonian is typically the sum of a free Hamiltonian H_0, involving each spin separately, and of an interaction potential V_0 :

$$H = H_0 + V_0 \qquad \text{where} \qquad H_0(N,[s]) = \sum_{j=1}^{n} h_0(s_j).$$

Individual spins s are grouped in k^d-blocks of linear size ka then each block is considered as a unique effective spin s'. Internal correlations between the elementary spins inside a block contribute to the coarse-grained free Hamiltonian $h_1(s')$ of the block. Correlations of range greater than ka, involving spins of different blocks, contribute to the effective interaction potential V_1 between spin-blocks.

The new idea[19,30] is to focus on the transformation $(h_0, V_0) \to (h_1, V_1)$ itself, if possible in an adequate space of parameters. Fixed points of this transformation are associated with an effective correlation length satisfying $k\xi = \xi$. Either $\xi = 0$ and the fixed point is associated with a system of free spins. Either $\xi = \infty$ and the fixed point is associated with a pure critical system, exhibiting an exactly self-similar structure since any coarse-grained system deduced from the initial one is identical to it.

A.3.2 - An exactly renormalizable example: the Ising model

One-dimensional Ising model illustrates in a convincing manner the spin-block renormalization sketched above. In fact, this model provides one of the few cases, if not the only one, where the complete renormalization analysis can be performed straightforwardly, involving only elementary calculations and giving *exact and explicit results*.

◇ *Background*

The *Ising model* refers to a system of spins
— fixed on the sites of a *one-dimensional regular lattice* of parameter a_0,
— taking *only the two values* $s = +1$ *and* $s = -1$,
— with a *ferromagnetic interaction* $-Js_js_{j+1}$ between nearest neighbors ($J>0$).

The microscopic state of the system is completely described by the spin configuration $[s] = (s_j)_{1\leq j\leq N} \in \mathcal{E}_n = \{-1,+1\}^N$. In order to have statistical translational invariance, a periodic boundary condition $s_1 = s_{N+1}$ is imposed. Thus, the Hamiltonian of the system is:

$$H_N(J,[s]) \equiv -J \sum_{j=1}^{N} s_j s_{j+1}$$

The dimensionless Hamiltonian $\mathcal{H}_N = \beta H_N$ (for inverse temperature β) depends only on the dimensionless coupling constant $K = \beta J$. I will denote $<>_{NK}$ the statistical average with respect to the Boltzmann-Gibbs distribution for N spins and a control parameter K. Analysis will have to prove the exponential dependence of the thermodynamic correlation function $\Gamma_\infty(K,j) = \lim_{N\to\infty} < s_i s_{i+j} >_{NK}$, expected to behave as $\exp[-ja/\xi(K)]$, to look for critical points K_c where the correlation length $\xi(K)$ diverges hence to determine the scaling behavior (if any) of the thermodynamic state functions with respect to $K - K_c$.

◇ *Scheme of the renormalization analysis*

Intending to fulfill the program presented above, renormalization analysis begins with a decimation that reduces the initial system of N_0 spins to a system of $N_1 = N_0/2$ spin-blocks. A partial sum over the odd-label spins in the partition function achieves the desired local statistical averaging replacing each pair (s_{2j}, s_{2j+1}) by an effective spin s'_j. A spatial rescaling of factor $1/2$ is performed in order to preserve the minimal scale a_0; as the spin density should be invariant, the effective spins still have modulus equal to 1, hence take exclusively the values $s' = \mp 1$. One thus writes:

$$\begin{aligned} Z(N_0,K_0) &= \sum_{[s_{2j}]_{0\leq j<N_1}} \sum_{[s_{2j+1}]_{0\leq j<N_1}} \prod_{j=0}^{N_1-1} e^{K_0 s_{2j+1}(s_{2j}+s_{2j+2})} \\ &= C^{N_1} \sum_{[s'_j]_{0\leq j<N_1}} \prod_{j=0}^{N_1-1} e^{K_1 s'_j s'_{j+1}} \\ &= C^{N_1}\ Z(N_1,K_1) \end{aligned}$$

with $\tanh K_1 = (\tanh K_0)^2$ and $C \equiv 2\,e^{K_1} = 2\sqrt{\cosh 2K_0}$. In the special case of the one-dimensional Ising model, the renormalized Hamiltonian has the same structure as the initial one so that the renormalization procedure amounts to a mere transformation $K_1 = r(K_0)$ in the space of parameters[8], defined by $\tanh r(K) = (\tanh K)^2$.

The "reduced" renormalization transformation r has two fixed points $K = 0$ and $K = \infty$. The value $K = 0$ is obtained either for infinite temperature or for zero coupling; it corresponds to *free particles*. The value $K = \infty$ is obtained either for zero temperature or for infinite coupling; it describes *totally correlated particles* and hence

[8] As it does not depend on the spin configuration, the constant C does not play any role in the thermodynamic properties.

corresponds to a critical system. Not surprinsingly since renormalization is designed to reduce critical behavior, $K = \infty$ is unstable whereas any trajectory under the action of r tends to the stable fixed point $K = 0$, since $r(K) < K$ on $]0, \infty[$. It is easy to derive the behavior of the correlation length near the critical point $K = \infty$:

$$\xi(K) \sim a e^{2K} \qquad \text{hence} \qquad \lim_{K \to \infty} \xi(K) = \infty$$

The correlation function satisfies $\Gamma_\infty[K, 2j] = \Gamma_\infty[r(K), j]$, which leads to the scaling behavior:

$$\Gamma_\infty(K, j) \ \sim \ \widetilde{\Gamma}\left(\frac{aj}{\xi(K)}\right) \qquad \text{since} \qquad \xi[r(K)] = \xi(K)/2$$

An analogous renormalization scheme can be used for the Ising model submitted to a *constant and uniform magnetic field* B i.e. when a contribution $-B\mu \sum_{j=1}^{N} s_j$ is added to the Hamiltonian. Again, the renormalized Hamiltonian has the same structure as the initial one, so that renormalization can be *exactly* implemented in the parameter space $\{K = \beta J, A = \beta B\mu\}$. The previous critical fixed point is recovered for $K = +\infty$ and $A = 0$ (zero temperature and zero field). It is then possible to show that the magnetic susceptibility diverges for $(K = +\infty,\ A = 0)$, which supports the critical character of the associated spin system: it is said that the *one-dimensional Ising model exhibits a critical phase transition at zero temperature.*

A.3.3 - Principles of renormalization analysis

The key idea of renormalization is to move the study from the phase space to a space of models or, in a restricted viewpoint explained below, to a space of Hamiltonians; renormalization is thus constructed as a transformation in this functional space, each element of which corresponds to a physical system for some fixed value of the control parameter(s) K. Instead of studying the equilibrium state of a given system $\mathcal{S}$ (within a prescribed model), for example computing the value of state functions and their variations with respect to variables and parameters, renormalization analysis focuses on the transformation of the model and of its parameters following a change of the scale of the description of $\mathcal{S}$; the benefit of this transfer is to obtain quantitative and universal results from the properties of the renormalization flow.

The typical steps common to all the renormalization procedures intending to describe critical behavior in the framework of statistical mechanics are the following[**14**].

1) The preliminary step is the *description of the renormalization action* on the model describing the physical system $\mathcal{S}$. It includes:

— a *reduction* of the number N of particles;

— a *rescaling* in the real space in order to keep constant the density of degrees of freedom, so that it makes sense to compare initial and renormalized problems;

— a *coarse-graining* that is, an average over the small scale structures in order to obtain their effective contribution in the model at larger scales. In the real space, it is more often implemented through a partial sum in the partition function; in the conjugate space, it is achieved through a cutoff $q < \Lambda$, the components $q \geq \Lambda$ being integrated out. It leads to replace the dimensionless Hamiltonian $\mathcal{H}$ with a *renormalized one* $R_k\mathcal{H}$;

— a *transformation* (more often a rescaling) of the *local order parameter* s and of the value of the *state functions* X. These transformations reproduce the consequences on the physical quantities X of the "geometrical" renormalization that is, direct and concrete transformation of the model composed of decimation, coarse-graining, action of a magnifying glass... Dimensional consistency may also be invoked in constructing these transformations. Another prescription is the group structure of the set of operators obtained when varying k. The remaining parameters, if any, are chosen in such a way that iterating the whole renormalization procedure ultimately leads to a fixed point associated to a non-trivial system.

To summarize, the action of the renormalization $\mathcal{R}_k$ expresses schematically:

$N \to Nk^{-d}$ which defines the label k of $\mathcal{R}_k$;
$\bar{x} \to \bar{x}/k$ in the real space $\mathbf{R}^d$;
$\bar{q} \to k\bar{q}$ in the conjugate space;
$\mathcal{H} \to R_k\mathcal{H}$ with $R_{k_1} \circ R_{k_2} = R_{k_1 k_2}$;
$\bar{s} \to \bar{s}/\sigma(k, \mathcal{H})$ with $\sigma(k_2, R_{k_1}\mathcal{H})\, \sigma(k_1, \mathcal{H}) = \sigma(k_1 k_2, \mathcal{H})$;
$X \to R_k^X(X)$ for the value of a state function X.

2) $\mathcal{R}_k$ amounts only to change the scales of the description of $\mathcal{S}$ or in other words, to change the analytical model, the values of the control parameters and the values of the extensive variables in such a way that the underlying physical system $\mathcal{S}$ is not modified. It thus makes sense to compare the initial situation with the renormalized one and to iterate the renormalization procedure. In particular, the macroscopic properties of $\mathcal{S}$ must be unaffected by the renormalization, which requires that:

$$Z(Nk^{-d}, R_k\mathcal{H}) = Z(N, \mathcal{H})$$

and, for any macroscopic state function (obtained through a statistical average):

$$\mathcal{R}_k[X(\mathcal{H})] = X[R_k(\mathcal{H})] \qquad \text{so that} \qquad X(Nk^{-d}, R_k\mathcal{H}) = R_k^X[X(N, \mathcal{H})]$$

For a field $\psi(N, \bar{x})$, the renormalization equation can be expressed in the form:

$$[\mathcal{R}_k\psi]\,(Nk^{-d}, \bar{x}/k) = R_k^\psi\,[\psi(N, \bar{x})]$$

More generally, for any state function, $\mathcal{R}_k[X(\mathcal{S})] \equiv X[\mathcal{R}_k(\mathcal{S})]$, which expresses analytically in:

$$R_k^X[X(\mathcal{H}, N, \bar{x}, \bar{s}] = X(R_k\mathcal{H}, Nk^{-d}, \bar{x}/k, \bar{s}/\sigma_k)$$

For example, $R_k^s(\bar{s}) = \bar{s}/\sigma_k$ for the order parameter $\bar{s}(\bar{x})$ so that $[\mathcal{R}_k(\bar{s})](\bar{x}) = \bar{s}(k\bar{x})/\sigma_k$. In consequence, $\sigma_k^2\ G(R_k\mathcal{H}, N, \bar{x}) = G(\mathcal{H}, Nk^d, k\bar{x})$ for the pair-correlation function

$G(\mathcal{H}, N, \bar{x}) =< \bar{s}(0)\bar{s}(\bar{x}) >_{\mathcal{H},N}$. It leads to the relation $\xi(R_k\mathcal{H}) = \xi(\mathcal{H})/k$ for the correlation length.

If X is a physical invariant of the system, its value must be preserved by the renormalization, which writes $R_k^X(X) = X$ or equivalently $X(Nk^{-d}, R_k\mathcal{H}) = X(N, \mathcal{H})$. If moreover, X is scale independent (intensive), its invariance writes $X(R_k\mathcal{H}) = X(\mathcal{H})$.

In fact, the relevant quantities are the thermodynamic limits $x = \lim_{N\to\infty} X(N)/N$ for an extensive state function, for example the dimensionless free energy per particle $f = -\lim_{N\to\infty}(\ln Z_N)/N$, and $y = \lim_{N\to\infty} Y(N)$ for an intensive one, for example the pair-correlation function. One then obtains:

$$f(R_k\mathcal{H}) = k^d f(\mathcal{H}) \qquad\qquad \sigma_k^2 \;\; G_\infty(R_k\mathcal{H}, \bar{x}) = G_\infty(\mathcal{H}, k\bar{x})$$

Physical invariance writes respectively $x(R_k\mathcal{H}) = k^d x(\mathcal{H})$ and $y(R_k\mathcal{H}) = y(\mathcal{H})$.

3) In some cases, an adequate parametrization $K \to \mathcal{H}_K$ allows to reduce, at least approximately, the renormalization action in the space of Hamiltonians to a transformation $K \to r(K)$ in the parameter space. Renormalization equations then read $Z[Nk^{-d}, r_k(K)] = Z(N, K)$ and $X[Nk^{-d}, r_k(K)] = R_k^X[X(N, K)]$. We shall see below (points 6 and 7) how to find such an "adapted" parametrization.

4) The main step is the *determination of the fixed points* $\mathcal{H}^*$ *of* R_k satisfying the fixed-point equation $R_k\mathcal{H}^* = \mathcal{H}^*$. If a renormalization analysis is to be fruitful, R_k must have some fixed points. From the relation $\mathcal{R}_k[X(\mathcal{H})] = X[R_k(\mathcal{H})]$ satisfied by an arbitrary state function X and from $R_k\mathcal{H}^* = \mathcal{H}^*$, it follows that the underlying model $\mathcal{S}^* \equiv \mathcal{S}(\mathcal{H}^*)$ is a fixed point of $\mathcal{R}_k$. From the very construction of $\mathcal{R}_k$, this property shows up an *exactly self-similar behavior*. In this sense, $\mathcal{S}^*$ appears as an "ideal" critical system. Indeed, as $\xi[R_k(\mathcal{H})] = \xi(\mathcal{H})/k$, then $\xi^* = \infty$ in case of a non-trivial fixed point, which shows the *critical character of* $\mathcal{S}^*$. One obtains $G^*(\bar{x}) \sim ||\bar{x}||^{-2\omega}$ for the thermodynamic pair-correlation function if $\sigma_k \sim k^{-\omega}$: it gives explicitely the power law dependence which is characteristic of a critical system. Denoting $X^* = X(\mathcal{H}^*)$, it follows from $R_k\mathcal{H}^* = \mathcal{H}^*$ that $X^*(Nk^{-d}) = R_k^X[X^*(N)]$. If $R_k^X(X) = k^\alpha X$, one obtains the scaling law $X^*(N) \sim N^{-\alpha/d}$. As a result, either $X^* = \infty$ if $-\alpha > d$ either $X^* = 0$ if $-\alpha_x < d$ in case of an extensive state function X. More generally, a state function X^* of the fixed point $\mathcal{S}^*$ satisfies:

$$R_k^X[X^*(N, \bar{x}, \bar{s})] = X^*(Nk^{-d}, \bar{x}/k, \bar{s}/\sigma_k)$$

If moreover X is a physical invariant, so that $R_k^X(X) = X$, then X^* satisfies:

$$X^*(N, \bar{x}, \bar{s}) = X^*(1, \bar{x}N^{-1/d}, \bar{s}/\sigma_{k=N^{1/d}})$$

For $\sigma_k \sim k^{-\omega}$:

$$X^*(N, \bar{x}, \bar{s}) = \Theta(\bar{x}N^{-1/d}, \bar{s}N^{\omega/d}) = \Psi(\bar{x}N^{-1/d}, \bar{s}\;||\bar{x}||^\omega$$

where Θ and Ψ are scaling functions depending on only two variables.

5) The next step is to perform a *linear and if possible nonlinear analysis of* R_k *around* $\mathcal{H}^*$. Due to the group-theoretic structure of the set $(R_k)_k$, the eigenvalues of $DR_k(\mathcal{H}^*)$ have the form k^{γ_j} and the eigenvectors do not depend on k. If a renormalization analysis is to be fruitful, R_k possesses a *hyperbolic fixed point*, that is a fixed point having both stable and unstable directions. The stable manifold $\mathcal{V}^s$ of $\mathcal{H}^*$ is the location of the critical systems: in case of a system whose modelling involves an Hamiltonian $\mathcal{H}_K$ depending on a control parameter K, the critical value K_c is given by the condition $\mathcal{H}_{K_c} \in \mathcal{V}^s$. Thus $\mathcal{H}^*$ appears as a universal function describing the typical and universal critical features. These properties are summarized in:

$$\forall\, \mathcal{H} \in \mathcal{V}^s, \qquad\qquad \lim_{n\to\infty} R_{k_0}^n \mathcal{H} = \lim_{k\to\infty} R_k \mathcal{H} = \mathcal{H}^*$$

The operators R_k are precisely designed to extract the universal large-scale behavior of a critical system. Each renormalization eliminates the specific small-scale details in order to highlight the self-similar properties common to all the systems of the same *universality class*; moreover, it shows that the universality class having $\mathcal{H}^*$ as a typical and "pure" representative is merely the stable manifold $\mathcal{V}^s$ of $\mathcal{H}^*$, which allows to construct it explicitely.

6) It is then possible to *determine the scaling behavior* of a system depending on a control parameter K. Let P be the projection onto the unstable direction (independent of k) of $DR_k(\mathcal{H}^*)$ associated to the maximal eigenvalue $k^{\gamma_1} > 1$. If the so-called "*condition of transversality*" $P.(\partial\mathcal{H}_K/\partial K) \neq 0$ is satisfied in the family $(\mathcal{H}_K)_K$, it is possible to write, for K close to K_c:

$$P[R_k(\mathcal{H}_K)] \approx P[\mathcal{H}_{r_k(K)}] \qquad \text{with} \qquad r_k(K) \approx K_c + k^{\gamma_1}(K - K_c)$$

Critical behavior of a thermodynamic state function x follows:

$$R_k^X[x(\mathcal{H}_K)] = x[R_k\mathcal{H}_K)] \approx x[\mathcal{H}_{r_k(K)}]$$

or, expressing $x(\mathcal{H}_K)$ as a function $\tilde{x}(K)$: $R^X[\tilde{x}(K)] \approx \tilde{x}[K_c + k^{\gamma_1}(K - K_c)]$.

7) An exhaustive description of the critical behavior is available from the knowledge of the renormalization flow close to an hyperbolic fixed point $\mathcal{H}^*$. A complete parametrization of the neighborhood of $\mathcal{H}^*$ in the space of Hamiltonians is obtained straightforwardly knowing the eigenvectors $(h_j)_{j\geq 1}$ of $DR_k(\mathcal{H}^*)$ (which are independent of k) and the associated eigenvalues $(k^{\gamma_j})_{j\geq 1}$; indeed, one can write at the leading order in $\mathcal{H} - \mathcal{H}^*$ which corresponds to the linear terms:

$$\mathcal{H} = \mathcal{H}^* + \sum_{j=1}^{\infty} a_j\, h_j \qquad \Longrightarrow \qquad R_k\mathcal{H} = \mathcal{H}^* + \sum_{j=1}^{\infty} k^{\gamma_j}\, a_j\, h_j + \mathcal{O}(a^2)$$

Hence, the renormalization of the Hamiltonian reduces in the linear approximation to a scale transformation r_k acting component-wise in the parameter space according to $r_k^{(j)}(a_j) = k^{\gamma_j}\, a_j$. The label $j \to a_j$ are chosen according to the ordering $\gamma_j \geq \gamma_{j+1}$.

The scaling behavior of a state function X follows. If for example $R_k^X(X) = k^\alpha\, X$, one writes for any $k > 0$:

$$X[N, (a_j)_{j\geq 1}] = k^{-\alpha}\; X[Nk^{-d}, (k^{\gamma_j} a_j)_{j\geq 1}]$$

$$x[(a_j)_{j\geq 1}] = k^{-\alpha-d}\; x[(k^{\gamma_j} a_j)_{j\geq 1}]$$

The signs of the exponents $(\gamma_j)_{j\geq 1}$ are crucial. The direction associated to γ_j is said to be:
— *relevant* if $\gamma_j > 0$,
— *marginal* if $\gamma_j = 0$,
— *irrelevant* if $\gamma_j < 0$.
Let us explain this terminology on a toy-model where $\gamma_1 > \gamma_2 > \gamma_3 = 0 > \gamma_4$. The coefficient a_1, associated to the maximal exponent $\gamma_1 > 0$, appears to be the control parameter of the phenomenon; in particular, critical behavior is observed for $a_1 \to 0$. Taking $k = a_1^{-1/\gamma_1}$ (if $a_1 > 0$) leads to[9]:

$$x(a_1, a_2, a_3, a_4) = a_1^{-(d+\alpha)/\gamma_1}\; x(1, a_2\, a_1^{-\gamma_2/\gamma_1}, a_3, a_4\, a_1^{-\gamma_4/\gamma_1})$$

The function x being continuous, the value of the parameter a_4 does not play any role on the limiting behavior $a_1 \to 0$. If it can be shown that $x(1, z, a_3, 0)$ behaves as a power $|z|^{\mu^\mp}$ for z tending to $\mp\infty$, the leading behavior of the thermodynamic function x when $a_1 \to 0^+$ is known: $x(a_1, 0, a_3, a_4) \sim a_1^{-\eta}$ where:

• if $a_2 = 0$, $x(a_1, 0, a_3, a_4) \sim a_1^{-(d+\alpha)/\gamma_1}\; x(1, 0, a_3, 0)$ hence $\eta = \frac{d+\alpha}{\gamma_1}$;

• if $a_2 > 0$, $x(a_1, a_2, a_3, a_4) \sim a_1^{-(d+\alpha+\gamma_2\mu^+)/\gamma_1}\; \Theta^+(a_2, a_3)$ hence $\eta = \frac{d+\alpha+\gamma_2\mu^+}{\gamma_1}$;

• if $a_2 < 0$, $x(a_1, a_2, a_3, a_4) \sim a_1^{-(d+\alpha+\gamma_2\mu^-)/\gamma_1}\; \Theta^-(a_2, a_3)$ hence $\eta = \frac{d+\alpha+\gamma_2\mu^-}{\gamma_1}$.

Analogous results are deduced for the various derivatives of x. Hence *the relevant directions control the exponent values*. In the example above, a *cross-over* between different[10] scaling laws, hence between different universality classes is observed when a_2 varies from negative values to positive ones, passing through $a_2 = 0$. The marginal directions[11] do not affect the exponent values but *intervene in the universal functions*. *The irrelevant directions have no influence at all on the leading scaling behavior*; they only provide corrections to the asymptotic scaling behavior ($a_1 \to 0$). Additional corrections come from the nonlinear terms in the renormalization action.

[9] Similar calculations can be carried over for $a_1 < 0$; it thus appears the reduced thermodynamic function $x(-1, a_2\,|a_1|^{-\gamma_2/\gamma_1}, a_3, a_4\,|a_1|^{-\gamma_4/\gamma_1})$; in general, it modifies the universal functions but not the values of the exponents.

[10] It is often the case that $a_2 = 0$ leads to a critical behavior, whereas standard values (for example mean-field values) are recovered for $a_2 \neq 0$.

[11] Let us note that a marginal direction is in most cases associated to a *continuous symmetry* $(S_b)_b$ of the physical system $\mathcal{S}$. Indeed, as renormalization preserves the symmetries of $\mathcal{S}$, R_k and S_b commute (for any $k > 0$, $b > 0$). Hence, if $\mathcal{H}^*$ is a fixed-point of R_k, $S_b\mathcal{H}^*$ is also a fixed point for any $b > 0$, so that $[\partial(S_b\mathcal{H}^*)/\partial b](b_0)$, if non-zero, is a marginal direction of $DR_k(S_{b_0}\mathcal{H}^*)$. It corresponds to a displacement along the curve $(S_b\mathcal{H}^*)_{b>0}$ of fixed points in the space of Hamiltonians.

Renormalization principles and their relevance for the prediction of scaling behavior can thus very clearly be asserted in the real space. Nevertheless, the explicit construction of R_k (acting in the space of Hamiltonians) is more often easier to implement *in the conjugate space*, by introducing an ultra-violet *cutoff* $q \leq \Lambda$ and by integrating out the components $q > \Lambda$. We refer to the reviews of Fisher[14] and of Wilson and Kogut[31] for technicalities, especially for perturbative expansions performed within a renormalization analysis and for their graphical formulation. This "conjugate" viewpoint is moreover well-suited to draw a parallel with renormalization methods used in quantum field theories.

We shall in the following Part B recognize an analogous scheme intending to describe the onset of deterministic chaos; spatial extension will be replaced by duration and the renormalization operator will act on the evolution law. More generally, the above procedure is the backbone of most of the renormalization methods. Let us list again, now in a general formulation, the main steps for implementing a renormalization analysis.

1) As previously, renormalization includes a *reduction* of the number of degrees of freedom (which defines the scale factor k labelling the transformation $\mathcal{R}_k$) together with suitable *rescalings* in real and phase spaces. It must preserve the physical invariants and the symmetries of the system.

2) A preliminary step in the analytical formulation of the renormalization action is the specification of a space of "structure rules" Φ. The "*structure rule*" of a system names the set of informations necessary to write down its equilibrium state or its evolution: it is the dimensionless Hamiltonian in statistical mechanics, the evolution map for a discrete dynamical system, the vector field for a continuous dynamical system or the transition probabilities for a random walk. Φ appears as a "*space of models*".

3) The main step is the *construction of a renormalization operator* R_k *acting in* Φ, involving a coarse-graining of the small scale details in the real space or a cutoff in the conjugate space, then rescalings in order to keep constant the apparent scales and the physical invariants of the system.

4) The core of the method lies in the *determination of fixed points* $R_k\phi^* = \phi^*$, hence in the *linear stability analysis* of R_k around these fixed points. All the results obtained in the framework of statistical mechanics still apply with only minor (mainly notational) changes.

Part B - Renormalization for dynamical systems and chaos

Hereafter," chaos" will refer to unpredictable and erratic asymptotic behavior encountered in the context of *differentiable dynamical systems*, thus induced by a deterministic and regular evolution in a *low dimensional phase space* $\mathcal{X}$. Such a chaotic motion[32] is different in nature both from *random motion*, since no stochasticity is included in its dynamics, and from *fully developed turbulence*, since only a finite number of degrees of freedom intervene.

My aim is to show that some of its main (and well-accepted) characteristics can be seen as critical features. It will thus be possible to relate it with the so-called critical phenomena encountered in statistical mechanics and to describe *transition to chaos as a critical phase transition*.

B.1 - Background for the study of deterministic chaos

B.1.1 - Asymptotic regime

First of all, let me point out that "pure" chaotic features and the associated singularities appear only in *asymptotic evolutions* of infinite duration. This keeps from any contradiction with the deterministic and regular behavior of finite evolutions, described by smooth functions of time, of initial conditions and of parameters of the system. This duality between finite and limiting behaviors is reminiscent of the "critical phenomena" encountered in statistical mechanics, which give rise to singularities in the observable state functions only in the thermodynamic limit. In both cases, critical singularities can be seen, from a mathematical viewpoint, as a mark of *the presence of two limits that cannot be taken in arbitrary order*. Namely, in statistical mechanics, the thermodynamic limit $N \to \infty$ cannot be inverted with the limit $\beta \to \beta_c$. Analogously, in dynamical systems framework, the asymptotic limit $T \to \infty$ cannot be indifferently taken after or before letting a control parameter μ increase to a critical value μ_c.

B.1.2 - Attractors and invariant measures

Since the relevant description of chaos involves only asymptotic regime, we shall only consider *the restriction of the dynamical system to its attractor*[11], that is a closed and indecomposable subset of its phase space $\mathcal{X}$ which is invariant and stable under the evolution. It is to note that the stationarity of this asymptotic regime is a *statistical stationarity* with respect to some suitable distribution m on $\mathcal{X}$, called an *invariant measure*; it means that m is a statistical weight on $\mathcal{X}$ which is unchanged during the evolution. In case of a discrete dynamical system of evolution map f, this invariance of m expresses equivalently:

i) $\forall A \subset \mathcal{X},\ m[f^{-1}(A)] = m(A)$

ii) $\forall \phi \in L_1(\mathcal{X}, dm),\ \int \phi \circ f(x) dm(x) = \int \phi(x) dm(x)$

The stationary regime being reached, the overall dynamics is entirely described by the invariant measure m, more precisely the measure, among all the possible invariant ones, which is spontaneously selected by the evolution from the initial data distribution. By definition of m, its support is the attractor $\mathcal{A}$.

Asymptotics of a dynamical system (chaotic or not) can thus be described in a *probabilistic framework*, in which the notion of trajectory in the phase space $\mathcal{X}$ is replaced by that of measure in $\mathcal{X}$, describing the probability of presence of the physical system in $\mathcal{X}$ at any time of the stationary regime. Thus, the question is no more to describe how the physical system evolves from any initial state (that is, from any point of $\mathcal{X}$), but to investigate the properties of the invariant measure m, as it gives a global description of the asymptotic dynamics. It is reminiscent from the statistical mechanics approach, in which individual deterministic trajectories are replaced by an adequate statistical ensemble.

B.1.3 - Ergodicity

The relevance of such a probabilistic approach arises when it can be supplemented by *ergodic properties* of the invariant measure m. Namely, m is *ergodic* with respect to the evolution map f if and only if any invariant subset B of $\mathcal{X}$ (such that $f^{-1}(B) = B$) has either a null m-measure or a full m-measure or, equivalently, if and only if any invariant measurable real function ϕ (such that $\phi \circ f = \phi$) is m-almost surely constant on $\mathcal{X}$. Besides, the notion of deterministic chaos is defined in the framework of ergodic theory. To explain this apparent restriction on m, I may invoke the *indecomposability of an attractor* which is densely visited by any generic asymptotic trajectory, and the intuitively *mixing character of chaotic dynamics* , which in turn implies ergodicity. I hope specialists of this domain will agree, or make precise, this rather naive justification.

◇ <u>*Statistical results*</u>

Whether in statistical mechanics or for a dynamical system, the aim of the theorist is to explain and, if possible, to predict the *macroscopic behavior from the microscopic structure* of the system. It is achieved by establishing some *statistical laws*, relating the macroscopically observable results with theoretically computable means.

In statistical mechanics (Part A), the spatial averages $A(N,[z]) = (1/N)\ \sum_{i=1}^{N} a(z_i)$ or $C(N,[z]) = [1/N(N-1)] \sum_{i \neq j} c(z_i, z_j)$ for a system $\mathcal{S}$ of $N \gg 1$ particles in a configuration $[z]$ may be identified with their statistical means $< a(z) >$ or $< c(z,z') >$ with respect to the Boltzmann-Gibbs distribution, provided $\mathcal{S}$ is non-critical.

For a dynamical system, the observable quantities are temporal means $T^{-1} \int_0^T a(z_t) dt$ along a given trajectory $(z_t)_{t \geq 0}$ that is, a configuration of the dynamical system. The analogy with statistical mechanics is formally exact for a discrete dynamical system, as the temporal mean writes down $(1/N) \sum_{i=1}^{N} a(z_{t_i})$ after N steps. This temporal mean

may be identified in the limit of an infinite duration $T \to \infty$ with the statistical mean of $a(z)$ with respect to the invariant measure m of the dynamical system, provided this measure is ergodic when associated with the evolution law of the dynamical system and finite ($m(\mathcal{X}) < \infty$). Thus, the apparent random character of chaotic trajectories is theoretically supported by the existence of *statistical laws* about the asymptotic motion, which show up the very nature of "deterministic chaos".

Such results stem from the *pointwise ergodic theorem of Birkhoff*[1] applied to the ergodic dynamics on $(\mathcal{X}, m)$. Given a discrete evolution map f, it asserts that for any integrable function $a \in L_1(\mathcal{X}, dm)$:

$$\exists \mathcal{X}_0 \subset \mathcal{X},\ m(\mathcal{X} - \mathcal{X}_0) = 0,\ \forall x_0 \in \mathcal{X}_0,\quad \lim_{T\to\infty} \frac{1}{T} \sum_{j=0}^{T-1} a[f^j(x_0)] = \frac{1}{m(\mathcal{X})} \int a(x) dm(x)$$

It exactly expresses that the (random) *temporal mean* of the successive values taken by a state function a is m-generically equal to its *statistical mean* in $(\mathcal{X}, m)$. In physical words, it means that the observation of a over a macroscopic time interval gives no longer a "random" result but can be predicted and computed in terms of the intrinsic statistical weight m generated by the dynamics itself.

◇ *Ergodicity and the Boltzmann ergodic hypothesis*

When using this measure-theoretic description for an ergodic dynamical system, the analogy with statistical mechanics is straightforward through a *comparison between the invariant measure and the Boltzmann-Gibbs distribution*. Indeed, the statistical approach for a system $\mathcal{S}$ composed of a large number N of particles, in thermal equilibrium at temperature T, amounts to replace the description of the deterministic trajectory of $\mathcal{S}$ in the microscopic phase space $\mathcal{E}_N$ as time flows by the description, in which time no more intervenes, of the relative frequency with which $\mathcal{S}$ visits any part of $\mathcal{E}_N$. According to the *ergodic hypothesis of Boltzmann*[16], this relative frequence coincide in the thermodynamic limit $N \to \infty$ with the Boltzmann-Gibbs distribution at T, which gives the statistical weight of each configuration $[z] \in \mathcal{E}_N$. This transfer from a temporal description to a probabilistic one, independent of time in equilibrium problems, is the basis of statistical mechanics since it allows to identify temporal averages over an infinite duration given by experimental observations with statistical means computed theoretically; hence analysis can be restricted to a statistical ensemble approach. In contemporary language, the Boltzmann hypothesis amounts to assume that *the microscopic evolution of $\mathcal{S}$ in $\mathcal{E}_N$ is ergodic* with respect to the Boltzmann-Gibbs distribution, which is supported by a more physical statement about "*molecular chaos*".

B.2 - Chaos is a critical phenomenon

B.2.1 - Bifurcations and routes to chaos

A striking advance in studying transition to chaos has been the experimental, numerical and analytical evidence of a few number of *generic scenarii* describing changes of asymptotics dynamics when some control parameters μ are varied[10]. Namely, a *scenario* is a qualitative sequence of elementary mechanisms which leads from a simple stationary motion (a fixed point) to a chaotic attractor. It is said to be *generic* if it is only shifted in the space of parameters but not destroyed when the evolution map is slightly modified. Typical steps of a scenario are *bifurcations* that is, qualitative changes of the attractor $\mathcal{A}_\mu$ when varying parameter(s) μ of the evolution map. For deterministic chaos, a scenario involves only a *finite dimensional subspace* of the phase space; *only one parameter* essentially controls it, otherwise genericity breaks down. Hence the qualitative features of a transition to chaos obey one of a few predetermined schemes; the unexpected result of this viewpoint is the discovery of some quantitative universal features, especially for the period-doubling scenario presented below.

Having in mind to compare dynamical systems and statistical mechanics, it is to note that a bifurcation exhibits common features with critical phase transitions; indeed, it appears typically as:

• *Jump discontinuities* of the branches $x_j(\mu)$ of the attractor $\mathcal{A}_\mu$ or *divergences* of their derivatives with respect to μ, which respectively correspond to first order and critical transitions.

• *Scaling laws* for average quantities with respect to the control parameter $\mu - \mu_0$.

• *Critical slowing down*, that is divergence of the time length necessary to reach stationary regime (the notion of attractor being replaced by that of thermal equilibrium) described by a stationary statistical distribution in the phase space (the notion of invariant measure being replaced by that of Boltzmann-Gibbs distribution).

• In the dynamic viewpoint for the configurations of the particle system, *ergodicity seems to break down* at the transition, the system staying an arbitrary long time in bounded parts of the phase space. In the Boltzmann-Gibbs distribution, exact compensation between thermal effects and coupling terms occurs at T_c, which may explain critical slowing down as the system hesitates between qualitatively different statistics. For dynamical systems, ergodicity seems to break down in the sense that the ergodic measures are not the same below and above the bifurcation value.

B.2.2 - Critical properties of chaos and transition to chaos

As chaos is a characteristic of the asymptotic dynamics[11], the key notions are the *invariant measure* m "spontaneously selected" by the evolution law and its support, which coincide with the *attractor* $\mathcal{A}$. Studying chaos and transition to chaos involves

only the *dynamics restricted to the attractor*. For the sake of simplicity, I shall restrict the discussion to a one parameter family $[(\mathcal{X}, f_\mu), m_\mu, \mathcal{A}_\mu]_\mu$ of discrete dynamical systems.

• To be chaotic, $\mathcal{A}_\mu$ must be more complex than a mere quasi-periodic attractor[12].

• Moreover, relevant definitions of chaos[11] involves ergodicity of the dynamics with respect to the invariant measure m_μ; in this sense, chaos is a *measure-theoretic notion*.

• The main characteristic of chaos is *sensitivity with respect to initial conditions* illustrated by Lorentz in the so-called *"butterfly effect"*: namely, the presence of a flying butterfly, if amplified by a chaotic flow, can eventually induce a hurricane. Such property originates in the *exponential enhancement of any initial disturbance* during a chaotic evolution. In particular, the trajectories stemming from two point $x_1 \approx x_2 \approx x$ that cannot be distinguished by an observer eventually have totally different behavior. In turn, it causes *impredictability* since knowing x, it is impossible to say which of the trajectories of x_1 and x_2 will be observed.

These properties are deduced from a single "mathematical" feature of the evolution law f_μ which is the *positivity of the maximal Lyapounov exponent* $L(\mu)$. For a one-dimensional map f, the Lyapounov exponent is defined as:

$$L(f,m) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \ln|f'(f^j(x_0))|$$

Depending on the initial position x_0 in its definition, $L(f,m,x_0)$ is in fact m-almost everywhere constant due to the ergodicity of m. It is to note that $L(f,m)$ depends on the invariant measure m, as the pointwise ergodic theorem of Birkhoff gives (m is supposed to be finite):

$$L(f,m) = \int_{\mathcal{X}} \ln|f'(x)|\, dm(x)$$

The relation asserted above between chaos and the positivity of this maximal Lyapounov exponent is an obvious consequence of its meaning, which is to be the rate of the exponential separation of initially close trajectories:

$$||f_\mu^n(x) - f_\mu^n(y)|| \sim e^{nL(\mu)}\ ||x-y|| \quad \text{if } n \text{ enough large and if } ||x-y|| < \epsilon(n,\mu)$$

• Another criterion, this one of topological nature, can be useful, especially for experimental analysis. It involves a measure of the impredictability. Let ϵ_0 be the accuracy on the initial data; admitting an error up to ϵ, the mean time length T_μ over which a trajectory can be predicted roughly obeys a scaling law $\epsilon \sim \epsilon_0\ \exp[T_\mu S_{top}(\mu)]$ which defines the *topological entropy* $S_{top}(\mu)$. Chaos is associated with $S_{top}(\mu) > 0$.

[12] A quasi-periodic motion is a combination of coupled periodic components.

• The statistical entropy of the distribution of probability $m_\mu/m_\mu(\mathcal{X})$ on $\mathcal{X}$ is called the *metric entropy* (or the "*information*") $s(\mu)$ of the ergodic finite measure m_μ; its positivity $s(\mu) > 0$ gives another metrical criterion of chaos.

• When the *nonlinearities* are strong enough (for $\mu > \mu_c$), chaos is likely to appear; indeed, the nonlinearities amplify any initial disturbance of the trajectory as time grows exactly as, in statistical mechanics, a collective particle state in strong coupling regime (for $\beta > \beta_c$) propagates and amplifies local perturbations of the configuration. In this sense, a chaotic system is critical.

• The maximal Lyapounov exponent $L(\mu)$, possibly smoothed in μ (then denoted $\bar{L}(\mu)$), provides a suitable order parameter for the transition to chaos, as it is strictly positive in the chaotic phase whereas it is zero in the other one. Just above the transition to chaos in $\mu = \mu_c$, the smoothed Lyapounov exponent $\bar{L}(\mu)$ is likely to satisfy a scaling law $\bar{L}(\mu) \sim (\mu - \mu_c)^\beta$ with $0 < \beta < 1$: it is continuous in μ_c but presents an infinite slope. S_{top} provides an equivalent order parameter.

• A significant quantity is the *characteristic time* $\tau(\mu)$, since it exhibits a singular scaling behavior at the transition. Below chaos ($\mu < \mu_c$), $\tau(\mu)$ is equal to the *largest period* of the periodic components which contribute to the asymptotic motion; it typically increases as such a quasi-periodic motion destabilizes into chaos. In "strongly chaotic" regime where the dynamics are mixing, the mixing property have a decorrelating effect along any m_μ-generic trajectory, which have thus shorter memory as the nonlinearities responsible of the exponential sensitivity to disturbances increase. Typically, $\tau(\mu)$ behaves acording to a singular power law:

$$\tau(\mu) \sim |\mu - \mu_c|^{-\nu} \qquad\qquad (\nu > 0)$$

in the neighborhood of μ_c, the exponent ν being in most cases the same on each side of μ_c. Of course, this scaling law is to be compared to the analogous behavior of the correlation length $\xi(K)$ around a critical point K_c.

• Let me point out that the analogy between bifurcations and phase transitions extends as far as to distinguish:

— *"first order" bifurcation* which does not lead to chaos in $\mu = \mu_0$: $L(\mu_0) = 0$, $L(\mu) \leq 0$ is continuous in μ_0 but its derivative $dL/d\mu$ has a jump discontinuity with limits from the right and from the left of opposite signs; besides, it indicates that $dL/d\mu$ is a better order parameter for this kind of bifurcation;

— *"second order" bifurcation* which leads to chaos in $\mu = \mu_c$: $\bar{L}(\mu_c) = 0$, $\bar{L}(\mu)$ is continuous in μ_c but has an infinite slope in μ_c, with $\bar{L}(\mu) \equiv 0$ if $\mu < \mu_c$ and $\bar{L}(\mu) > 0$ if $\mu > \mu_c$.

We shall pursue further the analogy in the case of period-doubling transition to chaos, where the parallel with critical phase transitions is quite amazing.

B.2.3 - How renormalization operates on chaotic motions

The evidence of universal qualitative and even quantitative characteristics, as well as scaling properties, for the transition to deterministic chaos is a clue to make use of renormalization methods. *The key-idea is to perform iterated coarse-grainings jointly to time-scale contractions* in order to capture the overall asymptotic dynamics into the trajectory of the evolution map under the renormalization action in an adequate functional space (whose elements are evolution maps).

A typical transformation is $R_k f(x) = [\lambda_{f,k}]^{-1}\ f^k(x\lambda_{f,k})$ where $\lambda_{f,k}$ is some scale factor in the phase space $\mathcal{X}$ which appears in order to preserve some physical properties, for example the value of $f(0)$ and to restrict the analysis to a neighborhood of $x = 0$. The action of R_k expresses according to the following relations.

- $L(R_k f) = kL(f)$ on the Lyapounov exponents, since:

$$|x - y|e^{nL(R_k f)} \sim |[R_k f]^n(x) - [R_k f]^n(y)| \qquad \text{with}$$

$$|[R_k f]^n(x) - [R_k f]^n(y)| \sim [\lambda_{f,k}]^{-1}|f^{kn}(x\lambda_{f,k}) - f^{kn}(y\lambda_{f,k})| \sim |x - y|e^{knL(f)}$$

- $S_{top}(R_k f) = kS_{top}(f)$ on the topological entropy.

- The measure $m_{R_k f}$ defined by $m_{R_k f}[B] = m_f[\lambda_{f,k}B]$ (for any subset B of the phase space) is invariant with respect to $R_k f$.

- $s(m_{R_k f}) = ks(m_f)$ on the statistical entropy (also called "information").

Let us note that the renormalization of L, S_{top} and s does not involve the scale factor $\lambda_{f,k}$ in $\mathcal{X}$ but only the time rescaling factor k.

B.3 - The example of the period-doubling transition to chaos

B.3.1 - Qualitative features

The characteristic feature of this route to chaos is the occurrence, when increasing a control parameter μ, of *a sequence of "period doublings"*, for bifurcation values $(\mu_j)_{j\geq 0}$ accumulating on a finite limiting value μ_c above which the asymptotic behavior is chaotic (at least for a non-countable infinite set of values $\mu > \mu_c$). The striking result is the evidence of a *universal number* δ governing the limiting behavior of the sequence $(\mu_j)_{j\geq 0}$: although the bifurcation values $(\mu_j)_{j\geq 0}$ and the value μ_c at the onset of chaos depend on the family $(f_\mu)_\mu$ of dynamical systems and even on parametrization $\mu \to f_\mu$, the rate of convergence takes a prescribed value $\delta \approx 4,66920\ldots$:

$$\lim_{j\to\infty} \delta^j(\mu_c - \mu_j) = A \neq 0 \qquad \text{which implies} \qquad \lim_{j\to\infty} \frac{\mu_{j+1} - \mu_j}{\mu_{j+2} - \mu_{j+1}} = \delta$$

The second formulation is easier to test experimentally as no preliminary knowledge of μ_c nor of δ is needed. This scenario can also be observed on the *power spectrum* of the asymptotic dynamics (restricted to the attractor); it then deserves the name of *sub-harmonic cascade*[12], as the successive period doublings correspond to the successive appearance (one new frequency each time the control parameter crosses a bifurcation value) of decreasing frequencies $(2^{-j}\omega_0)_{j\geq 0}$. Varying μ ultimately leads at $\mu = \mu_c$ to a *chaotic power spectrum*, which is characterized by an accumulation of non-zero components in $\omega = 0$.

This route to chaos was pointed out first numerically by Feigenbaum [13] and independently, at the same time, by Coullet and Tresser[7], when studying the asymptotics of the logistic map[25] $x \mapsto ax(1-x)$ on the interval $[0,1]$ when the control parameter a increases from $a = 0$ to $a = 4$. It was also well observed experimentally[13], for example in Rayleigh-Bnard convection[9,22] where the control parameter is the vertical gradient of temperature applied from below in a finite layer of liquid.

This scenario is specific to *discrete* dynamical systems and it is sufficient to consider one-dimensional[14] systems; so let $(f_\mu)_\mu$ be a one-parameter family of differentiable maps of the interval exhibiting this route to chaos. A typical example is provided by the family $[x \mapsto 1-\mu x^2]_{0<\mu<2}$ on the interval $[-1,1]$. Each of the successive qualitative steps leading from equilibrium for $\mu < \mu_0$, when the attractor is a mere fixed point, to chaos for $\mu > \mu_c$ is a stability exchange between a 2^j-cycle for $\mu < \mu_j$ and a 2^{j+1}-cycle for $\mu > \mu_j$. This doubling of the period of the attractor is the observable consequence of the occurrence of a *pitchfork bifurcation* in the iterate $f_\mu^{2^j}$ of the evolution law f_μ of the dynamical system in $\mu = \mu_j$. This generic bifurcation[28] corresponds to the following topological change of the attractor: a unique stable fixed point $x_0(\mu)$ for $\mu < \mu_0$ looses its stability in μ_0 and is replaced [15] by a stable 2-cycle $x^{\mp}(\mu)$ for $\mu > \mu_0$. Its mathematical formulation, which gives the bifurcation value μ_0, is:

$$g_{\mu_0}[x_0(\mu_0)] = x_0(\mu_0) \qquad \text{and} \qquad g'_{\mu_0}[x_0(\mu_0)] = 1$$

The branches are connected continuously in μ_0, but with an infinite slope:

$$x_0(\mu_0) = x^+(\mu_0) = x^-(\mu_0) \qquad \frac{dx_0(\mu)}{d\mu}(\mu_0 - 0) \text{ finite} \qquad \frac{dx_\mu^{\mp}}{d\mu}(\mu_0 + 0) = \infty$$

B.3.2 - Renormalization analysis

The power of renormalization applied to dynamical systems is demonstrated with its success in analysing this scenario; indeed, it achieves to *prove the scaling behavior* of the sequence of the bifurcation values, to predict the *numerical value* of the rate δ

[13] Due to noise, only a finite sequence of bifurcations can be observed experimentally.

[14] This does not contradict the assertion that chaos needs dimension $d \geq 3$ to appear, since the underlying continuous dynamics is embedded in a phase space of dimension at least 3.

[15] The fixed point $x_0(\mu)$ still exists for $\mu > \mu_0$ but is unstable.

and to describe the associated *universality class*. It additionally shows up the self-similarity of the bifurcation diagram, which ultimately evidences the fractal structure of the attractor $\mathcal{A}_{\mu_0}$ at the onset of chaos.

The first step of the renormalization procedure is to define the relevant space $\mathcal{F}$ of evolution maps on which renormalization will act:

$$\mathcal{F} = \left\{ \begin{array}{c} f \,:\, I \mapsto I = [-1,1], \;\; \text{even}, \;\; \text{analytical}, \;\; f(0) = 1 \\ f'(0) = 0 \;\; \text{and} \;\; f \;\; \text{is strictly increasing in} \;\; [-1,0[\\ Sf < 0 \;\; \text{in} \;\; [-1,0[\cup]0,1] \;\; \text{where} \;\; Sf = (f'''/f') \,-\, (3/2)\,(f''/f')^2 \end{array} \right\}$$

As renormalization analysis will show it, this space appears naturally when studying the period-doubling scenario for it is such that generically, *a one-parameter family belonging to it obeys the scenario*. Its elements are known as *unimodal maps*[6]. A typical one is $f_\mu(x) = 1 - \mu x^2$ with $\mu \in]0,2]$.

Let $\lambda_f = f(1)$, so that $f(I) = [\lambda_f, 1]$. If $\lambda_f \geq 0$, $f^n(I) = [a_n, b_n]$ shrinks monotonically either towards the unique fixed point $x_f \in]0,1[$ of f or towards a 2-cycle since $b_{n+1} = f(a_n) \geq a_{n+1} = f(b_n) \geq 0$. Hence, only the case $\lambda_f < 0$ can lead to a complex asymptotic behavior. A relevant renormalization transformation R for investigating *period-doublings* will *double the elementary step*. As it must moreover preserve the normalization $f(0) = 1$, imposed on the elements of $\mathcal{F}$ in order to match the position and the value of their peak with $(0,1)$, renormalization acts in $\mathcal{F}$ according to:

$$Rf(x) = \frac{1}{\lambda_f}\,[f \circ f]\,(x\lambda_f) \qquad\qquad f \in \mathcal{F}, \quad \lambda_f = f(1) < 0$$

To obtain a renormalized map $Rf \in \mathcal{F}$, R has to be restricted to a subset $\mathcal{D}_0 \subset \mathcal{F}$. Another argument supporting this definition of R is the fact that for $f \in \mathcal{F}$ with $\lambda_f < 0$, the graph of $f \circ f$ restricted to $[\lambda_f, -\lambda_f]$ is almost similar to the graph of f, up to a reversal of the space and a dilation of factor $1/|\lambda_f|$ to restore its extension $\Delta x = 2$. As $(Rf)^n(x) = (1/\lambda_f)\; f^{2^n}(\lambda_f x)$, the trajectory generated by the map Rf from an initial position $x_0 \in I$ interpolates every two time steps the trajectory stemming from $x_0\lambda_f$ and rescaled by the factor $1/\lambda_f$. Hence, the transformation $f \to Rf$ reproduces a "geometrical" renormalization of the trajectories, precisely adapted to show the self-similarity of the (finite) attractors associated with special sequences $(\hat{\mu}_j)_{j\geq 0}$ (among which the sequence of bifurcation values).

A major step in this functional analysis is to look for fixed points $R\varphi = \varphi$. It can be shown[8] that the fixed-point equation:

$$[\varphi \circ \varphi]\,[x\varphi^2(0)] = \varphi^2(0)\;\varphi(x) \qquad\qquad \varphi \in \mathcal{F}$$

where necessarily $\varphi(0) = 1$ admits a unique solution in $\mathcal{F}$. This equation expresses an *exact self-similarity* between φ and its iterate $\varphi \circ \varphi$, hence at all time scale between the trajectories they generate (even during the transients).

The following step is to investigate the linear stability of φ with respect to the renormalization action, in order to describe the flow generated by R in $\mathcal{F}$. It can

be shown that the linearized renormalization operator $DR(\varphi)$ has only one unstable eigenvalue $\delta > 1$, which moreover can be expressed explicitely in terms of φ hence computed approximately: $\delta \approx 4,66920\ldots$. The other eigenvalues are of modulus strictly smaller than 1, so that φ is an *hyperbolic fixed point of R*. Its stable manifold $\mathcal{V}^s$ is of codimension one. Both $\mathcal{V}^s$ and $\mathcal{V}^u$ can be characterized explicitely as graphs, at least approximately and in the neighborhood of φ.

B.3.3 - Universality of the scenario

Having constructed and analyzed the renormalization operator R, the last step (but not the least) is to relate the renormalization picture in the space $\mathcal{F}$ of unimodal maps with the period-doubling scenario observed in most one-parameter families of such maps. In the present situation, a *complete nonlinear analysis of renormalization equations is possible*, and it leads rigorously to the following results:

• The fixed point φ is the evolution map of the typical critical system exhibiting at all scales self-similar properties, as is shown by the fixed point equation $R\varphi = \varphi$.

• Among the maps of a family $(f_\mu)_\mu$ in $\mathcal{F}$, the map f_{μ_c} for which transition to chaos occurs is the unique one belonging to the stable manifold $\mathcal{V}^s$ of ϕ. From the very definition of the stable manifold, renormalization action expresses:

$$\lim_{n\to\infty} R^n f_{\mu_c} = \phi.$$

It gives rise to universal critical behavior as:

$$R^n f_{\mu_c} \approx \phi \qquad \text{hence} \qquad R(R^n f_{\mu_c}) \approx R^n f_{\mu_c}$$

for sufficiently large n, which means at sufficiently large time scale since $R^n f_{\mu_c}$ is associated to the time scale $2^n\tau$ if τ is the elementary time step.

• The neighborhood of the hyperbolic fixed point φ in $\mathcal{F}$ of evolution maps can be split into the stable manifold $\mathcal{V}^s$ and the unstable subspace $\mathcal{E}^u$ of dimension 1; for any $f \in \mathcal{F}$, one writes $f = f_c + g$ where $f_c \in \mathcal{V}^s$ and $g \in \mathcal{E}^u$, which defines the (nonlinear) projection P onto $\mathcal{E}^u$ (parallel to $\mathcal{V}^s$) through $Pf = g$. Ignoring the nonlinear contributions, the action of R then writes close to φ:

$$Rf \approx Rf_c + DR(\varphi).g \quad \text{with} \quad Rf_c \in \mathcal{V}^s \quad \text{and} \quad DR(\varphi).g = \delta\, g$$

Hence:

$$P(Rf) \approx \delta\, Pf$$

• It is possible to single out a family $(\mathcal{W}_j)_{j\geq 0}$ of manifolds of codimension 1 having the property that $\mathcal{W}_j$ is the location in $\mathcal{F}$ of the elements f such that f^{2^j} undergoes a pitchfork bifurcation. Hence, given a family $(f_\mu)_\mu$, the bifurcation value μ_j is determined by the condition $f_{\mu_j} \in \mathcal{W}_j$. It is moreover proved that the sequence $(\mathcal{W}_j)_{j\geq 0}$ accumulates on $\mathcal{V}^s$ for $j \to \infty$,

• From the very construction of R, it is obvious that $Rf_{\mu_{j+1}} \in \mathcal{W}_j$. The manifold $\mathcal{W}_j$ being roughly parallel to $\mathcal{V}^s$, it is possible to write at the leading order:

$$Pf_{\mu_j} \;\approx\; P(Rf_{\mu_{j+1}}) \;\approx\; \delta\, Pf_{\mu_{j+1}}$$

• Differentiating with respect to μ (around μ_c) and ignoring nonlinear corrections in $\mu_j - \mu_c$ in the limit $j \to \infty$ gives:

$$(\mu_j - \mu_c)\;\left[P_0\left(\frac{\partial f_\mu}{\partial \mu}\right)(\mu_c)\right] \approx \delta\;(\mu_{j+1} - \mu_c)\;\left[P_0\left(\frac{\partial f_\mu}{\partial \mu}\right)(\mu_c)\right]$$

where $P_0 = DP(\varphi)$. Provided $[P_0.(\partial f_\mu/\partial\mu)(\mu_c)] \neq 0$, which expresses the *transversality* of the family $(f_\mu)_\mu$, the above equation[16] leads to:

$$\lim_{j\to\infty}\;\frac{\mu_j - \mu_c}{\mu_{j+1} - \mu_c} = \delta$$

where δ is universal since it is the maximal eigenvalue of $DR(\varphi)$. As it is possible to compute the first terms of the analytical expansion of φ, it is possible to obtain an explicit value $\delta \approx 4,66920$.

To summarize, *the universality classes of the period doubling scenario is the set of all one-parameter families of unimodal maps which cross transversally the basin of attraction of ϕ with respect to the renormalization action.*

Part C - Analogies and further developments

C.1 - Comparison between Ising model and period-doubling scenario

In this section, I show that the "statistical mechanics renormalization" and the "dynamical systems renormalization" share not only the *same fundamental principles*, but also the *same technical steps*; this assertion is illustrated on the renormalization analysis for the Ising model and for the period-doubling transition to chaos.

C.1.1 - Background and critical features

I shall first unify the formalisms in which the *period-doubling scenario* (illustrated on the "normal form" $f_\mu(x) = 1 - \mu x^2$) and the *Ising model* are described.

[16] It needs much work to make it precise, taking into account all the nonlinear or quadratic corrections; see for example Collet and Eckmann[6].

Ising model	**Period-doubling scenario**
• Extensive variable ("real space") : discrete variable	
position $z_j = ja$, $j \in \mathbf{Z}$ microscopic length scale $\Delta z = a$ N sites, $N \gg 1$ macroscopic length scale $L = Na$	time $t_j = ja$, $j \in \mathbf{Z}$ microscopic time scale $\Delta t = \tau$ N steps, $N \gg 1$ macroscopic time scale $T = N\tau$
• Elementary state variable (labelled $j \in \mathbf{Z}$)	
local magnetization spin $s_j = s(z_j) \in \mathcal{E}_{el} = \{-1, +1\}$	position at a given time $x_j = x(t_j) \in \mathcal{X} = [-1, +1]$
• Collective state variable at the microscopic scale (N components)	
configuration $[s] = (s_j)_{1 \leq j \leq N} \in \mathcal{E}_N$	trajectory $[x] = (x_j)_{0 \leq j \leq N}$
• "Physical" coupling between nearest neighbors $(j, j+1)$	
pair Hamiltonian: $h_j(J) = -Js_j s_{j+1}$ short range a	evolution map: $x_{j+1} = f_\mu(x_j)$ short range τ
• "Structure rule" of the sytem	
dimensionless Hamiltonian	unimodal evolution law f_μ
• Control parameter (organization increases jointly with it)	
$K = \beta J$ (relative magnitude of coupling and thermal motion)	μ (measuring the size of the nonlinearities)
• Statistical description statistical averages $< >$ give a theoretic access to observable quantities	
Boltzmann-Gibbs distribution probability in $\mathcal{E}_N$ statistical equilibrium $< >$ = space average	invariant measure m_μ in $\mathcal{X}$ $dM_{\mu,N}([x]) = dm_\mu(x_0) \prod_{j=1}^{N-1} \delta[x_j - f_\mu(x_0)] dx_j$ stationary regime $< >$ = time average
• Critical behavior	
thermodynamic limit $N \to \infty$ ($L = Na \to \infty$) and $K \to K_c = \infty$ critical phase transition	asymptotic dynamics $N \to \infty$ ($t = N\tau \to \infty$) and $\mu \to \mu_c$ (finite) onset of deterministic chaos
• Divergence of the range ξ of statistical correlations which means criticality	
correlation length (typical size of domains of uniform magnetization)	characteristic time (period of the stable cycle if $\mu < \mu_c$)
• Order parameter which reveals the transition	
statistical average $< s >_K$	smoothed Lyapounov exponent $\bar{L}(\mu)$

C.1.2 - Renormalization analysis and scaling laws

It is now quite straightforward to put in parallel the renormalization procedures, which shows up the typical steps common to most renormalization methods, whatever the nature of the extensive variable(s) is.

• Decimation $N' = N/2$ block $(2j, 2j+1) \to j$	
N spins $\to N/2$ "macro-spins" $s'_j = s_{2j}$ partial sum in partition function	N steps $\to N/2$ "macro-steps" $x'_j = x_{2j}/\lambda_f$ where $\lambda_f = f^2(0)$ iteration of f and rescaling $x' = x/\lambda_f$
• Rescaling in the real space in order to leave invariant the minimum scale	
$z'_j = z_{2j}/2 \to \Delta z' \equiv z'_{j+1} - z'_j = a$	$t'_j = t_{2j}/2 = \to \Delta t' \equiv t'_{j+1} - t'_j = \tau$
• Physical invariants (constant under renormalization action)	
density $1/a$ of degrees of freedom spin density $\lvert s\rvert/a \Leftrightarrow \lvert s'\rvert = \lvert s\rvert = 1$	domain $[-1, 1]$, normalisation $f(0) = 1$ origin x_0 of a trajectory
• Internal consistency of the renormalisation action $\mathcal{R}$: changing the scale of the description does not change the physical system; renormalization hence preserves the macroscopic properties.	
$Z[2N, \mathcal{H}] = C(\mathcal{H})^N \; Z[N, \mathcal{R}\mathcal{H}]$	$f^{2N}(x) = \lambda(f) \; [\mathcal{R}f]^N(x/\lambda(f))$
• Reduction of the apparent range of correlations	
$\xi[\mathcal{R}\mathcal{H}] = \xi(\mathcal{H})/2$ (length)	$\xi[\mathcal{R}f] = \xi(f)/2$ (time)
• Renormalisation r expressed on the control parameter (within a given model)	
$\mathcal{R}[\mathcal{H}(K)] \equiv \mathcal{H}[r(K)]$	$\mathcal{R}(f_{\mu_j}) \approx f_{\mu_{j-1}}$ $\mu_{j-1} \approx r(\mu_j)$
• Critical point	
unstable fixed point $K_c = \infty$ $r(K_c) = K_c$ and $r'(K_c) = 1$	fixed map φ of R critical point if $f_{\mu_c} \in \mathcal{V}^s(\varphi)$ $r(\mu_c) = \mu_c$
• Scaling behavior Dimensionless parameter θ ($\theta_c = 0$) renormalized under the action of $\hat{r}$. Universal scaling laws outcome from the linearized transformation $\hat{r}(\theta) \approx \delta\, \theta$. $\delta = \hat{r}'(0) > 1$ is obtained through a linear analysis of $\hat{r}$ around its fixed point.	
$\theta = e^{-K}$ and $\hat{r}(e^{-K}) = e^{-r(K)}$ $e^{-r(K)} \sim \sqrt{2}\, e^{-K}$ gives $\delta = \sqrt{2}$	$\theta = \mu_c - \mu$ and $\hat{r}(\mu_c - \mu) = \mu_c - r(\mu)$ $\mu_c - r(\mu_{j+1}) \sim \mu_c - \mu_j \sim \delta(\mu_c - \mu_{j+1})$ where $\delta = 4,66920\ldots$
• Scaling law for the statistical correlation length is: $\xi(\theta) \sim \theta^{-\nu}$ where $\xi(\hat{r}(\theta)) = \xi(\theta)/2$ gives $\delta^\nu = 2$ that is, $\nu = (\ln 2)/(\ln \delta)$	
• For a state function X renormalized in $R^X(X)$: $R^X[X(\theta)] = X[\hat{r}(\theta)] \approx X(\delta\theta)$ if $R^X(X) \sim X/k$, then $X(\theta) \sim \theta^{-\alpha}$ where $\alpha = (\ln k)/(\ln \delta)$	

C.2 - Spatio-temporal renormalization

It is quite natural to take advantage of the unification of spatial and temporal renormalizations to introduce a general spatio-temporal renormalization. On the basis of the comparative scheme presented above (§C.1), it is in fact quite straightforward to put forth general principles for the study of large-scale properties of systems exhibiting some generalized critical features. This provides a promising tool for investigating both non-equilibrium statistical mechanics and extended dynamical systems.

C.2.1 - General principles of spatio-temporal renormalization

Spatio-temporal renormalization methods are relevant (and more often necessary):

— for the description of the *large scale dynamics* of a critical system $\mathcal{S}$, that is, of a spatially extended system whose thermodynamic equilibrium state exhibits critical features;

— to investigate their *scaling and universal properties* inside adequate classes. Scaling may involve parameters $K-K_c$, as in static critical phenomena, sizes L and T (finite-size scaling), but also space and time variables. A typical example of this kind of spatio-temporal scaling behavior is a *diffusion law*, or more generally any spatio-temporal phenomenon where growth (or motion) and time are related through scaling laws, at fixed parameter values;

— t *predict* their macroscopic features (for example critical exponents and universal functions) in terms of microscopic variables and parameters.

They give access to the dependence with respect to control parameters of *asymptotic thermodynamic quantities.* Their basic step is an *averaging over a prescribed set of degrees of freedom*, as any one of the following typical sets:

— details of spatial scale $a < \Delta x \leq ka$ and/or of time scales $\tau \leq \Delta t \leq k^\alpha \tau$;

— modes $S_{\bar{q}}(t)$ of characteristic times $\tau \leq \tau_{\bar{q}} \leq k^\alpha \tau$;

— modes $S_{\bar{q}}(t)$ of wave vectors $\Lambda/k < q \leq \Lambda$;

— Fourier components of wave vectors $\Lambda/k < q \leq \Lambda$ and/or of frequencies $\Omega/k^\alpha \leq \omega \leq \Omega$.

Renormalization intends to integrate out these degrees of freedom and to obtain effective contributions describing their effects at larger space-time scales. A key result is *to relate different models*, of increasing spatio-temporal minimum scale, hence to focus the subsequent analysis on this relation rather than on the properties specific to a given model. Another result is to reduce the number of degrees of freedom (number of elementary cells in position-time space or in the conjugate momentum-frequency space). Having fixed the resulting number N, renormalization gives access to larger and larger systems without increasing computational complexity. In that sense, iterating renormalization amounts to perform the limit $N \to \infty$ simultaneously modifying the parameters in such a way that the limiting system is non-trivial (non-trivial fixed point of the renormalization with exact self-similar and scaling properties).

More generally, renormalization can be conceived of as a constructive method for *the study of structural stability of large scale behaviors* that is, stability of the properties observed at a macroscopic scale when modifying the microscopic equations. Indeed, renormalization analysis is designed to describe explicitely, through quantitative relations, the large scale consequences of microscopic structures. As a result, it achieves to group into *universality classes* the microscopic models that induce the same large scale properties. Each element of a class is then structurally stable with respect to any perturbation such that the resulting model is still in the class.

C.2.2 - Out of equilibrium statistical mechanics

I have presented in Part A the main renormalization techniques used for predicting the *static critical properties* observed at large scales inside a system $\mathcal{S}$ in thermal equilibrium when the temperature T reaches a special value T_c. A natural extension is the spatio-temporal renormalization approach aiming at the description (being in fact rather necessary) of the *large scale dynamics* of $\mathcal{S}$ close to a critical point. Fruitful achievements [18,21,23] pave the way for a larger use of renormalization in *non-equilibrium statistical mechanics*.

◇ *Dynamic critical behavior*

Dynamic critical behavior shows up in the following spatio-temporal properties, observed in the thermodynamic limit when T reaches the critical point T_c.

• It first appears in the properties of the overall (spatio-temporal and multi-particle) equilibrium distribution, describing the statistical weight of any possible trajectory in the microscopic phase space of $\mathcal{S}$; here, "equilibrium" means that the distribution is statistically stationary or equivalently that the statistics is invariant under a change of the time origin. A typical feature is the divergence of the correlation time $\tau_{corr}(T_c)=\infty$, together to the previously encountered divergence of the correlation length $\xi(T_c)=\infty$; a scaling behavior is likely to exist for T close to T_c:

$$\xi(T) \sim |T-T_c|^{-\nu} \qquad\qquad \tau_{corr}(T) \sim |T-T_c|^{-\Delta_c}$$

• It also appears in the asymptotic behavior of an arbitrary non-equilibrium distribution. In T_c, its exponential relaxation towards the Boltzmann-Gibbs distribution at T, of leading behavior $e^{-t/\tau_r(T)}$, is generically replaced by a power law $t^{-\kappa}$ which means that the *relaxation time* $\tau_r(T)$ necessary for $\mathcal{S}$ to reach thermal equilibrium diverges in $T=T_c$. This property is known as *critical slowing down*. Moreover, a scaling behavior is likely to exist[17] for T close to T_c:

$$\tau_r(T) \ \sim \ |T-T_c|^{-\Delta_r}$$

[17] $\tau_{corr}(T)$ and $\tau_r(T)$ differ as they characterize respectively equilibrium (stationary) and non-equilibrium (transients) statistical properties; nevertheless, both $\tau_r(T)$ and $\tau_{corr}(T)$ diverge in T_c and it is interesting to compare the associated critical exponents Δ_r and Δ_c.

• Critical singularities for $T = T_c$ also appear at the origin ($q \to 0$, $\omega \to 0$) in the Fourier transform $\hat{f}(\bar{q}, \omega)$ of the linear *response fonctions*[18] (depending on T).

• *Transport coefficients* describing transport phenomena inside $\mathcal{S}$, which depend on the equilibrium temperature T, exhibit singularities in T_c if $\mathcal{S}$ is critical in T_c.

◇ Spatio-temporal renormalization techniques

Analysis is performed in the "semi-conjugate" space, the time variable being still the "real" one t. The relevant functions when describing the small scale dynamics of $\mathcal{S}$ are the *modes* $S_{\bar{q}}(t)$, defined as the components of the spatial Fourier transform of the order parameter field $\bar{s}(\bar{x}, t)$. In an efficient model, a preliminary coarse-graining has been performed in order to reproduce the lower bound on space and time scales. Hence, only modes $||\bar{q}|| \equiv q \leq \Lambda$ intervene, so that equations of evolution for the modes $S_{\bar{q}}(t)$ already contain a noise term reproducing the effective influence at scales $\Delta x > 2\pi/\Lambda$ of smaller scale mechanisms. These so-called *mode-coupling equations*[**23**] also include a phenomenological *damping term*. They describe the relaxation of each mode $S_{\bar{q}}(t)$ towards its equilibrium expression $S_{\bar{q}}(eq)$.

The basic renormalization idea is to eliminate the *fast modes* $S_{\bar{q}}(t)$ with short relaxation time $\tau(\bar{q})$, for the reason that they are not perceived in a macroscopic observation. As usual, their influence onto the evolution of the slow modes is taken into account in effective contributions which produces *renormalized parameters* and a *renormalized noise term*. Explicit procedure begins with a resolution of the evolution equations of the fast modes intending to express these modes in terms of the slow modes and of the initial noise term; it is followed up with a partial statistical averaging over the fast modes of the noise; it ends as usual with adequate rescalings in order to restore minimum scales and keep constant physical invariants.

One thus achieves to prove that relaxation times obey a universal scaling behavior:

$$\tau(\bar{q}, T) \ \sim \ \xi(T)^z \ \Theta[q\xi(T)] \qquad (q < \Lambda, \ \ T \to T_c)$$

where $\xi(T) \sim |T - T_c|^{-\nu}$ is the static correlation length and Θ a continuous function on $[0, +\infty[$. The real $z > 0$ is called the *dynamic exponent*. The limiting behavior of $\mathcal{S}$ under the iterated action of renormalization indicates that the overall relaxation time coincide with the correlation time:

$$\tau_r(T) \sim \tau_{corr}(T) \sim |T - T_c|^{-\nu z} \qquad (\text{hence} \quad \Delta_r = \Delta_c = \nu z)$$

which explains analytically why critical slowing down and strong temporal correlations always appear jointly. It appears that there is more *dynamic universality classes* than static ones. A standard reference on this subject is a review paper of Halperin and Hohenberg[**18**] and a book by Halperin[**17**].

[18] Let us recall that $\hat{f}(\bar{q}, \omega)$ is the multiplicative factor relating the Fourier components $\hat{A}(\bar{q}, \omega)$ of an applied field or excitation and those $\hat{s}(\bar{q}, \omega)$ of the local order parameter.

C.2.3 - Partial differential equations

Spatio-temporal renormalization is also well-suited when studying the *asymptotics of the solutions of partial differential equations* when a scaling behavior is likely to exist[3,5]. For example, renormalization turns to be helpful when investigating the *large scale dynamics* and the *scaling invariance*, if any, of the equation:

$$\partial_t u = \mathcal{D}(\mu, \partial_{\bar{x}})[u]$$

for a real or complex spatio-temporal function $u(\bar{x}, t)$; μ is a set of parameters and $\mathcal{D}$ a differential operator. It will jointly transform ($k > 1$ being a scaling factor):

— the unknown function: $u(\bar{x}, t) \longrightarrow [R_k u](\bar{x}, t) \equiv f_k[u(k\bar{x}, k^\alpha t)]$

— the parameters: $\mu \longrightarrow r_k \mu$

in such a way that $\partial_t[R_k u]$ is the closest possible to $\mathcal{D}(r_k\mu, \partial_{\bar{x}})[R_k u]$. Nevertheless, additional terms generally appear, so that the functional operator $\mathcal{D}$ must also be modified into $\mathcal{R}_k[\mathcal{D}]$ in order to give a consistent renormalized picture:

$$\partial_t[R_k u] = \mathcal{R}_k[\mathcal{D}](r_k\mu, \partial_{\bar{x}})[R_k u].$$

The renormalization procedure is carried over by looking at a fixed point $(u^*, \mu^*, \mathcal{D}^*)$ of the transform:

$$(u, \mu, \mathcal{D}) \quad \longrightarrow \quad (R_k u, r_k \mu, \mathcal{R}_k[\mathcal{D}])$$

The function u^* describes the self-similar asymptotic behavior of the solution. Performing a linear (and if possible nonlinear) stability analysis of the renormalization action around this fixed point in some adequate space $\{(u, \mu, \mathcal{D})\}$ will make precise the irrelevant modifications (associated with stable directions) of the initial problem hence will determine its universality class. The power of such an analysis is to study simultaneously the changes in the large scale behavior:

— when varying the initial conditions $u(\bar{x}, 0)$ (*nonlinear stability*),

— when varying the parameters μ (*bifurcation diagram*),

— when varying the differential operator $\mathcal{D}$ (*structural stability*).

It allows for example to describe which perturbations of $\mathcal{D}$ do not modify the asymptotics of the solution (*irrelevant* perturbations).

Extending the analysis to stochastic partial differential equations, a renormalization approach can be of great help to split the space of such equations into universality classes. It allows to describe the *irrelevant stochastic perturbations of a deterministic equation*, hence to investigate the *structural stability* of deterministic models with respect to noise, especially with respect to internal noise induced by microscopic fluctuations, that cannot be avoided nor even reduced. Renormalization intends *to extract the deterministic core of a stochastic partial differential equation*[15]. It will recursively include the stochastic contributions into deterministic parameters until the renormalized evolution is purely deterministic. It is hoped to construct the macroscopic deterministic

model embedding the microscopic stochastic description and to provide the whole class of stochastic models that can be replaced at a sufficient large scale by this deterministic model.

It is in fact possible to extend the scope of such an analysis to situations where stochasticity is essential instead of being introduced as a correction in a initially deterministic framework, namely *stochastic processes* and *random walks*[21]. Spatio-temporal renormalization methods, now acting on transition probabilities, are again a powerful tool for investigating *large-scale dynamics*, *self-similar properties* and *structural stability*.

Conclusion

Let me summarize the typical results that a renormalization analysis achieves for a physical system $\mathcal{S}$:

— it gives access to the limiting behavior $N \to \infty$ where N is the spatial or temporal size of $\mathcal{S}$. It describes its dependence with respect to the control parameters K, especially in case of long-range correlations which invalidate the usual approaches. It encompasses the difficulties that appear when the limits $\lim_{N\to\infty}$ and $\lim_{K\to K_c}$ do not commute;

— it predicts consequences at all scales of the microscopic structures or mechanisms and shows up their hierarchical organization, if any;

— it proves self-similarity properties of $\mathcal{S}$ and gives quantitative results about the associated scaling behavior (critical exponents, universal functions, universality classes);

— it determines perturbations of the microscopic model for $\mathcal{S}$ which are irrrelevant or on the contrary relevant with respect to its macroscopic properties;

— it allows to investigate structural stability of macroscopic models, especially with respect to internal noise induced by microscopic statistical fluctuations;

— it gives access to finite-size scaling (involving K and also N) and non-universal corrections to scaling behavior (non-universal scale factors, logarithmic corrections, nonlinear terms in the renormalization flow, ...).

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