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The adiabatic piston

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1 Introduction

1.1 Macroscopic problem

The "adiabatic piston" is an old problem of thermodynamics which has had a long and controversial history. It is the simplest example concerning the time evolution of an adiabatic wall, i.e. a wall which does not conduct heat. The system consists of a gas in a cylinder divided by an adiabatic wall (the piston). Initially the piston is held fixed by a clamp and the two gases are in thermal equilibrium characterized by $(p^{\pm}, T^{\pm}, N^{\pm})$ where the index $-/+$ refers to the gas on the left/right side of the piston and (p, T, N) denote the pressure, the temperature and the number of particles (Fig. 1). Since the piston is adiabatic the whole system remains in equilibrium even if $T^- \neq T^+$. At time $t = 0$, the clamp is removed and the piston is let free to move without any friction in the cylinder. The question is to find the final state, i.e. the final position X_f of the piston and the parameters $(p_f^{\pm}, T_f^{\pm})$ of the gases.

In the late 1950s, using the two laws of equilibrium thermodynamics (i.e. thermostatics), Landau and Lifshitz concluded that the adiabatic piston will evolve toward a final state where $p^-/T^- = p^+/T^+$. Later Callen [1] and others, realized that the maximum entropy condition implies that the system will reach mechanical equilibrium where the pressures are equal $p_f^- = p_f^+$; however nothing could be said concerning the final position X_f or the final temperatures $T_f^{\pm}$ which should depend explicitly on the viscosity of the fluids. It thus became a controversial problem since one was forced to accept that the two laws of thermostatics are not sufficient to predict the final state as soon as adiabatic movable walls are involved (see early references in [3]).

Experimentally the adiabatic piston has been used already before 1924 to measure the ratio c_p/c_v of the specific heats of gases. In 2000 new measurements have shown that one has to distinguish between two regimes, corresponding to weak damping or strong damping, with very different properties, e.g. for weak damping the frequency of oscillations corresponds to adiabatic oscillations, while for strong damping it corresponds to isothermal oscillations.

<Figure 1 near here>

1.2 Microscopic problem

The "adiabatic piston" was first considered from a microscopic point of view by Lebowitz who introduced in 1959 a simple model to study heat conduction. In this model the gas consists of point particles of mass m making purely elastic collisions on the wall of the cylinder and on the piston. Furthermore the gas is very dilute so that the equation of state $p = nk_BT$ is satisfied at equilibrium, where n is the density of particles in the gas and k_B the Boltzmann constant. The adiabatic piston is taken as a heavy particle of mass $M \gg m$ without any internal degree of freedom. Using this same model Feynman gave a qualitative analysis in *Lecture of Physics I* of 1965 [7]. He argued intuitively but correctly that the system should converge first toward a state of mechanical equilibrium where $p^- = p^+$ and then very slowly toward thermal equilibrium. This approach toward thermal equilibrium is associated with the "wiggles" of the piston induced by the random collisions with the atoms of the gas. Of course this stochastic behavior is not part of thermodynamics and the evolution beyond the mechanical equilibrium can not appear in the macroscopical framework assuming that the piston does not conduct heat.

From a microscopical point of view one is confronted with two different problems: the approach toward mechanical equilibrium in the absence of any a priori friction, where the entropy of both gases should increase and, on a different time scale the approach toward thermal equilibrium, where the entropy of one gas should decrease but the total entropy increase.

The conceptual difficulties of the problem beyond mechanical equilibrium come from the following intuitive reasoning. When the piston moves toward the hotter gas, the atoms of the hotter gas gain energy, while those of the cooler gas loose energy. When the piston moves toward the cooler side it is the opposite. Since on the average the hotter side should cool down and the cold side warm up we are led to conclude that on the average the piston should move toward the colder side. On the other hand, from $p = nk_BT$, the piston should move toward the warmer side to maintain pressure balance.

In 1996, Crosignani, Di Porto and Segev introduced a kinetic model to obtain equations describing the adiabatic approach toward mechanical equilibrium. Starting with the microscopical model introduced by Lebowitz, Gruber, Piasecki and Frachebourg, later joined by Lesne and Pache, initiated in 1998 a systematic investigation of the adiabatic piston within the framework of statistical mechanics, together with a large number of numerical simulations. This analysis was based on the fact that m/M is a very small parameter to investigate expansions in powers of m/M (see [6], [5] and reference therein). An approach using dynamical systems methods was then developed by Sinai together with Lebowitz and Chernov [9][2]. An extension to hard-disks particles was analyzed at the same time by Kestemont, Vandenbroeck and Malek-Mansour [8]. In the last five years several other authors have contributed to this subject.

The general picture which emerges from all the investigations is the following:

For an infinite cylinder, starting with mechanical equilibrium $p^- = p^+ = p$, the piston evolves to a stationary stochastic state with non-zero velocity toward the warmer side

$$\langle V \rangle = \frac{m}{M} \sqrt{\frac{\pi k_B}{8m}} \, (\sqrt{T^+} - \sqrt{T^-}) + o\left(\frac{m}{M}\right) \tag{1}$$

with relaxation time

$$\tau = \frac{M}{A} \sqrt{\frac{\pi k_B}{8m}} \, \frac{1}{p} \left(\frac{1}{\sqrt{T^-}} + \frac{1}{\sqrt{T^+}}\right)^{-1} \tag{2}$$

where M/A is the mass per unit area of the piston. In this state the piston has a temperature $T_P = \sqrt{T^+T^-}$ and there is a heat flux

$$j_Q = (\sqrt{T^-} - \sqrt{T^+}) \, \frac{m}{M} \sqrt{\frac{8k_B}{m\pi}} \, p + o\left(\frac{m}{M}\right) \qquad (p^- = p^+ = p) \tag{3}$$

For a finite cylinder and $p^+ \neq p^-$, the evolution proceeds in four different stages. The first two are deterministic and adiabatic. They correspond to the thermodynamic evolution of the (macroscopic) adiabatic piston. The last two stages, which go beyond thermodynamics, are stochastic with heat transfer across the piston. More precisely:

(1) in the first stage whose duration is the time needed for the shock wave to bounce back on the piston, the evolution corresponds to the case of the infinite cylinder (with $p^- \neq p^+$). If $R = Nm/M > 10$, the piston will be able to reach and maintain a constant velocity

$$V = (p^- - p^+)\sqrt{\frac{\pi k_B}{8m}}\,\frac{\sqrt{T^- T^+}}{p^+\sqrt{T^-} + p^-\sqrt{T^+}} + \mathcal{O}\left(\frac{m}{M}\right) \qquad \text{for} \quad |p^- - p^+| \ll 1 \tag{4}$$

(2) in the second stage the evolution toward mechanical equilibrium is either weakly or strongly damped depending on R. If $R < 1$, the evolution is very weakly damped, the dynamics takes place on a time scale $t' = \sqrt{R}\,t$, and the effect of the collisions on the piston is to introduce an external potential $\phi(X) = c_1/X^2 + c_2/(L - X)^2$. On the other hand if $R > 4$, the evolution is strongly damped (with two oscillations only) and does not depend on M nor R;

(3) after mechanical equilibrium has been reached, the third stage is a stochastic approach toward thermal equilibrium associated with heat transfer across the piston. This evolution is very slow and exhibits a scaling property with respect to $t' = mt/M$;

(4) after thermal equilibrium has been reached ($T^- = T^+$, $p^- = p^+$) in a fourth stage the gas will evolve very slowly toward a state with Maxwellian distribution of velocities, induced by the collision with the stochastic piston.

The general conclusion is thus that a wall which is adiabatic when fixed will become a heat conductor under a stochastic motion. However it should be stressed that the time involved to reach thermal equilibrium will be several order of magnitude larger than the age of the Universe for a macroscopical piston and no reasonable person will call such a wall a heat conductor. However for mesoscopic systems the effect of stochasticity may lead to very interesting properties, such as shown by Vandenbroeck, Meurs and Kawai in their investigations of Brownian (or biological) motors [10].

2 Microscopical model

The system consists of two fluids separated by an "adiabatic" piston inside a cylinder with x-axis, length L, area A. The fluids are made of $N^\pm$ identical light particles of mass m. The piston is a heavy flat disk, without any internal degree of freedom, of mass $M \gg m$, orthogonal to the x-axis, and velocity parallel to this x-axis. If the piston is fixed at some position X_0, and if the two fluids are in thermal equilibrium characterized by $(p_0^\pm, T_0^\pm, N^\pm)$, then they will remain in equilibrium forever even if $T_0^+ \neq T_0^-$: it is thus an "adiabatic piston" in the sense of thermodynamics. At a certain time $t = 0$, the piston is let free to move and the problem is to study the time evolution. To define the dynamics, we consider that the system is purely Hamiltonian, i.e. the particles and the piston move without any friction according to the laws of mechanics. In particular, the collisions between the particles and the walls of the cylinder, or the piston, are purely elastic and the total energy of the system is conserved. In most studies, one considers that the particles are point particles making purely elastic collisions. Since the piston is bound to move only in the x-direction, the velocity components of the particles in the transverse directions play no role in this problem. Moreover since there is no coupling between the components in the x and transverse directions, one can simplify the model further by assuming that all probability distributions are independent of the transverse coordinates. We are thus led to a formally one-dimensional problem (except for normalizations). Therefore in this review we consider that the particles are non-interacting and all velocities are parallel to the x-axis. From the collision law, if v and V denote the velocities of a particle and the piston before a collision, then under the collision on the piston:

$$v \to v' = 2V - v + \alpha(v - V) \qquad V \to V' = V + \alpha(v - V) \tag{5}$$

where:

$$\alpha = \frac{2m}{M + m} \tag{6}$$

Similarly, under a collision of a particle with the boundary at $x = 0$ or $x = L$:

$$v \to v' = -v \tag{7}$$

Let us mention that more general models have also been considered, e.g. the case where the two fluids are made of point particles with different masses $m^\pm$, or two-dimensional models where the particles are hard-disks. However no significant differences appear in these more general models and we restrict this review to the simplest case.

One can study different situations: $L = \infty$, L finite, and $L \to \infty$. Furthermore, taking first M and A finite, one can investigate several limits:

a) thermodynamic limit for the piston only. In this limit, L is fixed (finite or infinite) and $A \to \infty$, $M \to \infty$, keeping constant the initial densities $n^\pm$ of the fluids and the parameter

$$\gamma = \frac{2mA}{M+m} = \alpha A \sim 2m\ \frac{A}{M} \tag{8}$$

If L is finite, this means that $N^\pm \to \infty$ while keeping constant the parameters

$$R^\pm = \frac{mN^\pm}{M} = \frac{M^\pm_{gas}}{M} \tag{9}$$

b) thermodynamic limit for the whole system, where $L \to \infty$ and $A \sim L^2$, $N^\pm \sim L^3$. In this limit, space and time variables are rescaled according to $x' = x/L$ and $t' = t/L$. This limit (b) can be considered as a limiting case of (a) where $R^\pm \sim \sqrt{A} \to \infty$ (and time is scaled).

c) continuum limit where L and M are fixed and $N^\pm \to \infty$, $m \to 0$ keeping constant $M^\pm_{gas}$, i.e. $R^\pm = cte$.

The case L infinite and the limit (a) have been investigated using statistical mechanics (Liouville or Boltzmann's equations). On the other hand, the limit (b) has been studied using dynamical system methods, reducing first the system to a billiard in a $(N^+ + N^- + 1)$-dimensional polyhedron. The limit (c) has been introduced to derive hydrodynamical equations for the fluids.

In this review, we present the approach based on statistical mechanics. Although not as rigorous as (b) on a mathematical level, it yields more informations on the approach toward mechanical and thermal equilibrium. Moreover, it indicates what are the open problems which should be mathematically solved. In all investigations, advantage is taken of the fact that m/M is very small and one introduces the small parameter

$$\epsilon = \sqrt{m/M} \ll 1 \tag{10}$$

Let us note that ϵ measures the ratio of thermal velocities for the piston and a fluid particle, while $\alpha \sim \epsilon^2$ measures the ration of velocity change during a collision.

3 Starting point: exact equations

Using the statistical point of view, the time evolution is given by Liouville's equation for the probability distribution on the whole phase space for $(N^+ + N^- + 1)$ particles, with L, A, $N^\pm$ and M finite. Initially ($t \le 0$) the piston is fixed at $(X_0, V_0 = 0)$ and the fluids are in thermal equilibrium with homogeneous densities $n_0^\pm$, velocity distributions $\varphi_0^\pm(v) = \varphi_0^\pm(-v)$ and temperatures

$$T_0^\pm = m \int_{-\infty}^{\infty} dv\ n_0^\pm\ \varphi_0^\pm(v)\, v^2 \tag{11}$$

Integrating out the irrelevant degrees of freedom, the Liouville's equation yields the equations for the distribution $\rho^\pm(x, v; t)$ of the right and left particles:

$$\partial_t \rho^\pm(x, v; t) + v \partial_x \rho^\pm(x, v; t) = I^\pm(x, v; t) \tag{12}$$

The collision term $I^{\pm}(x,v;t)$ is a functional of $\rho_{\pm,P}(X,v;X,V;t)$, the two-point correlation function for a right (resp. left) particle at $(x=X,v)$ and the piston at (X,V). Similarly, one obtains for the velocity distribution of the piston

$$\begin{aligned}\partial_t\Phi(V;t) = \quad & A \int_{-\infty}^{\infty}(V-v)\,[\theta(V-v)\rho^-_{surf}(v';V';t)+\theta(v-V)\rho^-_{surf}(v;V;t)]\,dv \\ - \quad & A\int_{-\infty}^{\infty}(V-v)\,[\theta(v-V)\rho^+_{surf}(v';V';t)+\theta(V-v)\rho^+_{surf}(v;V;t)]\,dv \end{aligned} \tag{13}$$

where (v',V') are given by Eq. 5 and

$$\rho^{\pm}_{surf}(v;V;t) = \int_{-\infty}^{\infty} dX\ \rho_{\pm,P}(X,v;X,V;t) \tag{14}$$

We thus have to solve Eqs. 12-13 with initial conditions:

$$\begin{cases} \rho^-(x,v;t=0) = n_0^-\ \varphi_0^-(v)\theta(x)\theta(X_0-x) \\ \rho^+(x,v;t=0) = n_0^+\ \varphi_0^+(v)\theta(L-x)\theta(x-X_0) \\ \Phi(V;t=0)=\delta(V) \end{cases} \tag{15}$$

Using the fact that $\alpha = 2m/(M+m) \ll 1$, we can rewrite Eq. 13 as a formal series in powers of α

$$\partial_t\Phi(V;t) = \gamma\sum_{k=1}^{\infty}\frac{(-1)^k\alpha^{k-1}}{k!}\left(\frac{\partial}{\partial V}\right)^k\ \widetilde{F}_{k+1}(V;t) \tag{16}$$

$$\tilde{F}_k(V;t) = \int_V^{\infty}(v-V)^k\rho^-_{surf}(v;V;t)dv - \int_{-\infty}^{V}(v-V)^k\rho^+_{surf}(v;V;t)dv \tag{17}$$

from which one obtains the equations for the moments of the piston velocity:

$$\frac{1}{\gamma}\ \frac{d\langle V^n\rangle}{dt} = \sum_{k=1}^{n}\alpha^{k-1}\ \frac{n!}{k!(n-k)!}\int_{-\infty}^{\infty} dV\ V^{n-k}\ \tilde{F}_{k+1}(V;t) \tag{18}$$

However we do not know the two-point correlation functions.

If the length of the cylinder is infinite, the condition $M \gg m$ implies that the probability for a particle to make more than one collision on the piston is negligible. Alternatively, one could choose initial distributions $\varphi_0^{\pm}(v)$ which are zero for $|v| < v_{min}$, where v_{min} is taken such that the probability of a recollision is strictly zero. Therefore, if $L=\infty$ one can consider that before a collision on the piston the particles are distributed with $\varphi_0^{\pm}(v)$ for all t, and the two-point correlation functions factorize, i.e.

$$\begin{cases} \rho^-_{surf}(v;V;t) = \rho^-_{surf}(v;t)\ \Phi(V;t) & \text{if}\quad v>V \\ \rho^+_{surf}(v;V;t) = \rho^+_{surf}(v;t)\ \Phi(V;t) & \text{if}\quad v<V \end{cases} \tag{19}$$

where for $L=\infty$, $\rho^{\pm}_{surf}(v;t) = n_0^{\pm}\varphi_0^{\pm}(v)$ and thus the conditions to obtain Eq. 18 are satisfied.

If L is finite, one can show that the factorization property Eq. 19 is an exact relation in the thermodynamic limit for the piston ($A\to\infty$, $M/A = cte$). For finite L and finite A we introduce

Assumption 1: (Factorization condition)

Before a collision the two-point correlation function have the factorization property Eq. 19 to first order in α.

Under the factorization condition, we have

$$\tilde{F}_k(V;t) = F_k(V;t)\ \Phi(V;t) \tag{20}$$

with

$$F_k(V;t) = \int_V^{\infty} dv(v-V)^k \rho^-_{surf}(v;t) - \int_{-\infty}^{V} dv(v-V)^k \rho^+_{surf}(v;t) = F_k^-(V;t) - F_k^+(V;t) \tag{21}$$

and from Eq. 18

$$\left(\frac{M}{A}\right)\frac{d}{dt}\langle V\rangle \quad = \quad M\alpha\ \langle F_2(V;t)\rangle_\Phi \tag{22}$$

$$\left(\frac{M}{A}\right)\frac{d}{dt}\langle V^2\rangle \quad = \quad M\alpha\ \left[\langle V F_2(V;t)\rangle_\Phi + \alpha\langle F_3(V;t)\rangle_\Phi\right] \tag{23}$$

Moreover from Eq. 12 and Eq. 20 follows that the (kinetic) energies satisfy

$$\frac{d}{dt}\left(\frac{< E^\pm >}{A}\right) = \pm\ M\alpha\left[\langle F_2^\pm(V;t)\rangle_\Phi \bar{V} + \langle (V-\bar{V})F_2^\pm(V;t)\rangle_\Phi + \frac{\alpha}{2}\langle F_3^\pm(V;t)\rangle_\Phi\right] \tag{24}$$

which implies conservation of energy.

From the first law of thermodynamics,

$$\frac{d}{dt}\left(\frac{< E^\pm >}{A}\right) = \frac{1}{A}\ \left[P_W^{P\to\pm} + P_Q^{P\to\pm}\right] \tag{25}$$

where $P_W^{P\to\pm}$ and $P_Q^{P\to\pm}$ denote the work- and heat-power transmitted by the piston to the fluid, we conclude from Eq. 22 and Eq. 25 that the heat flux is

$$\frac{1}{A}P_Q^{P\to\pm} = \pm\ M\alpha\left[\langle (V-\bar{V})F_2^\pm(V;t)\rangle_\Phi + \frac{\alpha}{2}\langle F_3^\pm(V;t)\rangle_\Phi\right] \tag{26}$$

Since $\alpha \ll 1$, it is interesting to introduce the irreducible moments (see Sec. 6)

$$\Delta_r = \langle (V-\bar{V})^r\rangle_\Phi \tag{27}$$

and the expansion around $\bar{V} = \langle V\rangle_t$

$$F_n^\pm(V;t) = \sum_{r=0}^{\infty}\frac{1}{r!}\ F_n^{(r,\pm)}(\bar{V})(V-\bar{V})^r \tag{28}$$

from which one obtains equations for $d\Delta_r/dt$. In particular, using the identity

$$F_3^{(r+1,\pm)} = -3F_2^{(r,\pm)} \qquad\qquad F_2^{(r+2,\pm)} = 2F_0^{(r,\pm)} \tag{29}$$

in (22) and (24), we have

$$\langle F_2^\pm(V;t)\rangle_\Phi = F_2^\pm(\bar{V};t) + \sum_{r\geq 0}\frac{2}{(2+r)!}\ F_0^{(r,\pm)}\ \Delta_{2+r} \tag{30}$$

$$\frac{d}{dt}\left(\frac{< E^\pm >}{A}\right) = \pm\ M\alpha\ [\langle F_2^\pm(V;t)\rangle_\Phi \bar{V} + \frac{\alpha}{2}F_3^\pm(\bar{V};t) + \frac{1}{2}\sum_{r\geq 2}\frac{1}{r!}\ (2r-3\alpha)\ F_2^{(r-1,\pm)}(\bar{V};t)\ \Delta_r\ ] \tag{31}$$

Depending on the questions or approximations one wants to study, either the distribution $\Phi(V;t)$ (Sec. 4), or the moments $\langle V^n \rangle_t$ (Secs. 5,6) will be the interesting objects. Finally with the condition (19) one can take Eq. 12 for $x \neq X_t$ and impose the boundary conditions at X_t:

$$\begin{cases} \rho^-(X_t, v; t) = \rho^-(X_t, v'; t) & \text{if} \quad v < V_t \\ \rho^+(X_t, v; t) = \rho^+(X_t, v'; t) & \text{if} \quad v > V_t \end{cases} \tag{32}$$

and similarly for $x = 0$ and $x = L$ with $v' = -v$.

Let us note that this factorization condition is of the same nature as the molecular chaos assumption introduced in kinetic theory, and with this condition the equation (13) yields the *Boltzmann's equation* for this model.

In the following, to obtain explicit results in function of the initial temperatures $T_0^\pm$, we shall take Maxwellian distributions $\varphi_0^\pm(v)$ and initial conditions $(p_0^\pm, T_0^\pm, n_0^\pm)$ such that the velocity of the piston remains small (i.e. $|\langle V \rangle_t| \ll |\langle v^\pm \rangle_0|$).

4 Distribution $\Phi(V;t)$ for the infinite cylinder ($L = \infty$)

To lowest order in $\epsilon = \sqrt{m/M}$, and assuming $|1 - p^+/p^-|$ is of order ϵ, one obtains from Eq. 16 the usual Fokker-Planck equation whose solution gives

$$\Phi_0(V;t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\Delta(t)} e^{-\dfrac{(V - \bar{V}(t))^2}{2\Delta^2(t)}} \tag{33}$$

with

$$\begin{cases} \bar{V}(t) = (p^- - p^+) \sqrt{\dfrac{\pi k_B}{8m}} \left[\dfrac{p^+}{\sqrt{T^+}} + \dfrac{p^-}{\sqrt{T^-}} \right]^{-1} (1 - e^{-\lambda t}) \\ \lambda = \dfrac{A}{M} \sqrt{\dfrac{8m}{\pi k_B}} \left[\dfrac{p^+}{\sqrt{T^+}} + \dfrac{p^-}{\sqrt{T^-}} \right] \\ \Delta^2(t) = \dfrac{k_B}{M} \sqrt{T^- T^+} \dfrac{p^+\sqrt{T^+} + p^-\sqrt{T^-}}{p^+\sqrt{T^-} + p^-\sqrt{T^+}} (1 - e^{-2\lambda t}) \end{cases} \tag{34}$$

where we have dropped the index "zero" on the variable $T^\pm$, $n^\pm$and used the equation of state $p^\pm = n^\pm k_B T^\pm$.

In conclusion in the thermodynamic limit for the piston ($M \to \infty$, M/A fixed) Eq. 33 shows that the evolution is deterministic, i.e. $\Phi(V;t) = \delta(V - \bar{V}(t)$, where the velocity $\bar{V}(t)$ of the piston tends exponentially fast toward stationary value $V_{stat} = \bar{V}(\infty)$ with relaxation time $\tau = \lambda^{-1}$.

Let us note that for $p^+ = p^-$, we have $\bar{V}(t) \equiv 0$ and the evolution (33) is identical to the Ornstein-Uhlenbeck process of thermalization of the Brownian particle starting with zero velocity and friction coefficient λ.

The analysis of (16) to first order in ϵ yields then

$$\Phi(V;t) = [1 + \epsilon \sum_{k=0}^{3} a_k(t)(V - \bar{V}(t))^k] \; \Phi_0(V;t) \tag{35}$$

where $a_k(t)$ can be explicitly calculated and $a_0(t) = -\Delta^2(t) a_2(t)$ because of the normalization condition. Moreover $a_2(t) \sim (p^- - p^+)$, i.e. $a_2(t) = 0$ if $p^- = p^+$. From (35) one obtains

$$\langle V \rangle_t = \sqrt{\frac{\pi k_B}{8m}} \frac{\sqrt{T^- T^+}}{p^+\sqrt{T^-} + p^-\sqrt{T^+}} \times$$

$$
\begin{aligned}
&\{(p^- - p^+)\,(1-e^{-\lambda t}) \\
&+(p^- - p^+)^2\,\frac{\pi}{8}\,\frac{(p^-T^+ - p^+T^-)}{(p^+\sqrt{T^-}+p^-\sqrt{T^+})^2}\,(1-2\lambda t e^{-\lambda t}-e^{-2\lambda t}) \\
&+\frac{m}{M}\,\frac{1}{\sqrt{T^-T^+}}\,(p^-T^+ - p^+T^-)\,\left(\frac{p^+\sqrt{T^+}+p^-\sqrt{T^-}}{p^+\sqrt{T^-}+p^-\sqrt{T^+}}\right)\,(1-e^{-\lambda t})^2\}
\end{aligned}
\tag{36}
$$

and

$$
\langle V^2\rangle_t - \langle V\rangle_t^2 = \Delta^2(t)\left[1+\sqrt{\frac{m}{M}}\,2\Delta^2(t)a_2(t)\right] \tag{37}
$$

From Eq. 36, we now conclude that for equal pressures $p^- = p^+$, the piston will evolve stochastically to a stationary state with nonzero velocity toward the warmer side

$$
\begin{cases}
\langle V\rangle_{stat} = \dfrac{m}{M}\sqrt{\dfrac{\pi k_B}{8m}}\,(\sqrt{T^+}-\sqrt{T^-}) \\[2ex]
\langle V^2\rangle_{stat} - \langle V\rangle_{stat}^2 = \dfrac{k_B}{M}\,\sqrt{T^-T^+}
\end{cases}
\qquad \text{if} \quad p^- = p^+ \tag{38}
$$

Let us remark that we have established Eq. 35 under the condition that $|1-p^+/p^-| = \mathcal{O}(\epsilon)$, but as we shall see in Sec 5.2, the stationary value V_{stat} obtained from Eq. 36 remains valid whenever $|(1-p^+/p^-)(1-\sqrt{T^+/T^-})| \ll 1$.

5 Moments $\langle V^n\rangle_t$: Thermodynamic limit for the piston

5.1 General equations: Adiabatic evolution

In this limit $M \to \infty$, $\alpha \to 0$, $\gamma = \alpha A$ is fixed and Eq. 16 reduces to

$$
\partial_t \Phi(V;t) = -\,\gamma\,\frac{\partial}{\partial V}\tilde{F}_2(V;t) \tag{39}
$$

Integrating (39) with initial condition $\Phi(V;t=0) = \delta(V)$ yields

$$
\Phi(V,t) = \delta(V-\bar{V}(t)), \qquad \text{i.e.} \quad \langle V^n\rangle_t = \langle V\rangle_t^n \tag{40}
$$

where

$$
\frac{d}{dt}V(t) = \gamma\,F_2(V(t);t), \qquad\qquad V(t=0)=0 \tag{41}
$$

Moreover

$$
\tilde{F}_2(V;t) = F_2(V;t)\,\Phi(V;t) \tag{42}
$$

and

$$
\rho_{\pm,P}(X,v;X,V;t) = \rho^{\pm}(x,v;t)\delta(X-X(t))\delta(V-V(t)) \tag{43}
$$

(where $dX(t)/dt = V(t)$, $X(t=0) = X_0$).

In conclusion, as mentioned in Sec. 3, *in this limit the factorization condition Eq. 19 is an exact relation.* Let us note that $\rho^{\pm}_{surf}(v;t) = \rho^{\pm}_{surf}(2V-v;t)$ if $v > V(t)$ (on the right) or $v < V(t)$ (on the left). Let us also remark that $2mF_2^{\pm}(V(t);t)$ represents the effective pressure from the right/left exerted on the piston. Moreover, since for any distribution $\rho^{\pm}_{surf}(v;t)$, the functions $F_2^-(V;t)$ and $-F_2^+(V;t)$ are monotonically decreasing, we can introduce the decomposition:

$$
p^{\pm}_{surf} = 2mF_2^{\pm}(V;t) = \hat{p}^{\pm}\;\pm\;\left(\frac{M}{A}\right)\,\lambda^{\pm}(V;t)\,V \tag{44}
$$

where the static pressure at the surface is $\hat{p}^{\pm}(t) = p^{\pm}_{surf}(V = 0; t)$ and the friction coefficients $\lambda^{\pm}(V; t)$ are strictly positive. The evolution (41) is thus of the form

$$\frac{d}{dt}V(t) = \frac{A}{M}\,(\hat{p}^{-} - \hat{p}^{+}) - \lambda(V)\,V \tag{45}$$

and involves the difference of static pressure and the friction coefficient $\lambda(V) = \lambda^{-}(V) + \lambda^{+}(V)$. Finally from Eq. 12 we obtain the evolution of the (kinetic) energy per unit area for the fluids in the left and right compartments:

$$\frac{d}{dt}\left(\frac{< E^{\pm} >}{A}\right) = \pm\, 2mF_2^{\pm}(V; t)\,V \tag{46}$$

Therefore, from (40), (46), and the first law of thermodynamics, we recover the conclusions obtained in Sec. 4, i.e. *in the thermodynamic limit for the piston, the evolution (41, 12; 35) is deterministic and adiabatic* (i.e. in (46) only work and no heat is involved).

5.2 Infinite cylinder ($L = \infty$, $M = \infty$)

As discussed in Sec. 3, for $L = \infty$ we can neglect the recollisions. Therefore in $F_2^{\pm}$ the distribution $\rho^{\pm}(v; t)$ can be replaced by $n_0^{\pm}\varphi_0^{\pm}(v)$ and $F_2^{\pm}(V)$ is independent of t. In this case, the evolution of the piston is simply given by the ordinary differential equation

$$\frac{d}{dt}V(t) = \frac{A}{M}\,2mF_2(V) \qquad\qquad V(t = 0) = 0 \tag{47}$$

where $F_2(V)$ is a strictly decreasing function of V. If $p_0^{+} = p_0^{-}$, then $V(t) = 0$, i.e. the piston remains at rest and the two fluids remain in their original thermal equilibrium. If $p_0^{+} \neq p_0^{-}$, i.e. $n_0^{+}k_BT_0^{+} \neq n_0^{-}k_BT_0^{-}$, the piston will evolve monotonically to a stationary state with constant velocity V_{stat} solution of $F_2(V_{stat}) = 0$. From (34) follows that V_{stat} is a function of n_0^{+}/n_0^{-}, T_0^{-}, T_0^{+}, but does not depend on the value M/A. Moreover, the approach to this stationary state is exponentially fast with relaxation time $\tau_0 = 1/\lambda(V = 0)$. For Maxwellian distributions $\varphi_0^{\pm}(v)$, V_{stat} is solution of

$$k_B(n_0^{-}T_0^{-} - n_0^{+}T_0^{+}) - V_{stat}\sqrt{\frac{8k_Bm}{\pi}}\,\left(n_0^{-}\sqrt{T_0^{-}} - n_0^{+}\sqrt{T_0^{+}}\right) + V_{stat}^2 m(n_0^{-} - n_0^{+}) + \mathcal{O}(V_{stat}^3) = 0 \tag{48}$$

Moreover,

$$\tau_0^{-1} = \frac{A}{M}\sqrt{\frac{8k_Bm}{\pi}}\,\left(n_0^{-}\sqrt{T_0^{-}} + n_0^{+}\sqrt{T_0^{+}}\right) \tag{49}$$

which implies that the relaxation time will be very small either if $M/A \ll 1$, or if $n_0^{\pm} = \xi\,\tilde{n}_0^{\pm}$ with $\xi \gg 1$. In this case, the piston acquires almost immediately its final velocity V_{stat} and one can solve Eq. 12 to obtain the evolution of the fluids.

5.3 Finite cylinder ($L < \infty$, $M = \infty$)

For finite L, introducing the average temperature in the fluids,

$$T_{av}^{\pm} = \frac{2\langle E^{\pm}\rangle_t}{k_B\,N^{\pm}} \tag{50}$$

we have to solve (41), (46), i.e.

$$\begin{cases} \dfrac{d}{dt}V(t) = \dfrac{A}{M}\, 2m\, [F_2^-(V;t) - F_2^+(V;t)] \\ \\ k_B \dfrac{d}{dt}T_{av}^{\pm} = \pm\, 4m \dfrac{A}{N^{\pm}}\, F_2^{\pm}(V;t)\, V \end{cases} \tag{51}$$

where $F_2^{\pm}(V;t)$ is a functional of $\rho_{surf}^{\pm}(v;t)$ which we decompose as

$$F_2^{\pm}(V;t) = \hat{n}^{\pm}(t)\, k_B \hat{T}^{\pm}(t) \pm \left(\frac{M}{A}\right)\, \lambda^{\pm}(V;t)\, V \tag{52}$$

with

$$\hat{n}^-(t) = \int_0^{\infty} dv\, \rho_{surf}^-(v;t) \qquad\qquad \hat{n}^+(t) = \int_{-\infty}^{0} dv\, \rho_{surf}^+(v;t) \tag{53}$$

and

$$\hat{n}^{\pm}\, k_B \hat{T}^{\pm} = \hat{p}^{\pm} \tag{54}$$

For a time interval $\tau_1 = L\sqrt{m/k_B T}$ which is the time for the shock wave to bounce back, the piston will evolve as discussed in Sec. 5.2. In particular, if $R^{\pm}$ is sufficiently large, then after a time $\tau_0 = \mathcal{O}((R^{\pm})^{-1})$ the piston will reach the velocity $\bar{V}$ given by $F_2(\bar{V}, t) = 0$, Eq. 47. For $t > \tau_1$, $F_2^{\pm}(V;t)$ depends explicitly on time. For $R^{\pm}$ sufficiently large we can expect that for all t the velocity $V(t)$ will be a functional of $\rho_{surf}^{\pm}(v;t)$ given by $F_2[V(t); \rho_{surf}^{\pm}(.;t)] = 0$, and thus the problem is to solve Eq. 12 with the boundary condition Eq. 32. Since $V(t)$ so defined is independent of M/A the evolution will be independent of M/A if $R^{\pm}$ is sufficiently large. This conclusion, which we can not prove rigorously, will be confirmed by numerical simulations.

To give a qualitative discussion of the evolution for arbitrary values of $R^{\pm}$ we shall use the following assumption already introduced in the experimental measurement of c_p/c_v.

Assumption 2: (average assumption)

The surface coefficients $\widehat{n}^{\pm}(t)$ and $\widehat{T}^{\pm}(t)$, Eqs. 52-53, coincide to order 1 in α with the average value of the density and temperature in the fluids, i.e.

$$\hat{n}^- = \frac{N^-}{AX(t)}, \qquad \hat{n}^+ = \frac{N^+}{A(L - X(t))}, \qquad \hat{T}^{\pm} = T_{av}^{\pm}(t) \tag{55}$$

We still need an expression for the friction coefficients. From

$$F_2^{\pm}(V;t) = \hat{p}^{\pm}(t) - 4mVF_1^{\pm}(V = 0;t) + mV^2\hat{n}^{\pm}(t) + \mathcal{O}(V^3) \tag{56}$$

then, assuming that to first order in α, $F_1^{\pm}(V = 0;t)$ is the same function of $\widehat{T}^{\pm}(t)$ as for Maxwellian distributions we have

$$\lambda^{\pm}(V) = \left(\frac{A}{M}\right)\, m\hat{n}^{\pm}\, \left[\sqrt{\frac{8k_B\hat{T}^{\pm}}{m\pi}} \pm V\right] + \mathcal{O}(V^2) \tag{57}$$

Therefore choosing initial condition such that $V(t)$ is small for all time, Eq. 51 yields

$$\sqrt{\hat{T}^-}\, X - \sqrt{\hat{T}^+}\, (L - X) = C = \sqrt{\hat{T}_0^-}\, X_0 - \sqrt{\hat{T}_0^+}\, (L - X_0) \tag{58}$$

We thus obtain the *equilibrium point* for the adiabatic evolution ($M = \infty$)

$$\left(\frac{N^-}{A}\right) T_f^- = \frac{2E_0}{Ak_B} \frac{X_f}{L} \tag{59}$$

$$\left(\frac{N^+}{A}\right) T_f^+ = \frac{2E_0}{Ak_B} \left(1 - \frac{X_f}{L}\right) \tag{60}$$

where

$$\frac{2E_0}{Ak_B} = \left(\frac{N^-}{A}\right) T_0^- + \left(\frac{N^+}{A}\right) T_0^+ \tag{61}$$

and

$$\sqrt{\left(\frac{A}{N^-}\right) X_f^3} - \sqrt{\left(\frac{A}{N^+}\right) (L - X_f)^3} = \sqrt{\frac{AL}{2E_0 k_B}}\, C \tag{62}$$

Solving (58-62) gives the equilibrium state $(X_f, T_f^{\pm})$ which is a state of mechanical equilibrium $p_f^- = p_f^+$, but not thermal equilibrium $T_f^- \neq T_f^+$. Moreover this equilibrium state does not depend on M. Having obtained the equilibrium point we can then investigate the evolution close to the equilibrium point. Linearizing Eq. 51 around $(X_f, T_f^{\pm})$ yields

$$\frac{d}{dt} V = k_B \left[\left(\frac{N^-}{M}\right) \frac{T_f^- X_f^2}{X^3} - \left(\frac{N^+}{M}\right) \frac{T_f^+ (L - X_f)^2}{(L - X)^3} \right] - \lambda(V = 0) V \tag{63}$$

In other words, the effect of collisions on the piston is to induce an external potential of the form $[c_1 |X|^{-2} + c_2 (L - X)^{-2}]$ and a friction force. It is a damped harmonic oscillator with

$$\begin{cases} \omega_0^2 = 6 \left(\dfrac{E_0}{M}\right) \dfrac{1}{X_f (L - X_f)} \\[2ex] \lambda = 4\sqrt{\dfrac{1}{\pi}} \sqrt{\dfrac{E_0}{ML}} \left[\sqrt{\dfrac{R^-}{X_f}} + \sqrt{\dfrac{R^+}{(L - X_f)}} \right] \end{cases} \tag{64}$$

(recall that $R^{\pm} = mN^{\pm}/M$). For the case $N^- = N^+$ to be considered in the simulations, Eq. 64 implies that the motion is weakly damped if

$$R < R_{max} = \frac{3\pi}{2} \left[\sqrt{\frac{X_f}{L}} + \sqrt{1 - \frac{X_f}{L}} \right]^{-2} \tag{65}$$

with period

$$\tau = \frac{2\pi}{\omega_0} \frac{1}{\sqrt{R - R_{max}}} \tag{66}$$

and strongly damped if $R > R_{max}$, in agreement with experimental observations (Sec. 1.1).

6 Moments $\langle V^n \rangle_t$: Piston with finite mass

6.1 Equation to first order in $\alpha = 2m/(M + m)$

If the mass of the piston is finite with $M \gg m$, then the irreducible moments Δ_r are of the order $\alpha^{[\frac{r+1}{2}]}$ where $[\frac{r+1}{2}]$ is the integral part of $\frac{r+1}{2}$. If the factorization condition (19) is satisfied, to first order in α we have:

$$\langle V^n \rangle_t = V^n(t) + \frac{n(n-1)}{2} V^{n-2}(t) \Delta_2(t) \tag{67}$$

where $V(t) = \langle V \rangle_t$ and $\Delta_2(t) = \langle V^2 \rangle_t - \langle V \rangle_t^2$ are solutions of

$$\begin{cases} \frac{1}{\gamma}\frac{d}{dt} V(t) = F_2 + \Delta_2 F_0 \\ \\ \frac{1}{\gamma}\frac{d}{dt} \Delta_2(t) = -4\Delta_2 F_1 + \alpha F_3 \\ \\ \frac{1}{\gamma}\frac{d}{dt} \langle E^{\pm} \rangle_t = \pm \left\{ M[F_2^{\pm} + \Delta_2 F_0^{\pm}]V + (M/2)[4\Delta_2 F_1^{\pm} - \alpha F_3^{\pm}] \right\} \end{cases} \tag{68}$$

and $\Delta_2 \doteq k_B T_P / M$ defines the temperature of the piston.

6.2 Infinite cylinder: Heat transfer

For the infinite cylinder, the factorization assumption is an exact relation and in this case the functions $F_k(V;t)$ are independent of t. The solution of the the autonomous system (68) with $F_k = F_k(V)$ shows that the piston evolves to a stationary state with velocity $\bar{V}$ given by

$$F_2(\bar{V}) + \frac{\alpha}{4} \frac{F_3(\bar{V}) F_0(\bar{V})}{F_1(\bar{V})} = 0 \tag{69}$$

The temperature of the piston is

$$\bar{\Delta}_2 = \frac{k_B T_P}{M} = \frac{\alpha}{4} \frac{F_3(\bar{V})}{F_1(\bar{V})} \tag{70}$$

and the heat flux from the piston to the fluid is

$$\frac{1}{A} P_Q^{P \to -} = \frac{m^2}{2M} \left[\frac{F_3^+ F_1^- - F_3^- F_1^+}{F_1^- - F_1^+} \right] \tag{71}$$

If we choose initial conditions such that $|V(t)| \ll 1$ for all t, and Maxwellian distributions $\varphi^{\pm}(v)$, the solutions $V(t)$, $\Delta_2(t)$ coincide with the solutions previously obtained Eq. 36, 37 and

$$\frac{1}{A} P_Q^{P \to -} = (T^+ - T^-) \frac{m}{M} \sqrt{\frac{8k_B}{m\pi}} \frac{p^- p^+}{(p^+ \sqrt{T^-} + p^- \sqrt{T^+})} \tag{72}$$

In conclusion to first order in m/M there is a heat flux from the warm side to the cold one proportional to $(T^+ - T^-)$, induced by the stochastic motion of the piston.

6.3 Finite cylinder ($L < \infty$, $M < \infty$)

6.3.1 Singular character of the perturbation approach

Whereas the leading order is actually the "thermodynamic behavior" $M = \infty$ in the two first stages of the evolution (fast relaxation toward mechanical equilibrium), the fluctuations of order $\mathcal{O}(\alpha)$ rule the slow relaxation toward thermal equilibrium. It is thus obvious that a naive perturbation approach cannot give access to *both* regimes. This difficulty is reminiscent of the *boundary-layer problems* encountered in hydrodynamics, and the perturbation method to be used here is the exact temporal analog of the *matched perturbative expansion method* developed for these boundary layers. The idea is to implement two different perturbation approaches:
— one at short times, with time variable t describing the fast dynamics ruling the fast relaxation toward mechanical equilibrium;

— one for longer times, with a rescaled time variable $\tau = \alpha t$. This second perturbation approach is supplemented with a "*slaving principle*", expressing that at each time of the slow evolution, i.e. at fixed τ, the still present fast dynamics has reached a local asymptotic state, slaved to the values of the slow observables. The initial conditions are set on the first-stage solution. The initial conditions of the second regime match the asymptotic behavior of the first-stage solution ("*matching condition*").

The slaving principle is implemented by interpreting an evolution equation of the form

$$\frac{da}{dt} \equiv \alpha \frac{da}{d\tau} = A(\tau, a) \qquad \text{where} \quad A = \mathcal{O}(1) \tag{73}$$

as follows: it indicates that a is in fact a fast quantity relaxing at short times ($\ll \tau$) toward a stationary state $a_{eq}(\tau)$ slaved to the slow evolution and determined by the condition

$$A[\tau, a_{eq}(\tau)] = 0 \tag{74}$$

(at lowest order in α, actually $A[\tau, a_{eq}(\tau)] = \mathcal{O}(\alpha)$ which prescribes the leading order of $a_{eq}(\tau)$); the following-order terms can be arbitrarily fixed as long as only the first order of perturbation is implemented. Physically, such a condition arises to express that an instantaneous mechanical equilibrium takes place at each time τ of the slow relaxation to thermal equilibrium.

6.3.2 Equations for the fluctuation-induced evolution of the system

Following this procedure we arrive at explicit expressions for the rescaled quantities (of order $\mathcal{O}(1)$) $\widetilde{V} = V/\alpha$, $\widetilde{\Delta}_2 = \Delta_2/\alpha$, and $\widetilde{\Pi} = (p^- - p^+)/\alpha$:

$$\begin{cases} \widetilde{V} &= \dfrac{m}{3}\left(\dfrac{AL}{E_0}\right)\left(\dfrac{F_3^- F_1^+ - F_3^+ F_1^-}{F_1}\right) + \mathcal{O}(\alpha) \\[2ex] \dfrac{\widetilde{\Pi}}{2m} &= \dfrac{2m}{3}\left(\dfrac{AL}{E_0}\right)(F_3^- F_1^+ - F_3^+ F_1^-) \;-\; \dfrac{F_3 F_1}{4F_1} + \mathcal{O}(\alpha) \\[2ex] \widetilde{\Delta_2} &= \dfrac{F_3}{4F_1} + \mathcal{O}(\alpha) \end{cases} \tag{75}$$

We then introduce a (dimensionless) rescaled position for the piston

$$\xi = \frac{1}{2} - \frac{X}{L} \in [-1/2, 1/2] \tag{76}$$

which satisfies

$$\frac{d\xi}{d\tau} = -\,k_B(T^- - T^+)\left(\frac{2A}{3E_0}\right)\frac{F_1^- F_1^+}{F_1} \tag{77}$$

To discuss Eq. 77, a third assumption has to be introduced.

<u>Assumption 3</u> (Maxwellian identities)

In the regime when $V = \mathcal{O}(\alpha)$, the relations between the functional F_1, F_2 and F_3 are the same at lowest order in α as if the distributions $\rho^{\pm}_{surf}(v; V; t)$ were Maxwellian in v:

$$F_1^{\pm}(V) \approx \mp\, \rho^{\pm} \sqrt{\frac{k_B T^{\pm}}{2m\pi}} \qquad \text{and} \qquad F_3^{\pm}(V) \approx \left(\frac{2k_B T^{\pm}}{m}\right) F_1^{\pm}(V) \;-\; V\, F_2^{\pm}(V) \tag{78}$$

Using these identities and the (dimensionless) rescaled time

$$s = \tau\, \frac{2}{3L}\sqrt{\frac{k_B}{m\pi}}\sqrt{\frac{2(N^- T_0^- + N^+ T_0^+)}{N}} \tag{79}$$

where $N = N^+ - N^-$, we obtain a deterministic equation describing the piston motion [5]:

$$\frac{d\xi}{ds} = -\left[\sqrt{\frac{N}{2N^+}(1+2\xi)} - \sqrt{\frac{N}{2N^-}(1-2\xi)}\right] \qquad \xi(0) = \frac{1}{2} - \frac{X_{ad}}{L} \tag{80}$$

where X_{ad} is the piston position at the end of the adiabatic regime (i.e. X_f, Eq. 62). The meaningful observables straightforwardly follow from the solution $\xi(s)$:

$$X(s) = L\left(\frac{1}{2} - \xi(s)\right) \qquad T^{\pm}(s) = [\,1 \pm 2\xi(s)\,]\left(\frac{N^-T_0^- + N^+T_0^+}{2N^{\pm}}\right) \tag{81}$$

The first-order perturbation analysis using a single rescaled time $t_1 = \alpha t_0$ is valid in the regime when $V = \mathcal{O}(\alpha)$ and it gives access to the relaxation toward thermal equilibrium up to a temperature difference $T^+ - T^- = \mathcal{O}(\alpha)$. For the sake of technical completeness (rather that physical relevance, since the above first-order analysis is enough to get the observable, meaningful behavior), let us mention that the perturbation analysis can be carried over at higher orders; using further rescaled times $t_2 = \alpha^2 t_0, \ldots, t_n = \alpha^n t_0$, it would allow to control the evolution up to a temperature difference $|T^+ - T^-| = \mathcal{O}(\alpha^n)$; however one could expect that the factorization condition does not hold at higher orders.

7 Numerical simulations

As we have seen the results were established under the condition that m/M is a small parameter. Moreover for finite systems ($L < \infty$, $M < \infty$) it was assumed that before collisions and to first order in m/M the factorization and the average assumptions are satisfied. The numerical simulations are thus essential to check the validity of these assumptions, to determine the range of acceptable values m/M for the perturbation expansion, to investigate the thermodynamic limit, and to guide the intuition.

In all simulation we have taken, $k_B = 1$, $m = 1$, $T^- = 1$ and usually $T^+ = 10$. For L finite, we have taken $L = 60$, $X_0 = 10$, $A = 10^5$ and $N^+ = N^- = N$, i.e. $p^- = R(M/A)(1/10)$ and $p^+ = 2p^-$. The number of particles N was varied from a few hundreds to one or several millions; the mass M of the piston from 1 to 10^5. We give below some of the results which have been obtained for $L = \infty$ (Figs. 2,3) and for $L < \infty$ approach to mechanical equilibrium (Figs. 4-6) and to thermal equilibrium (Figs. 7,8).

<Figures 2-8 near here>

8 Conclusions and open problems

In this review the adiabatic piston has been investigated to first order in the small parameter m/M, but no attempt has been made to control the remainder terms. For an infinite cylinder no other assumptions were necessary and the numerical simulations (Figs. 2,3) are in perfect agreement with the theoretical prediction in particular for the stationary velocity V_{stat}, the friction coefficient $\lambda(V)$ and the relaxation time τ.

For a finite cylinder ($L < \infty$) and in the thermodynamic limit ($M = \infty$) we were forced to introduce the average assumption to obtain a set of autonomous equations. As we have seen when initially $p^- \neq p^+$, this limiting case also describes the evolution to lowest order during the first two stages characterized by a time of the order $t_1 = L\sqrt{m/k_BT}$, where the evolution is adiabatic and deterministic. In the first stage, i.e. before the shock wave bounces back on the piston, the simulations confirm the theoretical predictions. In particular, they show that if $R > 4$, the piston will be able to reach and maintain for some time the velocity V_{stat}, while this will not be the case for $R < 1$ (Fig. 4b). In the second stage of the evolution, the simulations (Fig. 4) exhibit damped oscillations toward mechanical equilibrium which are in very good agreement with the predictions for the final state $(X_{ad}, T_{ad}^{\pm})$, the frequency of oscillations and the existence of weak and strong damping depending on $R < 1$ or $R > 4$. Moreover the general behavior of the evolution observed in the simulations in function of the parameters involved as predicted. However the damping coefficient of these oscillations is wrong by one or several order of magnitude. To understand this discrepancy we note that using the average assumption we have related the damping to the friction coefficient. However the simulations clearly show that those two dissipative effects have totally different origins. Indeed, as one can see with $L = \infty$, the friction is associated with the fact that the density of the gas in front and in the back of the piston are not the same as in the bulk, and this generates a shock wave which propagates in the fluid. For finite L, when $R > 4$, the stationary velocity V_{stat} is reached and the effect of friction is to transfer in this first stage more and more energy to the fluid on one side and vice-versa on the other side. However to stop the piston and reverse its motion only a certain amount of the transferred energy is necessary and the rest remains as dissipated energy in the fluid leading to a strong damping. On the other hand for $R < 1$, the value V_{stat} is never reached and all the energy transferred is necessary to revert the motion. In this case very little dissipation is involved and the damping will be very small. This indicates that the mechanism responsible for damping is associated with shock waves bouncing back and forth and the average assumption, which corresponds to an homogeneity condition throughout the gas, can not describe the situation. In fact the simulations (Fig. 5b) indicate that the average assumption does not hold in this second stage. In conclusion one is forced to admit that to describe correctly the adiabatic evolution it is necessary to study the coupling between the motion of the piston and the hydrodynamic equations of the gas. Preliminary investigations have been initiated, but this is still one of the major open problems. Another problem would be to study the evolution in the case of interacting particles. However investigations with hard disks suggest that no new effects should appear. To investigate adiabatic evolution a simpler version of the problem, without any controversy, has been introduced to investigate adiabatic piston: this is the model of a standard piston with a constant force acting on it.

In the third stage, i.e. the very slow approach to thermal equilibrium, another assumption was necessary, namely the factorization condition. The simulations (Fig. 7) show a very good agreement with the prediction, and in particular the scaling property with $t' = t/M$ is perfectly verified. It appears that the small discrepancy between simulations and theoretical predictions could be due to the fact that to compute explicitly the coefficients in the equations of motion, we have taken Maxwellian relations for the velocities of the gas particles, and this is clearly not the case (Fig. 8a).

The fourth stage of the evolution, i.e. the approach to Maxwellian distributions (Fig. 8b) is still another major open problem. Some preliminary studies have been conducted, where one investigates the stability and the evolution of the system when initially the two gases are in the same equilibrium state, but characterized by a distribution function which is not Maxwellian.

Finally let us mention that the relation between the piston problem and the Second Law of thermodynamics is one more major problem. The question of entropy production out equilibrium, and the validity of the Second Law, are still highly controversial. Again preliminary results can be found in the literature. Among other things, this question has led to a model of heat conductivity in the gas which reproduces the correct behavior [4].

Cross-references within the Encyclopedia:

Non-equilibrium statistical mechanics: Overview;
Boltzmann's equation;
Dynamical system approach to non-equilibrium
Billiards in bounded convex domains
Multi-scale approach
Hamiltonian
Fluid dynamics

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Figure captions

Figure 1: The adiabatic piston problem

Figure 2: Evolution of the piston for $L = \infty$, $M = 10^4$ and $p^- = p^+ = 1$ as observed in simulations (stochastic line in (a), dots in (b)) compared with prediction.

(a) Position $X(t)$ for $T^+ = 10$.

(b) Stationary velocity for $T^+ = 10$ (continuous line) and $T^+ = 100$ (dotted line), as a function of M.

Figure 3: Evolution of the piston for $L = \infty$, $M = 10^4$ and $p^+ \neq p^-$ as observed in simulations (continuous line) compared with predictions (dotted line).

(a) $p^- = 1$, $p^+ = p^- + \Delta p$, from top to bottom $\Delta p/p^- = 0.05,\ 0.1,\ 0.2,\ 1,\ 2,\ 3$.

(b) $p^- = \zeta$, $p^+ = 2\zeta$, $\Delta p/p^- = 1$; $X' = \zeta X$, $t' = \zeta t$, $\zeta = 10^{-3},\ 10^{-2},\ 10^{-1},\ ,1,\ 10,\ 10^2,\ 10^3,\ 10^4$.

Figure 4: "Deterministic" evolution toward mechanical equilibrium for $L < \infty$, $M = 10^5$.

(a) Position $X(t)$; one finds $X_{ad}^{sim} = 8.3$ whereas $X_{ad}^{th} = 8.42$.

(b) Velocity $V(t)$; one finds $\bar{V}^{sim} = -0.343$ whereas $\bar{V}^{th} = -0.3433$.

From top to bottom:

$R = 12$: strong damping, independent of R and M for $R > 4$ and $M > 10^3$.

$R = 2$: critical damping.

$R = 0.1$: weak damping; damping coefficient increases with R and $\omega_0 \sim \sqrt{R}$ for $R < 1$ but is independent of M for $M > 10^3$.

Figure 5: Same conditions as Fig. 4, $R = 12$

(a) Average pressure and temperature in the fluid: $p_{av}^{\pm}(t) = 2E^{\pm}n^{\pm}/N^{\pm}$, $T_{av}^{\pm} = E^{\pm}/N^{\pm}k_B$.

(b) Pressure and temperature at the surface of the piston.

Prediction: $T_{ad}^- = 1.54$, $T_{ad}^+ = 9.46$, $p_{ad}^- = p_{ad}^+ = 2.2$. Simulations: $T_{ad}^- = 1.52$, $T_{ad}^+ = 9.48$, $p_{ad}^- = p_{ad}^+ = 2.2$.

Figure 6: Velocity distribution in the left compartment. Same conditions as Fog. 4, $R = 12$. Dotted line corresponds to Maxwellian with T^-=1.52

(a) $t = 12,\ ,24,\ ,36,\ 48,\ 60,\ 92,\ 144,\ 240$.

(b) $t = 276$ to 460.

Figure 7: Approach to thermal equilibrium, $N^{\pm} = 3.10^4$. The smooth curves correspond to the predictions, the stochastic curves to simulations.

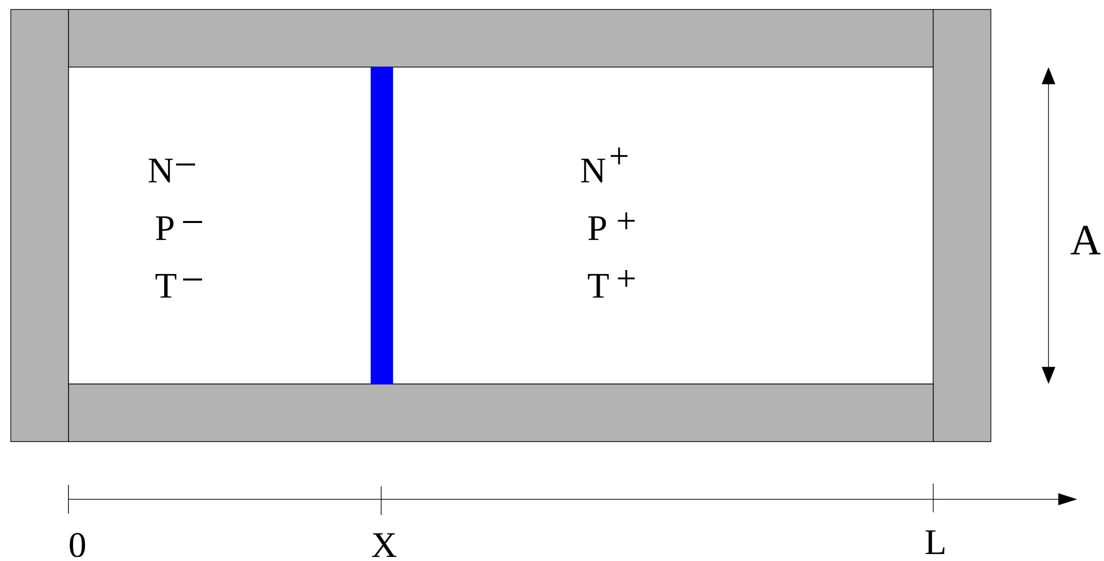
(a) Position $X(\tau)$, $\tau = \alpha t$, no visible difference for $M = 100,\ 200,\ 1000$.

(b) Average temperatures $T^{\pm}(\tau)$, $\tau = \alpha t$, $M = 200$.

Figure 8: Velocity distribution functions on the left $M = 200$, $N^{\pm} = 3.10^4$.

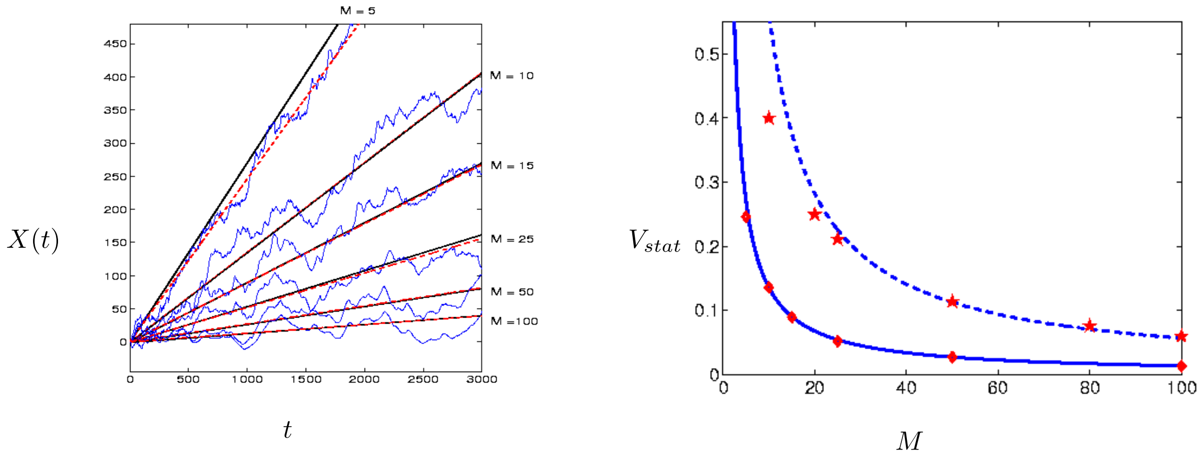
(a) Approach to thermal equilibrium from $T_{ad}^- = 1.54$ to $T_f^- = 5.5$ for $\tau < 144$.

(b) Approach to Maxwellian distribution $\tau > 445$.



N^-
P^-
T^-
N^+
P^+
T^+
A
0
X
L

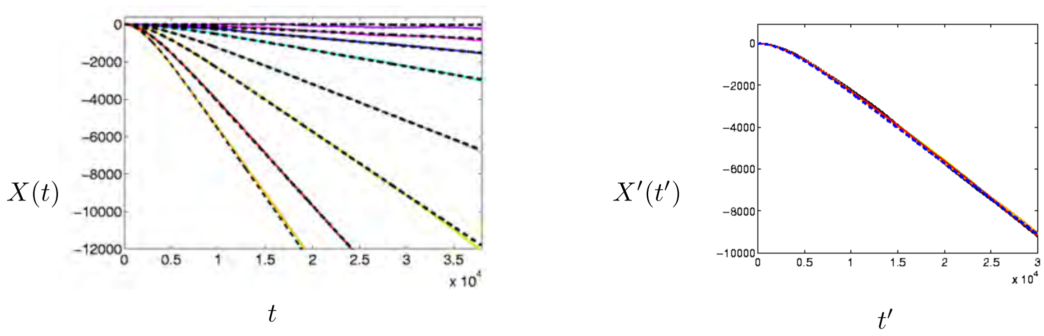
Figure 1



(a)

(b)

Figure 2

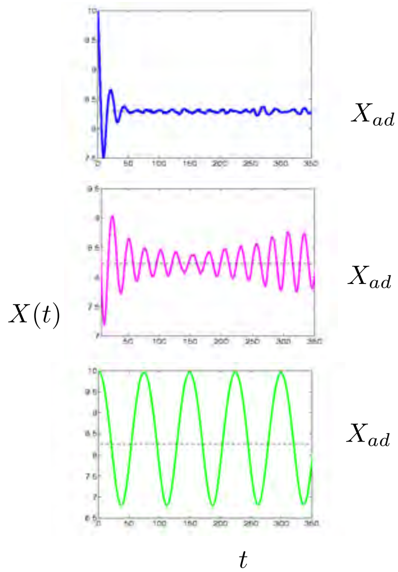


0
-2000
-4000
-6000
-8000
-10000
-12000
$X(t)$
0 0.5 1 1.5 2 2.5 3 3.5
x 10^4
t
0
-2000
-4000
-6000
-8000
-10000
$X'(t')$
0 0.5 1 1.5 2 2.5 3
x 10^4
t'

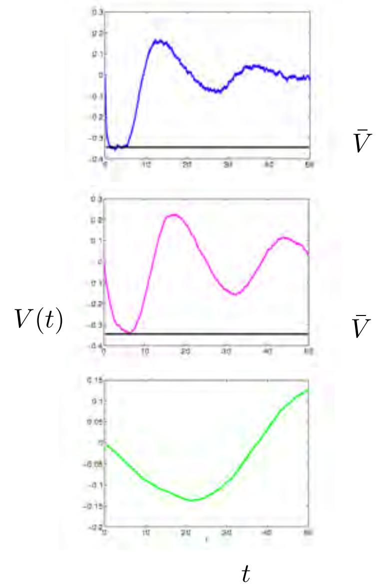
(a) (b)

Figure 3

Figure 3

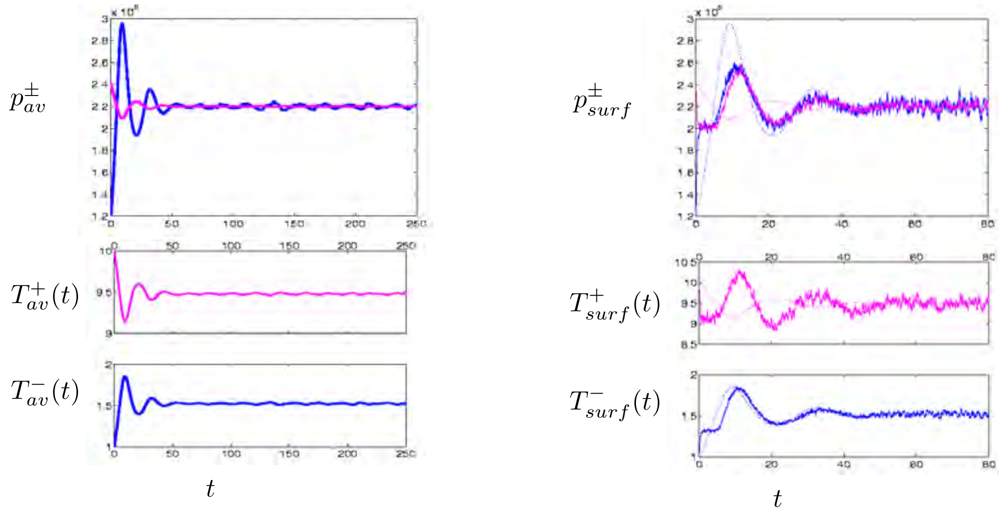


(a)



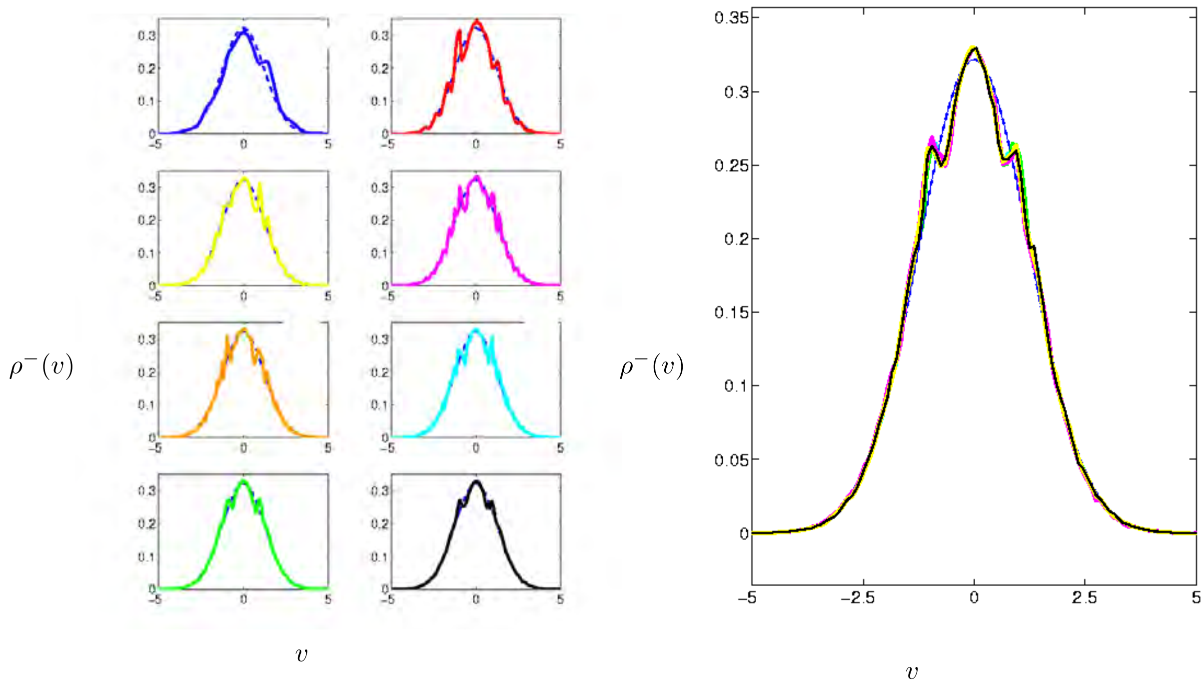
(b)

Figure 4



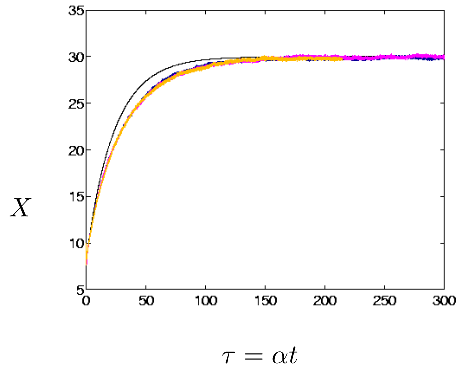
(a) (b)

Figure 5

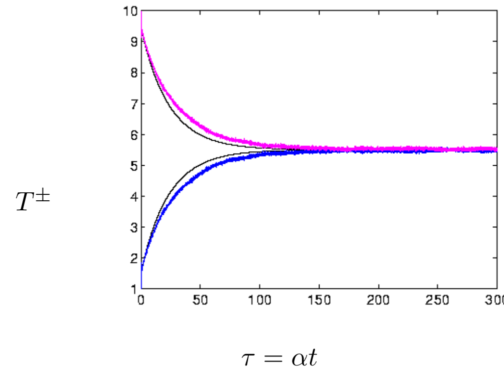


(a) (b)

Figure 6

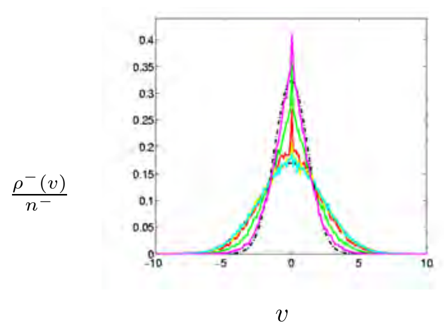


(a)



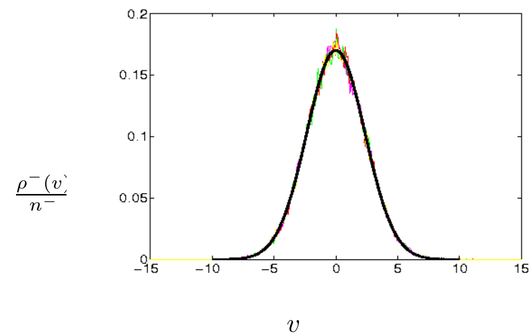
(b)

Figure 7



$\frac{\rho^-(v)}{n^-}$
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-10
-5
0
5
10
v

(a)



$\frac{\rho^-(v)}{n^-}$
0.2
0.15
0.1
0.05
0
−15
−10
−5
0
5
10
15
v

(b)

Figure 8