Adiabatic piston - Draft for MEMPHYS

14th November 2006

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1 Macroscopic problem

The adiabatic piston is an old problem of thermodynamics which has received considerable attention during the last decade. The problem is the following: a gas contained in an insulated cylinder is divided in two parts by a massive movable adiabatic wall (the piston), i.e. it does not conduct heat (Fig. 1). Initially the piston is held fixed by a clamp at the position X_0 and the two gases on the left and the right of the piston are in thermal equilibrium characterized respectively by their pressure $p_0^{\pm}$, temperature $T_0^{\pm}$ and number of particles $N^{\pm}$, where the symbols $-/+$ refer to the left/right compartments. Since the piston is adiabatic the system remains in equilibrium even if $T_0^- \neq T_0^+$. At time $t = 0$ the clamp is removed and the piston is let free to move without any friction. The question is to find the evolution of the system, especially the final state.

The problem became controversial because it is the simplest example where the two laws of thermostatics (conservation of energy E and maximum of entropy S) are not sufficient to predict the final equilibrium state [3]. Indeed, from these two laws one obtains only the necessary conditions

$$p_f^+ = p_f^- = p_f, \qquad E_f^+ + E_f^- = E_f^0 + E_f^0, \qquad S_f^{\pm} \geq S_0^{\pm} \tag{1}$$

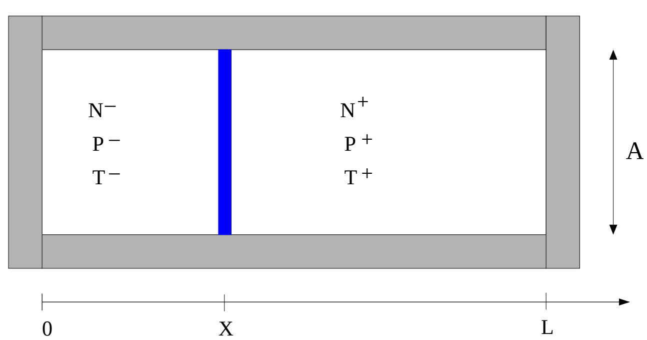

Figure 1: The adiabatic piston problem

Experimentally, the adiabatic piston has been used already before 1924 to measure the ratio $\gamma = c_p/c_v$ of the specific heats of gases. However in 2000 new measurements have shown two regimes with very different properties depending on the ratio $R = M_{gas}/M$ where M is the mass of the piston: the first one exhibits weakly damped oscillations of period corresponding to adiabatic oscillations of the gas while the other one exhibits strongly damped oscillations of period corresponding to isothermal oscillations.

2 Microscopic description and results

A qualitative microscopic discussion was first given by Feynman [2]. His idea was later confirmed by numerical simulations as well as by analytical methods based either on kinetic theory or statistical mechanics, together with some simplifying assumptions [6]. The gas is usually described as a system of point particles with mass $m \ll M$, making purely elastic collisions on the walls of the cylinder and on the piston. Two-dimensional gas of hard-disks have also been studied and led to similar conclusions as regards the piston motion [7].

Under the random collisions with the particles the piston acquires a stochastic motion. An important conclusion of all these investigations is that the parameter

$$R = \frac{m(N^+ + N^-)}{M} \tag{2}$$

plays a fundamental role. Macroscopic properties are recovered in the thermodynamic limit where M, $N^\pm$, and the area A of the cylinder cross-section tend to infinity keeping constant M/A, $R^\pm = N^\pm m/M$, and the cylinder length L. One can show that in this thermodynamic limit the evolution is deterministic (no fluctuations) and adiabatic (no heat transfer). If the initial pressures are different the system will evolve to a final equilibrium state with equal pressures on both sides but different temperatures. For small R, typically $R < 1$, the evolution of the piston is a weakly damped oscillation with relaxation time proportional to $1/R$. In the limit $R \to 0$ the motion is purely oscillatory (no damping) with period proportional to $1/\sqrt{R}$ and constant entropy in each compartment. For large R, typically $R > 4$, the evolution in the thermodynamic limit is a strongly damped oscillation which is independent of R.

To understand qualitatively such an influence of R on the evolution one can consider a cylinder of infinite length: if $p_0^- \neq p_0^+$ the piston reaches a constant velocity $\bar{V}$ (independent of M) given approximately by

$$\bar{V} \approx (p_0^- - p_0^+) \sqrt{\frac{\pi k_B}{8m}} \left[\frac{p_0^+}{\sqrt{T_0^+}} + \frac{p_0^-}{\sqrt{T_0^-}} \right]^{-1} \qquad (L = \infty) \tag{3}$$

for $p_0^- - p_0^+$ small enough. In a finite cylinder of length L, the time necessary for the piston to reach the velocity $\bar{V}$ is $t_1 \approx R v_{th}/L$ (with v_{th} the thermal velocity of the gas particles) whereas the time necessary for a particle to make a recollision on the piston is $t_2 \approx v_{th}/L$. Therefore if $t_1 > t_2$, i.e. $R > 1$, the piston will reach the velocity $\bar{V}$ which builds up a shock wave and leads to strong damping. On the contrary if $R < 1$ the piston never reaches the velocity $\bar{V}$, the fluid will remain homogeneous and the whole energy given by the piston to the fluid during a compression will be recovered during the expansion, so that the dissipation hence the damping are weak.

For a piston with large but finite mass, the evolution proceeds in two stages with very different time scales. In a first stage the evolution is identical to the one discussed in the thermodynamic limit, i.e. it is deterministic and adiabatic, and the system evolves to a state of quasi equilibrium where the pressures are equal but the temperatures different; furthermore this adiabatic evolution is weakly or strongly damped depending on R, and does not depend on R when R is sufficiently large. In the second stage, the system evolves very slowly and stochastically, with pressures on both sides approximately equal, to a final equilibrium state where both the pressures and temperatures are equal; furthermore, in this second stage, the approach to equilibrium scales with the very slow time $\tau = (m/M)t$. For a real macroscopic piston the time necessary to reach thermal equilibrium will be several orders of magnitude the age of the Universe ($m/M \approx 10^{-25}$, $\tau_{Universe} \approx 3.10^{17}$ s). However for mesoscopic systems the effects of stochasticity lead to very interesting properties, that might become especially relevant for nano-devices since a scale where the piston problem is no longer purely academic is now reachable.

To understand the second stage of the piston evolution, several authors have considered initial conditions where the pressures are equal ($p^+ = p^- = p$) but $T^+ > T^-$. The simplest case is the one where the length of the cylinder is infinite. It was shown that the piston will evolve to a stationary state with non-zero average velocity toward the warmer side [4]

$$\langle V \rangle = \frac{m}{M} \sqrt{\frac{\pi k_B}{8m}} \left(\sqrt{T^+} - \sqrt{T^-} \right) \qquad (L = \infty) \tag{4}$$

with relaxation time

$$t_{relax} = \frac{M}{A} \sqrt{\frac{\pi k_B}{8m}} \frac{1}{p} \left(\frac{1}{\sqrt{T^-}} + \frac{1}{\sqrt{T^+}} \right)^{-1} \tag{5}$$

and heat flux across the piston

$$j_Q = \frac{m}{M} \, p \sqrt{\frac{8k_B}{m\pi}} \left(\frac{T^- - T^+}{\sqrt{T^-} + \sqrt{T^+}} \right) \tag{6}$$

In conclusion the important consequence of stochasticity is that a wall which is adiabatic when fixed will conduct heat when it is let free to move. We present below the main technical steps leading to the above results [6].

3 Derivation of the evolution equations for the piston

In statistical mechanics, the time evolution follows from Liouville equation for the closed system composed of the piston and all the gas particles. The equation for the n-th moment $\langle V^n \rangle(t)$ of the piston velocity involves the velocity distribution function $\rho^{\pm}_{surf}(v;t)$ for the particles at $x = X$ and $\Phi(V;t)$ for the piston, together with the correlation function $\rho^{\pm}_{surf}(v;V;t)$ for the velocities of the particles and the piston. Since this correlation function is unknown, we introduce an

Assumption (Factorization condition)

$$\begin{cases} \rho^-_{surf}(v;V;t) = \rho^-_{surf}(v;t)\,\Phi(V;t) & \text{if} \quad v > V \\ \rho^+_{surf}(v;V;t) = \rho^+_{surf}(v;t)\,\Phi(V;t) & \text{if} \quad v < V \end{cases} \tag{7}$$

This assumption is satisfied either when the length L of the cylinder is infinite, or in the thermodynamic limit when $L < \infty$, provided that the initial velocity distributions are zero for $|v| < v_{min}$, with v_{min} such that there is no recollision of the particles on the piston. Moreover for $L < \infty$ and M, $N^{\pm}$ very large but finite, this assumption is reasonable since there will be a very large number of collisions on the piston before a given particle can make a recollision. Introducing

$$V(t) = \langle V \rangle_t \tag{8}$$
$$\Delta_2(t) = \langle V^2 \rangle_t - \langle V \rangle_t^2 \tag{9}$$

where $\Delta_2(t)$ is related to the piston temperature T^P, namely $\Delta_2(t) = k_B T^P / M$, and the temperatures $T^{\pm}(t)$ given by the average energy of the gas, $\langle E^{\pm} \rangle_t = (1/2) N^{\pm} k_B T^{\pm}(t)$, we obtain to first order in the very small parameter $\alpha = 2m/(M+m)$:

$$\begin{cases} \dfrac{1}{\gamma}\dfrac{d}{dt}V(t) = F_2 + \Delta_2 F_0 \\ \\ \dfrac{1}{\gamma}\dfrac{d}{dt}\Delta_2(t) = -4\Delta_2 F_1 + \alpha F_3 \\ \\ \dfrac{1}{\gamma}k_B\dfrac{d}{dt}T^{\pm}(t) = \pm\,\dfrac{2M}{N^{\pm}}\ \left\{[F_2^{\pm} + \Delta_2 F_0^{\pm}]V + (1/2)[\alpha F_3^{\pm} - 4\Delta_2 F_1^{\pm}]\right\} \end{cases} \tag{10}$$

where $\gamma = \alpha A = 2mA/(M+m) = \mathcal{O}(1)$ and

$$F_k(V;t) = \int_V^{\infty} dv(v-V)^k \rho^-_{surf}(v;t) - \int_{-\infty}^{V} dv(v-V)^k \rho^+_{surf}(v;t) = F_k^-(V;t) - F_k^+(V;t) \tag{11}$$

We can easily check that the total energy $\langle E^+\rangle_t + \langle E^-\rangle_t + (1/2)M\langle V^2\rangle_t$ is conserved. Moreover from the first law of thermodynamics,

$$\frac{d}{dt}\left(\frac{< E^{\pm} >}{A}\right) = \frac{1}{A}\ \left[P_W^{P\to\pm} + P_Q^{P\to\pm}\right] \tag{12}$$

where $P_W^{P\to\pm}$ and $P_Q^{P\to\pm}$ denote the work- and heat-power transmitted by the piston to the fluid, we conclude that the heat flux is given by

$$j_Q^{\pm} = \frac{1}{A}P_Q^{P\to\pm} = \pm\,\frac{M}{M+m}\ [\alpha F_3^{\pm} - 4\Delta_2 F_1^{\pm}] \tag{13}$$

It should be noted that by definition for $V = 0$, $F_2^{\pm}$ is the pressure $\hat{p}^{\pm}(t)$ exerted by the incoming particles on the piston and

$$2mF_2^{\pm} = \hat{p}^{\pm}\ \pm\ \left(\frac{M}{A}\right)\ \lambda^{\pm}(V)\ V \tag{14}$$

where the friction coefficients $\lambda^{\pm}(V)$ are strictly positive. The second term of Eq. 10, $\Delta_2 F_0$, gives the first-order correction in α to the pressure and the friction force.

3.1 Infinite cylinder ($L = \infty$, $N^{\pm} = \infty$)

If $L = \infty$, then $\rho^{\pm}_{surf}(v;t) = \rho^{\pm}_0(v)$ and thus $p^{\pm}$, $T^{\pm}$ and $F_k^{\pm}$ are independent of t, given by $\rho_0^{\pm}(v)$. The solution of the autonomous equations (10) shows that the piston evolves to a stationary state with temperature $\bar{T}^P$ and velocity $\bar{V}$. For Maxwellian distributions $\rho_0^{\pm}(v)$ we thus obtain Eqs. 3, 4, 6 and the piston temperature is

$$\bar{T}^P = \sqrt{T^- T^+}\ \left(\frac{p^+\sqrt{T^+} + p^-\sqrt{T^-}}{p^+\sqrt{T^-} + p^-\sqrt{T^+}}\right) \tag{15}$$

3.2 Finite cylinder: Thermodynamic limit ($L < \infty$, $M = \infty$)

In the thermodynamic limit $\alpha = 0$ and $\gamma = 2mA/M$. Eq. 10 implies $\Delta_2 = 0$ and

$$\frac{d}{dt}V(t) = \frac{A}{M}\ (\hat{p}^- - \hat{p}^+) - [\lambda^-(V) + \lambda^+(V)]\ V \tag{16}$$

$$k_B\frac{d}{dt}T^{\pm}(t) = \pm\,2\,\frac{A}{N^{\pm}}\hat{p}^{\pm}V + 2\,\frac{M}{N^{\pm}}\lambda^{\pm}(V)V^2 \tag{17}$$

Since $\lambda^\pm = \mathcal{O}(R)$, in the limit as $R \to 0$ (i.e. $N^\pm$ fixed, $M \to \infty$) the velocity V tends to zero and the fluid is homogeneous with $\hat{p}^+ = \hat{p}^- = N^\pm k_B T^\pm / A X^\pm$ (with $X^- = X$ and $X^+ = L - X$). In this limit, Eq. 3 implies $\sqrt{T^\pm} X^\pm = cte$, i.e. $S^\pm = cte$, and the motion of the piston is purely oscillatory

$$\frac{d^2X}{dt^2} = \frac{E_0^-}{A}\,\frac{X_0^2}{X^3} - \frac{E_0^+}{A}\,\frac{(L-X_0)^2}{(L-X)^3} \tag{18}$$

This last result was also rigorously established by Sinai [8]. For $R > 0$ and $V(t)$ sufficiently small for all t, we can assume that the distributions are Maxwellian and $\hat{p}^\pm \approx p^\pm$ which gives

$$\lambda^\pm(V) = \left(\frac{A}{M}\right) p^\pm \sqrt{\frac{8m}{\pi k_B T^\pm}} + \mathcal{O}(V) \tag{19}$$

From Eq. 3 we now have $\sqrt{T^-}\,X - \sqrt{T^+}\,(L-X) = cte$, i.e. $S^\pm > 0$. In this case the motion is damped. The piston evolves to the final equilibrium position X_f given by

$$\sqrt{\left(\frac{N}{N^-}\right) X_f^3} - \sqrt{\left(\frac{N}{N^+}\right)(L-X_f)^3} = \sqrt{\frac{NLk_B}{2E_0}}\,\left[\sqrt{T_0^-}\,X_0 - \sqrt{T_0^+}\,(L-X_0)\right] \tag{20}$$

where $N = N^+ + N^-$, with frequency

$$\omega_0^2 = 6\,\left(\frac{E_0}{M}\right)\,\frac{1}{X_f(L-X_f)} \tag{21}$$

3.3 Finite cylinder: Finite piston mass ($L < \infty$, $M < \infty$)

If $p_0^+ \neq p_0^-$, the solution of Eq. 10 proceeds in two stages. At short time the variable t describes a fast relaxation toward mechanical equilibrium where $p^+(t) = p^-(t)$ according to the discussion of section 3.2. Once this quasi-equilibrium state has been reached, the evolution takes place on a much longer time scale, with the rescaled time $\tau = \alpha t$. In this second stage we have to follow another perturbation approach. Introducing the scale variable $\xi = \frac{1}{2} - \frac{X}{L}$ and assuming the distributions are Maxwellian, which can be justified since the motion is very slow, one finally obtains [5]

$$\frac{d\xi}{ds} = -\left[\sqrt{\frac{N}{2N^+}(1+2\xi)} - \sqrt{\frac{N}{2N^-}(1-2\xi)}\right] \tag{22}$$

where

$$s = \tau\,\frac{2}{3L}\,\sqrt{\frac{k_B}{m\pi}}\,\sqrt{\frac{2(N^-T_0^- + N^+T_0^+)}{N}} \tag{23}$$

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