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Transitions to Chaos in the Presence of an External Periodic Field: Cross-Over Effect in the Measure of Critical Exponents.

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Abstract. – The influence of noise on the accumulation of the period-doubling cascade has been analysed mostly for independent Gaussian noise. Universality extends to much more general situations including the case of periodic noises although some differences occur in the value of the critical exponents. We propose an explanation for these apparent discrepancies based on numerical and theoretical results. We report on simulations of discrete systems which show a cross-over in the behaviour of the Lyapunov exponent as a function of the amplitude of the periodic noise. Within a renormalization group approach, we discuss this phenomenon in terms of the competition between the usual deterministic fixed point and a noise-dependent fixed point.

We first recall the notion of iteration in the presence of noise. We shall consider mappings of the interval statisfying the standard hypothesis of regularity and unimodality (see for example [1]). A good example to keep in mind is the family of maps of the interval

$$x \to 1 - \mu |x|^z, \tag{1}$$

where the parameter μ is between 0 and 2, and $z > 1$. This family presents the well-known phenomenon of accumulation of period-doubling bifurcations (see references in [2]) for a critical value μ_c of the parameter μ. Let f denote such a map. In the classical iterations, one generates a sequence of number $\{x_n\}_{n \in \mathbf{N}}$ by the formula

$$x_{n+1} = f(x_n) . \tag{2}$$

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For random iterations, one considers a sequence $\{\xi_n\}$ of random variables and defines the sequence $\{x_n\}_{n\in\mathbf{N}}$ by

$$x_{n+1} = f(x_n) + g(x_n, \xi_n) , \tag{3}$$

where g is a given function of two variables. Note that the sequence of random variables $\{\xi_n\}$ may be strongly correlated (see Example 2 below). We now give two examples of the above situation:

Example 1: $\{\xi_n\}_{n\in\mathbf{N}}$ is a sequence of real independent identically distributed random variables, and the iteration is given by [3]

$$x_{n+1} = 1 - \mu|x_n|^z + \varepsilon\xi_n \tag{4}$$

(*i.e.* $g(x, y) = \varepsilon y$).

Example 2: the sequence $\{\xi_n\}$ is generated by a rotation of the circle [4]. One chooses a rotation number Ω and defines

$$\begin{cases} \xi_0 = 0 , \\ \xi_{n+1} = \xi_n + \Omega \qquad \mathrm{mod}\, 2\pi , \end{cases} \tag{5}$$

and

$$x_{n+1} = 1 - \mu|x_n|^z + \varepsilon \sin \xi_n \tag{6}$$

(*i.e.* $g(x, y) = \varepsilon \sin y$). In general $\Omega/2\pi$ is chosen irrational although this is not necessary. This is not what is usually called «noise» in the physical literature. However, it is covered by our previous definition and we shall adopt this convention throughout this paper.

In both examples, the parameter ε measures the amplitude of the noise, and in particular, if we set $\varepsilon = 0$ we recover the standard iteration scheme. A natural question is: how to measure the influence of the noise on the accumulation of period-doublings? There are essentially two ways of doing this. The first method consists in measuring the Lyapunov exponent L which describes the instability of the trajectories [3]. More precisely one sets μ equal to the critical value μ_c and computes L defined by

$$L = \lim_{N\to+\infty} 1/N \sum_{n=0}^{N} \ln|f'(x_n)| . \tag{7}$$

For $\varepsilon = 0$,it is known that $L = 0$. If ε is small one gets a scaling law for the envelope $\bar{L}(\varepsilon)$ of the Lyapunov exponent

$$\bar{L}(\varepsilon) = \varepsilon^{\chi}, \tag{8}$$

where χ is a universal exponent which does not depend on the aspecific shape of f, but only on the order z of its local maximum.

Another method (which is not very well suited for accurate numerical computations) is to consider the bifurcation diagram of the mapping. This diagram is obtained by plotting the attractor (the asymptotic attracting set for almost any trajectory) as a function of the parameter μ. If $\varepsilon = 0$ one observes the well-known infinite sequence of windows corresponding to period-doublings which accumulate at the critical value μ_c [1]. In the presence of noise the smallest windows are of course washed out. However, if we let ε tend to zero, we should recover the original picture. A natural question is: by which quantity should one divide the amplitude of the noise (ε) to see another level of windows? For the case of a noise generated by Gaussian-independent random variables (4), this number turns out to be equal to a universal constant $\varkappa$. This constant is related to the universal exponent χ by the equation

$$\varkappa = 2^{1/\chi}. \tag{9}$$

This relation can be deduced from a renormalization group analysis independently of the nature of noise [5-7]. In fig. 1 we have computed the z-dependence of the rescaling factor $\varkappa$; in the generic quadratic case $\varkappa(z=2) = 6.61903...$. However, for periodic noises the situation seems to be more involved [4, 8, 9], and we will now describe this case in more details.

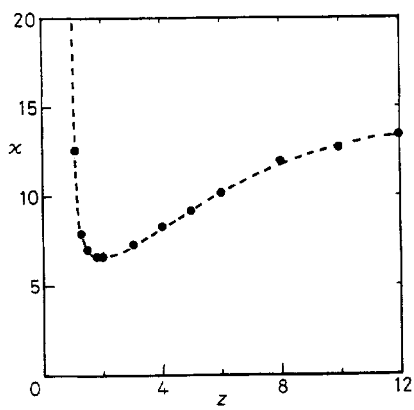

Fig. 1. – The renormalization group prediction for the universal constant $\varkappa(z)$ (see formula (9)) as a function of the order z of the local maximum of $f(x)$.

In [10] a more general renormalization method is proposed to study the scaling limit ($\varepsilon \to 0$, $\mu \to \mu_c$). The universal numbers appear as Lyapunov exponents of a skew product renormalization. This method allows to draw some conclusions about the noise dependence of the universal numbers. For example, if one considers a perturbation which is a trigonometric polynomial evolving under a rotation of angle Ω, then one finds the same universal number for a set of values of Ω of full measure. As explained below, numerical results indicate that this number is about equal to the universal number $\varkappa(z)$ for independent noise. This fact is illustrated in fig. 2 using (5) and (6) in the generic quadratic case $z = 2$ with $\Omega = 3$. Following the second method of analysis discussed above, we plot $\alpha^p f^{2^p}(x)$ ($x \in [-1/2\alpha^p, 1/2\alpha^p]$) as a function of $\lambda^p(\mu_c - \mu)$, where $\alpha = -2.5029...$ and $\lambda = 4.6692...$ are well-known scaling factors [1, 2]. For each p one observes the same trends in the bifurcations diagram, namely the same number of period-doubling bifurcation when rescaling ε by a factor $\varkappa^p$ with $\varkappa = 6.61903...$. Computing the Lyapunov exponent as defined in (7) with (5) and (6) for z different from 2, we show in fig. 3 that this result extends to the whole set of universality classes. From the measure of the exponent χ entering the scaling laws (8) and (9), one recovers the theoretical predictions $\varkappa(z)$ for random noise as described in

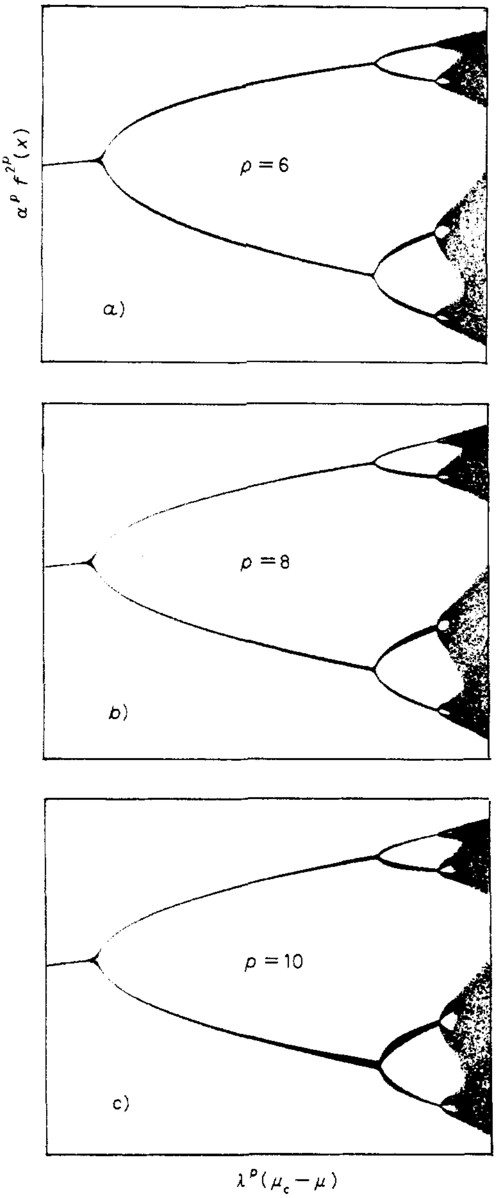

Fig. 2. – Bifurcation diagrams computed with the mapping (6) with $z=2$ and $\Omega=3$. When plotting $\alpha^p f^{2^p}(x)$ *vs.* $\lambda^p(\mu_c-\mu)$, one observes the same number of period-doubling bifurcations if one rescales ε by a factor $\varkappa^p$ in good agreement with the renormalization group predictions: $\alpha=-2.5029$, $\lambda=4.6692$, $\varkappa=6.6190$ ($x\in[-1/2\alpha^p, 1/2\alpha^p]$).

fig. 1 independently of the value of the winding number Ω ($\Omega/2\pi$ irrational). Note, however, that the Lyapunov exponent might well be slightly different than the noise exponent; actually one can prove that it is in between α^2 and $\varkappa$ [10].

This kind of «superuniversality» of the critical exponent is not true for every value of Ω. If one takes for example $\Omega=0$, the universal number $\varkappa$ is the usual one λ, *i.e.* $\varkappa=4.6692\ldots$ for

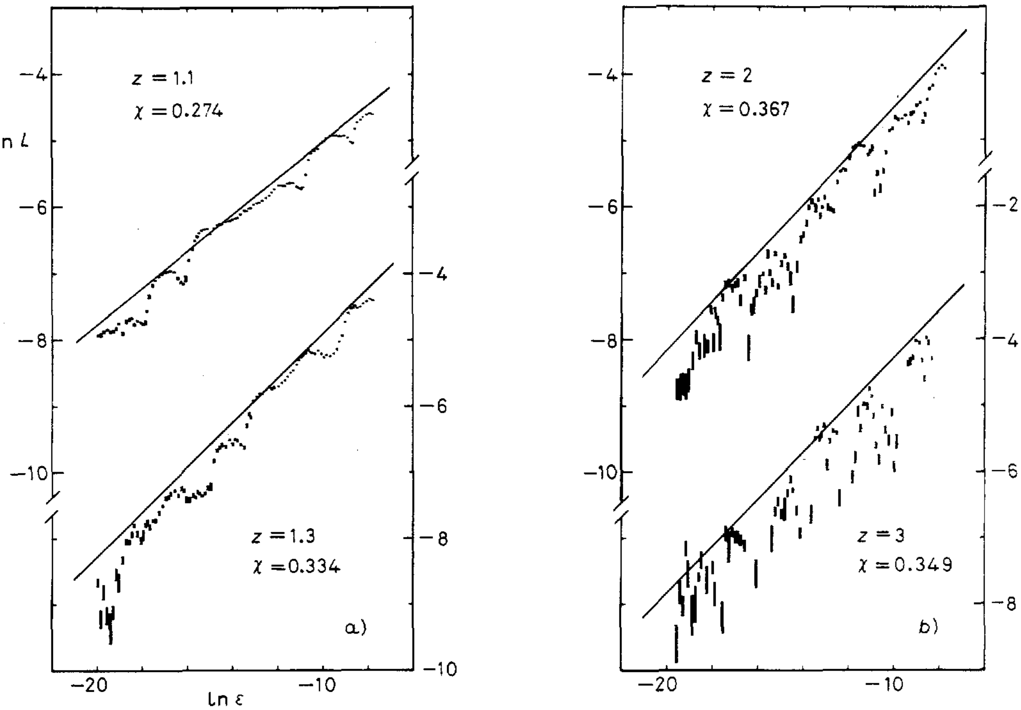

Fig. 3. – A log-log plot of L *vs.* ε. L is calculated with $N = 10^5$ iterations of (6) for $\mu = \mu_c(z)$, $\Omega = 3$. The error bars indicate the dispersion in the computed Lyapunov exponent for the last $2 \cdot 10^4$ iterations. The continuous lines correspond to the predictions (8) given by the renormalization group analysis (9) (fig. 1): *a*) $z = 1.1$, $\varkappa = 12.597$; $z = 1.3$, $\varkappa = 7.951$; *b*) $z = 2$, $\varkappa = 6.619$; $z = 3$, $\varkappa = 7.307$.

$z = 2$. The critical number depends now on $\Omega/2\pi$. An example is shown in fig. 4. Moreover, as proved in [10], if one denotes by h the mapping of the interval [0, 1]

$$x \to h(x) = 2x \qquad \mathrm{mod}\ 1 \tag{10}$$

two rational values of $\Omega/2\pi$ which are preimages under h of the same periodic orbit (of h) will give rise to the same universal number. We illustrate this result in fig. 4.

This renormalization analysis is, however, valid only in the scaling limit regime *i.e.* the universal number $\varkappa(z)$ measures the rate of escape from the usual fixed point due to the presence of the new degrees of freedom associated to the noise. Somewhat different results were observed [4, 8] if one computes the Lyapunov exponent in different regimes. For a given (small) ε, the number of iterations one can use to compute L (see formula (7)) in the scaling regime is limited by the fact that one has to stay (from the renormalization point of view) near the classical deterministic fixed point. If N is taken too large, one will drift away and eventually reach a new region in the space of mappings where the dependence upon the noise is nontrivial. We expect this cross-over to take place for a number of iterations N satisfying

$$\varepsilon N^{1/\chi} = O(1)\ , \tag{11}$$

where χ is defined in (9). This comes roughly as follows: p steps of renormalization expand ε by a factor $\varkappa^p$; on the other hand, since each renormalization corresponds to a doubling of the

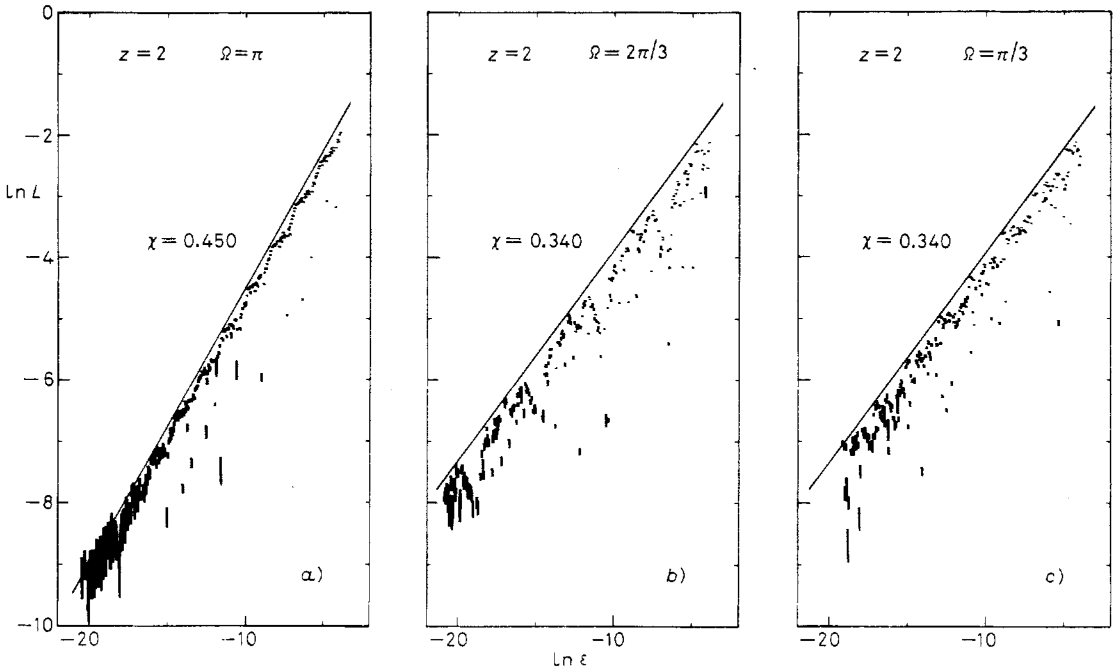

Fig. 4. – A log-log plot of L *vs.* ε. L is calculated with $N = 10^5$ iterations of (6) for $\mu = \mu_c$ and rational forcings. The continuous lines correspond to the predictions given by the renormalization group analysis: *a*) $\Omega = \pi$, $\varkappa = \lambda = 4.669$; *b*) $\Omega = 2\pi/3$, $\varkappa = 7.678$; *c*) $\Omega = 2\pi/6$, $\varkappa = 7.678$.

unit of time, p steps of renormalization need $N = 2^p$ iterations. The numerical results presented in fig. 5 show this phenomenon very clearly for the particular choice $z = 3/2$ in (6). We have used a fixed number of iterations $N = 10^5$ of (6) to compute a numerical approximation of L in (7). If ε is not too small the slope of the curve $\log \bar{L}$ *vs.* $\log \varepsilon$ appears to be very sensitive to Ω, whereas for smaller values of ε one gets a different result in good agreement with the previous universal exponent $\chi = 0.3564...$ as given by (9) with $\varkappa(z = 3/2) = 6.9935...$ (see fig. 1). Notice that in this new regime (ε bigger than the value given by (11)), the envelope of the Lyapunov exponent still has a power law behaviour as a function of ε, although the exponent depends strongly on the rotation number Ω. This of course suggests that this «nontrivial noisy regime» is associated to another renormalization depending on the noise.

In a previous work [4, 8], Ω-dependent critical exponents have been obtained (in the generic case $z = 2$) when linearizing the renormalization operator in the neighbourhood of an invariant cycle of the general form

$$\Gamma_\Omega(x, \theta) = \rho_\Omega(x, \theta) \exp[i\Psi_\Omega(x, \theta)], \tag{12}$$

where a complex notation has been used in order to account for the actual bidimensionality of the problem. If we approximate (12) by the invariant circle $\rho(x) \exp[i\theta]$, where $\rho(x)$ is the usual fixed point, the modulus of an appropriate perturbation was shown to expand by a factor $\varkappa_\Omega$ when averaging over successive iterations of the renormalization operation. In fig. 6 we have computed the Ω-dependence of this average dilation factor $\varkappa_\Omega$ using the universal constant $\varkappa$ (see fig. 1) as unit scale for different values of the order z of the local maximum of

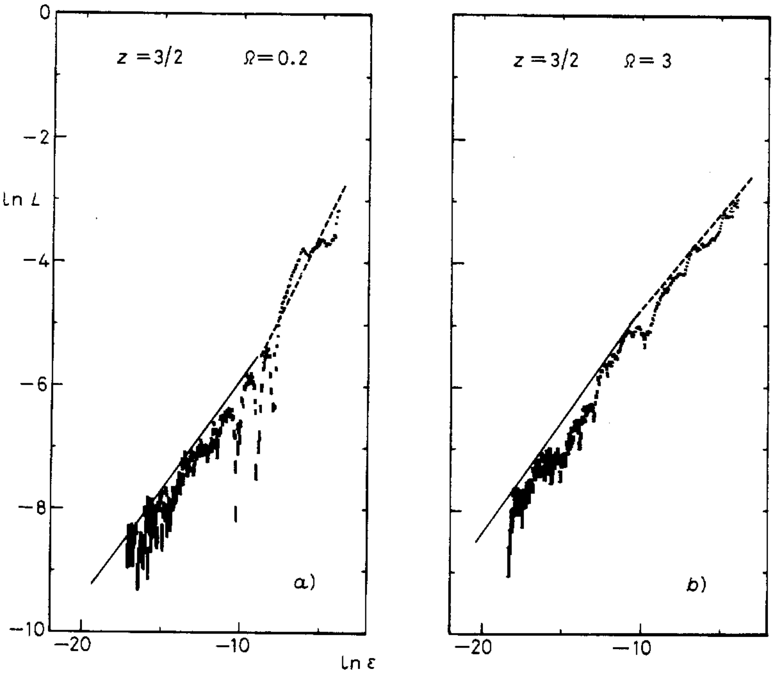

Fig. 5. – A log-log plot of L *vs.* ε. L is calculated with $N = 10^5$ iterations of (6) for $\mu = \mu_c$, $z = 1.5$. The continuous lines correspond to the prediction (8) given by the renormalization group analysis (9) near the usual fixed point: $\chi = \ln 2/\ln \varkappa$ (see fig. 1); the dashed lines correspond to the predictions given by the renormalization group analysis near the invariant cycle (12): $\chi_\Omega = \ln 2/\ln \varkappa_\Omega$ (see fig. 6). *a*) $\Omega = 0.2$, $\varkappa = 6.9935$, $\varkappa_\Omega = 3.7756$; --- $\chi_\Omega = 0.522$, —— $\chi = 0.356$. *b*) $\Omega = 3$, $\varkappa = 6.9935$, $\varkappa_\Omega = 9.4253$; --- $\chi_\Omega = 0.309$, —— $\chi = 0.356$.

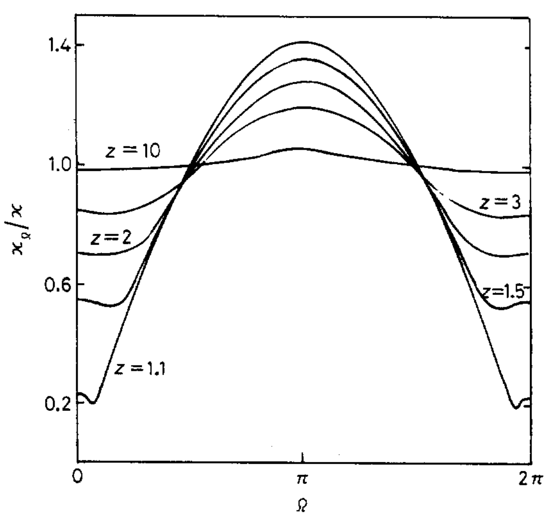

Fig. 6. – The Ω-dependence of the universal constant $\varkappa_\Omega$ predicted by the renormalization group analysis near the invariant cycle (12) as compared to $\varkappa(z)$ derived from the renormalization group analysis near the usual fixed point (see fig. 1). Here Ω takes only irrational values.

$f(x)$. $\varkappa_\Omega$ is a symmetric function of Ω with respect to π; this is a consequence of the global invariance of the problem under the change $\Omega \to -\Omega$. Note that there exists a range of Ω-values around π such that $\varkappa_\Omega > \varkappa$. The graph of $\varkappa_\Omega$ becomes flatter when one increases z. The result obtained for the universality class $z = 10$ strongly suggests that in the asymptotic limit $z \to +\infty$, the ratio $\varkappa_\Omega/\varkappa$ converges to 1. In order to test the relevance of (12) one thus needs to investigate universality classes associated with small values of z. In fig. 5 we have represented in dashed lines the theoretical predictions $\chi_\Omega = \ln 2/\ln \varkappa_\Omega$ obtained from fig. 6 for the universality class $z = 3/2$. For both the considered values of Ω, these predictions are in good agreement with the scaling behaviour observed in the «nontrivial noisy regime». Let us note that our particular choices for Ω are quite representative of the cross-over phenomenon we are discussing; for $\Omega = 0.2$, $\varkappa_\Omega < \varkappa$ implies an increase of the slope of $\log \bar{L}$ *vs.* $\log \varepsilon$ as seen in fig. 5*a*); whereas for $\Omega = 3$, $\varkappa_\Omega > \varkappa$ implies a decrease of this slope as seen in fig.
5*b*). Such an agreement between simulation and theory is not specific to a particular universality class; we have performed numerical experiments for different values of z which all confort our theoretical argumentation. It is thus very likely that the cross-over phenomenon observed when measuring the critical exponent associated to a periodic forcing of the period-doubling cascade is the consequence of a competition between the deterministic fixed point and a noise-dependent fixed point (*i.e.* the invariant cycle (12)).

In conclusion, let us emphasize that if one extends such an approach to type-I intermittency in the presence of a periodic noise, both the skew product renormalization in the scaling regime and the noise-dependent renormalization in the «nontrivial noisy» regime lead to critical exponent [9, 11]

$$\chi = \ln 2/\ln \varkappa = z - 1 \tag{13}$$

as deduced from the universal constant $\varkappa = 2^{z-1}$. Notice that this critical exponent is different from the exponent obtained in the presence of a random noise $\chi = (z+1)/2(z-1)$ [12-15].

In fig. 7 we illustrate the result of a numerical investigation of the two-dimensional mapping

$$\begin{cases} x_{n+1} = \mu + x_n + a|x_n|^z + \varepsilon \sin \xi_n \qquad \bmod 1\,, \\ \xi_{n+1} = \xi_n + \Omega \qquad \bmod 2\pi\,. \end{cases} \tag{14}$$

For $\varepsilon = 0$, this mapping reduces to the simplest model exhibiting type-I intermittency [16, 17], *i.e.* the family of iterations

$$x_{n+1} = \mu + x_n + a|x_n|^z, \tag{15}$$

where μ accounts for a displacement from the tangent bifurcation value ($\mu_c = 0$) and $-1 \leq x \leq 1$ ensures the reinjection process.

For small ε-values, the envelope of the Lyapunov exponent has still a power law behaviour in good agreement with the theoretical prediction (13). Contrary to the period-doubling scenario there is no cross-over effect when plotting L *vs.* ε on a full logarithmic scale (fig. 7). However, one should mention that the scaling regime is rather poorly investigated in this numerical experiment, since according to our statistics ($N = 10^5$ iterations), the convergence of the Lyapunov exponent becomes questionable for ε-values

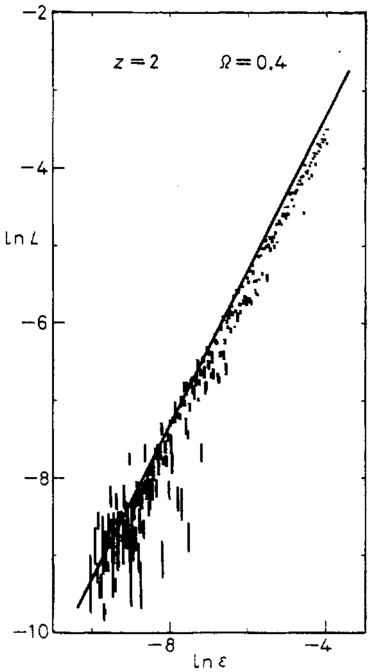

Fig. 7. – Log-log plot of L *vs.* ε for type-I intermittency in the presence of a periodic noise. L is calculated with $N = 10^5$ iterations of (14) for $\mu = \mu_c = 0$, $a = 1$, $\Omega = 0.4$, $z = 2$. The continuous line corresponds to the prediction (13) (—— $\chi = 1$).

below 10^{-4}. Moreover, it is not certain that on increasing the number of iterations one would be able to decide whether or not such a cross-over phenomenon exists, since from (11) it is expected to shift to smaller values of ε.

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