Phase transition-like behavior in a low-pass filter

H. Krivine[a)]
Laboratoire de Physique Théorique et Modèles Statistiques, Bât. 100, 91405 Orsay Cedex, France

A. Lesne
Laboratoire de Physique Théorique des Liquides, Université Pierre et Marie Curie, Case Courrier 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

(Received 11 March 2002; accepted 23 August 2002)

We discuss an iterative electric circuit for which the limits of infinite number of elements and zero dissipation in each element do not commute. The circuit is taken from the Feynman lectures, where it was argued on physical considerations that an infinite circuit made only of inductances and capacitances would behave as a dissipative system with nonvanishing resistance below a threshold frequency. The understanding of this behavior requires that the two limits be taken in the appropriate order. This simple example illustrates that caution in multiple limiting procedures is necessary to obtain the correct physical behavior. A close analogy with the standard ferromagnetic transition of the Ising model is drawn.
[DOI: 10.1119/1.1514233]

I. INTRODUCTION

In physics courses, caution when taking infinite limits is often considered as a useless subtlety by students (not to speak of their teachers). Many interesting physical properties can however be missed because of the improper use of mathematical techniques. One of the most famous examples is the incorrect treatment of phase transitions by Mayer,[1] which was corrected by Lee and Yang in 1953.[2] The latter proved that in the thermodynamic limit (that is, when the number N of atoms tends to infinity), a rigorous treatment of the partition function is sufficient to explain the possible discontinuity of the specific heat and compressibility of a real fluid[3] without any assumptions beyond the usual postulates of statistical mechanics. Unfortunately, this proof is a bit tedious and is not included in undergraduate (or even graduate) courses. In this paper we address a simpler problem considered by Feynman[4] that illustrates the same point.

II. AN ITERATIVE ELECTRICAL CIRCUIT

Let z_n be the impedance of an electric circuit made of n sections as indicated in Fig. 1. It is easy to show that

$$z_{n+1}=Z_1+\frac{1}{\dfrac{1}{Z_2}+\dfrac{1}{z_n}}. \tag{1}$$

Namely, the circuit with $n+1$ sections can be viewed as a circuit composed of an impedance Z_1 in series with two impedances Z_2 and z_n in parallel. The problem is to find the limiting value of z_n when n goes to infinity. Mathematically, this problem amounts to investigating the convergence of the recursion relation (1), given the initial condition

$$z_1=Z_1+Z_2. \tag{2}$$

The limit, if any, is necessarily a fixed point of the map:

$$z\to f(z)=Z_1+\frac{1}{\dfrac{1}{Z_2}+\dfrac{1}{z}}. \tag{3}$$

The fixed-point equation $z=f(z)$ reads

$$z^2-Z_1z-Z_1Z_2=0, \tag{4}$$

which admits the roots:

$$z^*=\frac{Z_1}{2}\pm\sqrt{\frac{Z_1^2}{4}+Z_1Z_2}. \tag{5}$$

The sequence in Eq. (3) converges to the fixed point z^* only if the map f is contracting, which requires that $|f'(z)|<1$ in the neighborhood of z^*.[5] We thus have to study the modulus of

$$f'(z^*)=\left(\frac{Z_2}{z^*+Z_2}\right)^2. \tag{6}$$

We now consider the case where each section is composed of an inductance L and a capacitance C (see Fig. 2). In this case the impedances are $Z_1=iL\omega$ and $Z_2=1/iC\omega$, where ω is the frequency of the electric current. A consequence of Eq. (5) is that there are two cases separated by the critical frequency $\omega_c=2/\sqrt{LC}$.

$\omega>\omega_c$. In this case, the roots given by

$$z^*=i\frac{L\omega}{2}\left(1\pm\sqrt{1-\frac{\omega_c^2}{\omega^2}}\right) \tag{7}$$

are purely imaginary.

$\omega<\omega_c$. In this second case, z^* is given by

$$z^*=\frac{L\omega}{2}\left(i\pm\sqrt{\frac{\omega_c^2}{\omega^2}-1}\right), \tag{8}$$

which contains a *nonvanishing real part*.

It is easy to recover, by recursion, that z_n is purely imaginary for all n. Therefore, the sequence cannot converge for $\omega<\omega_c$, because the limit is necessarily a fixed point and the fixed points in this case contain a real part. A straightforward calculation shows that $|f'(z^*)|=1$, for all $\omega<\omega_c$, which confirms the nonconvergence of the sequence.

If $\omega>\omega_c$, we obtain

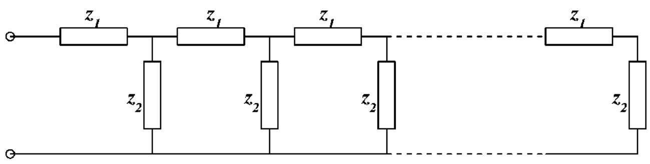

Fig. 1. Ladder (electric) circuit composed of n sections.

$$f'(z^*) = \left[\frac{1}{1 - \frac{2\omega^2}{\omega_c^2}\left(1 \pm \sqrt{1 - \frac{\omega_c^2}{\omega^2}}\right)} \right]^2. \tag{9}$$

The study of the function $y = 2x(1 \pm \sqrt{1 - 1/x})$ with $x > 1$ shows that $|f'(z^*)|$ is always < 1 for the choice of the $+$ sign and > 1 for the other choice. The first fixed point is then the limit of the sequence (z_n).

III. RESOLUTION OF PARADOX

In the second case, where $\omega < \omega_c$, we are faced with the paradox of a purely imaginary sequence having a limit with a nonvanishing real part. From a mathematical point of view the situation is clear: in the second case the sequence diverges, whereas it converges in the first case. Nevertheless, Feynman states that even in the second case, where one still expects a purely imaginary limit, the fixed point, despite its real part, is the correct value of the impedance of the infinite ladder network. According to Feynman, this real part originates in the fact that in an *infinite* circuit, "energy can be continually absorbed from the generator at a constant rate and flows constantly out into the network,"[4] thus creating a dissipation described by the real part of z^*. But this explanation cannot be exempted from a correct mathematical justification. It only means that it should be possible to explain the paradox.

A possible way to resolve the paradox is to take into account that inductances and capacitances are never perfect. To simplify, let us add, as indicated in Fig. 3, a small resistance r in series to each inductance and a large resistance r_0^2/r in parallel to each capacitance (a standard description of the imperfect, hence dissipative character of inductances and capacitances). If $\omega < \omega_c$, a straightforward, although tedious calculation gives to first-order in r (z^* is the fixed point with $+$ sign):

$$|f'(z^*)| = 1 - \frac{r}{2r_0^2} \frac{L\omega_c + r_0^2 C\omega_c}{\sqrt{1 - \frac{\omega^2}{\omega_c^2}}} + O(r^2). \tag{10}$$

Equation (10) implies that for all $r \neq 0$, the transformation f is contracting, hence z^* is also the limit. This limit is given by Eq. (5):

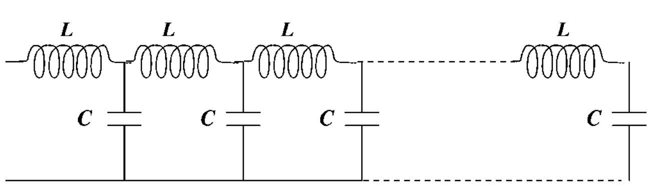

Fig. 2. Ladder circuit composed of pure impedances and capacitances.

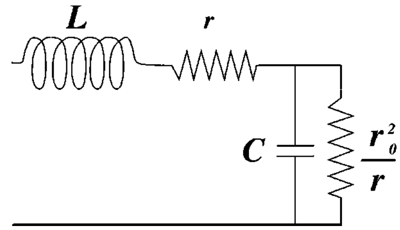

Fig. 3. A section of the ladder composed of imperfect impedances.

$$z^* = \frac{iL\omega + r}{2} + \sqrt{\frac{(iL\omega + r)^2}{4} + \frac{iL\omega + r}{iC\omega + \frac{r}{r_0^2}}}. \tag{11}$$

Equation (11) contains a real part (a dissipative term) that obviously depends on r. But the limit of the real part of z^* when $r \to 0$ *does not vanish* and leads rigorously to Feynman's limit[4] of z_n,

$$\frac{L\omega}{2}\left[i + \sqrt{\frac{\omega_c^2}{\omega^2} - 1} \right].$$

In other words, Feynman's limit is recovered by taking $\lim_{r\to 0}\lim_{n\to\infty} z_n(r)$ and not $\lim_{n\to\infty}\lim_{r\to 0} z_n(r)$, which does not exist.

If perfect capacitances did exist, a peculiar behavior would be expected in this kind of low-bass filter for $\omega < \omega_c$. Indeed, for $r = 0$, there is no stable limiting behavior for $\omega < \omega_c$. It is enlightening to consider the recursion relation $z_{n+1} = f(z_n, r = 0, \omega)$ in the framework of discrete dynamical systems. A bifurcation occurs at $\omega = \omega_c$, where the pair of purely imaginary fixed points (respectively, stable and unstable) collapse and is replaced by a pair of centers (neutrally stable focuses) for $\omega < \omega_c$. In the latter case, the trajectory of the dynamical system (that is, the impedance of a ladder having an increasing number of elements), which theoretically sticks to the imaginary axis, will then be extremely sensitive to any minute fluctuation, shifting from one side to the other of the imaginary axis, thus making large excursions around the centers. The physical implication for an actual system is that the observed response of the ladder would be highly sensitive to any surrounding noise.

This example teaches us two things. First, as stated by Feynman, it points out that dissipation might occur in a system not because of a local mechanism dissipating energy or matter, that is, friction of some kind, but merely because energy may be dissipated into more and more spatially distant degrees of freedom (and not by feeding degrees of freedom at lower and lower scales). In the above example, the limit $N \to \infty$ has to be taken before the limit $r \to 0$ to recover the actual behavior of a real circuit. Another illuminating example of this mechanism is provided by the behavior of a system consisting of a single heavy mass embedded in an infinite square lattice of springs and masses.[6]

The circuit example reminds us of a crucial caveat ubiquitous in critical phenomena. The behavior observed at a macroscopic scale is identified with the behavior of an idealized system obtained after taking various limits: infinite size, critical temperature, and zero field for instance. But often, these limits do not commute, and hence care has to be taken in order that the idealized limit system actually fits reality. The basic mechanism of the spontaneous magnetiza-

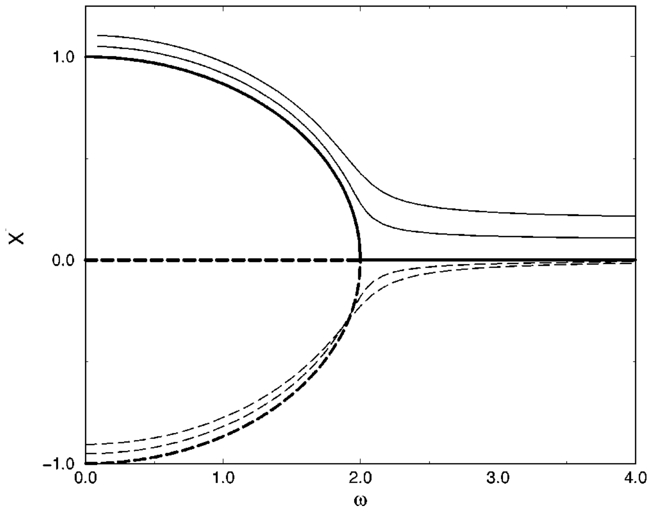

Fig. 4. Bifurcation diagram of a ladder composed of imperfect impedances and conductances. The real parts x^* of the fixed-points z^* of Eq. (1) are drawn vs the frequency ω for $L=1$, $C=1$, $r_0=1$, and different values of r. Two values of $r>0$, namely $r=0.1$ and $r=0.2$ are shown (thin lines) together with the singular limiting case $r=0$ (bold line). Dashed lines correspond to the unstable branches. The branches are smooth for any $r>0$, whereas a bifurcation occurs in $\omega_c=2$ when $r=0$. Note the analogy with the phase diagram of the Ising model: x^* corresponds to the order parameter M (magnetization), ω to the control parameter T (temperature), and r to the symmetry-breaking magnetic field H.

tion of an iron sample is captured by the Ising model.[7] The order in which the limits are taken is again crucial: one first has to introduce a small external magnetic field H, and then take the limit $N\to\infty$ (N being the number of atoms of the sample) and finally the limit $H\to 0$ in that order. The magnetization plays a role analogous to the dissipative part of the impedance and the Curie temperature T_c, above which any spontaneous magnetization is destroyed, a role analogous to the cutoff frequency ω_c.

The analogy between the ladder behavior and ferromagnetism can be pursued further. Let us consider again the recursion relation in Eq. (1) as a discrete dynamical system. For $r>0$, there are two branches of fixed points, one stable and one unstable. The limiting behavior of these branches is singular and is not the same as the picture obtained for $r=0$. The singularity of the limiting behavior $r\to 0$ is shown in Fig. 4, where the real part x^* of the fixed points z^* is plotted as a function of the control parameter ω for different values of r ($r>0$ and $r=0$). Such a bifurcation diagram is quite similar to the phase diagram $M(H,T)$ plotted as a function of T for different values of H ($H>0$ and $H=0$). The importance of the order in which the limits $N\to\infty$ (evolution toward the stable fixed point, if any) and $r\to 0$ is thus graphically recovered. Figure 4 reveals a deep analogy between bifurcations and phase transitions.

An interesting problem would be to develop for the circuit example a finite-size scaling analysis allowing us to reconcile the analytical finite-size behavior and the singular limit behavior, and hence to reconcile experimental or numerical studies and theoretical analysis.

In general, much care should be taken whenever the vanishing of a small parameter qualitatively modifies the asymptotic behavior (number of particles $N\to\infty$ for a many-body system at equilibrium or time $t\to\infty$ for a dynamical system), which prevents a naive series expansion and perturbative analysis from giving the correct physical behavior.[8] The difficulty is currently overcome by resorting to scaling theory or more powerful renormalization-group methods.[9] Such mathematical breakdowns hint at profound physical peculiarities, such as dissipation of a new kind, criticality,[10] bifurcations,[11] or catastrophes.[12] This point is best stated by Michael Berry: ‘‘These noanalyticities, obstructions to naive reduction... should not be regarded as a nuisance. On the contrary, they are pointers to new physics, important features of the world like turbulence and critical behavior, inhabiting the asymptotic borderland between theories.’’ [13]

ACKNOWLEDGMENTS

It is a pleasure to thank P. Leboeuf for suggesting this problem to us and providing a critical reading of the manuscript and O. Bohigas and J. Treiner for the improvements they proposed. We also greatly acknowledge anonymous referees for their careful reading and helpful criticisms.

[a]Present address: Unité de recherche de l’Université de Paris XI associée au CNRS; electronic mail: krivine@ipno.in2p3.fr

[1]J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (Wiley, New York, 1940), Chap. 13.
[2]C. N. Yang and T. D. Lee, ‘‘Statistical theory of equations of state and phase transitions,’’ Phys. Rev. **87**, 404–409 (1952).
[3]An observable signature of this discontinuity is critical opalescence.
[4]R. P. Feynman, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1964), Vol. II, p. 22-12.
[5]The additional condition, not discussed here, requires that the sequence actually reaches the neighborhood of z^*, where $|f'(z)|<1$. In other words, the initial condition should belong to the basin of attraction (with respect to the transformation f) of the stable fixed-point z^*.
[6]R. J. Rubin, ‘‘Statistical dynamics of simple cubic lattices. Model for the study of Brownian motion,’’ J. Math. Phys. **1**, 309–318 (1960); **2**, 373–386 (1961).
[7]E. Ising, ‘‘Beitrag zur Theorie des Ferromagnetismus,’’ Z. Phys. **31**, 252–258 (1925).
[8]A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973).
[9]A. Lesne, *Renormalization Methods* (Wiley, New York, 1998).
[10]N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison–Wesley, Reading, MA, 1992).
[11]J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields* (Springer, Berlin, 1983).
[12]R. Thom, *Structural Stability and Morphogenesis* (Benjamin, Reading, MA, 1975).
[13]M. Berry, ‘‘Some quantum-to-classical asymptotics,’’ in Les Houches 1989, Session LII, *Chaos and Quantum Physics*, edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, Amsterdam, 1990), pp. 251–304.