Table 1.2: Main properties of coherent states. ψ_α denotes a complex variable for bosons ($\zeta = 1$) and a Grassmann number for fermions ($\zeta = -1$). $\mathcal{N} = 2i\pi$ for bosons and $\mathcal{N} = 1$ for fermions.

Definition	$\|\psi\rangle = e^{\zeta \sum_\alpha \psi_\alpha \hat{\psi}_\alpha^\dagger} \|\text{vac}\rangle$ $\langle\psi\| = \langle\text{vac}\| e^{\sum_\alpha \psi_\alpha^* \hat{\psi}_\alpha}$
	$\hat{\psi}_\alpha \|\psi\rangle = \psi_\alpha \|\psi\rangle, \qquad \langle\psi\| \hat{\psi}_\alpha^\dagger = \langle\psi\| \psi_\alpha^*$
Overlap	$\langle\psi\|\psi'\rangle = e^{\sum_\alpha \psi_\alpha^* \psi'_\alpha}$
Closure relation	$\displaystyle\int \prod_\alpha \frac{d\psi_\alpha^* d\psi_\alpha}{\mathcal{N}} e^{-\sum_\alpha \|\psi_\alpha\|^2} \|\psi\rangle\langle\psi\| = 1$
Trace	$\displaystyle \text{Tr}(\hat{A}) = \int \prod_\alpha \frac{d\psi_\alpha^* d\psi_\alpha}{\mathcal{N}} e^{-\sum_\alpha \|\psi_\alpha\|^2} \langle\zeta\psi\|\hat{A}\|\psi\rangle$
Normal-ordered operator	$\langle\psi\| A(\hat{\psi}_\alpha^\dagger, \hat{\psi}_\alpha) \|\psi'\rangle = e^{\sum_\alpha \psi_\alpha^* \psi'_\alpha} A(\psi_\alpha^*, \psi'_\alpha)$

Thus each α must belong to both $\{\alpha_i\}$ and $\{\beta_i\}$ or be absent from both sets for (1.256) to be nonzero. In this case $n = m$ and $\beta_i = \alpha_{P(i)}$ with $P \in S_n$ a permutation, and Eq. (1.256) reduces to

$$\begin{aligned} &\int \prod_\alpha d\psi_\alpha^* d\psi_\alpha \prod_\alpha (1 - \psi_\alpha^* \psi_\alpha) \psi_{\alpha_n} \cdots \psi_{\alpha_1} \psi^*_{\alpha_{P(1)}} \cdots \psi^*_{\alpha_{P(m)}} \\ = &\int \prod_\alpha d\psi_\alpha^* d\psi_\alpha \prod_\alpha (1 - \psi_\alpha^* \psi_\alpha) (-1)^P \psi_{\alpha_n} \cdots \psi_{\alpha_1} \psi^*_{\alpha_1} \cdots \psi^*_{\alpha_m} = (-1)^P. \end{aligned} \tag{1.258}$$

This result should be compared with

$$\langle\alpha_1 \cdots \alpha_n | \beta_1 \cdots \beta_m\rangle = \delta_{n,m} \sum_{P \in S_n} (-1)^P \delta_{\alpha_i, \beta_{P(i)}}, \tag{1.259}$$

which proves the closure relation (1.254).

To conclude this section, we note that fermion coherent states are not physical states of the system; they do not belong to the Fock space. For instance, the expectation value of the particle number operator $\hat{N} = \sum_\alpha \hat{\psi}_\alpha^\dagger \hat{\psi}_\alpha$ in the coherent state $|\psi\rangle$,

$$\frac{\langle\psi|\hat{N}|\psi\rangle}{\langle\psi|\psi\rangle} = \sum_\alpha \psi_\alpha^* \psi_\alpha, \tag{1.260}$$

is not a real number.

1.4 Functional integral in many-particle systems

We are now in a position to derive a functional integral representation of the partition function of a many-particle system following the general procedure outlined in Sec. 1.1.

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