Tutorial class 5: Fermi-liquid theory – Answers

Fermi-liquid theory describes the low-energy properties of an interacting Fermi gas, e.g. electrons in metals, Helium 3, nuclear matter and ultracold fermion gases. In a Fermi liquid, the elementary excitations (quasi-particles and quasi-holes) are in direct correspondence with the (particle or hole) excitations of the ideal Fermi gas; they carry the same quantum numbers and satisfy the Fermi-Dirac statistics. This correspondence can be made explicit by means of an adiabatic switching-on of the interactions.

Let us add a particle with momentum $\mathbf{k}$ ($|\mathbf{k}| > k_F$) and spin σ to the ground state of the ideal Fermi gas. According to the adiabatic continuity assumption, as the interaction is slowly turned on we generate an (excited) eigenstate of the interacting system. However, because of the interactions the state under study is damped and acquires a finite lifetime. Central to Fermi-liquid theory is the fact that the lifetime becomes larger and larger at low energies ($|\mathbf{k}| \to k_F$) and can often be considered as effectively infinite. This property is a consequence of the Pauli principle which makes interactions ineffective near the Fermi surface. Thus, the state obtained by adding a low-lying ($|\mathbf{k}| \gtrsim k_F$) particle to the non-interacting Fermi sea evolves into a quasi-eigenstate of the interacting system, which is referred to as a quasi-particle. Similarly, one can define a quasi-hole by removing a particle with momentum $|\mathbf{k}| \lesssim k_F$ from the non-interacting Fermi sea and switching on the interactions adiabatically. Since the total momentum and spin are conserved, quasi-particles and quasi-holes can be labeled by the same quantum numbers as in the non-interacting case, namely the momentum $\mathbf{k}$ and the spin projection σ along a given axis.[1]

Near the Fermi surface, the quasi-particle dispersion can be expanded as

$$\epsilon_{\mathbf{k}} = \epsilon_{k_F} + v_F^*(|\mathbf{k}| - k_F) + \mathcal{O}\left((|\mathbf{k}| - k_F)^2\right), \qquad v_F^* = \frac{k_F}{m^*}, \tag{1}$$

which defines the (renormalized) Fermi velocity v_F^* and the effective mass m^*. $\epsilon_F = \mu(T = 0)$ is the Fermi energy. The quasi-particle group velocity is defined by

$$\mathbf{v}_{\mathbf{k}}^* = \boldsymbol{\nabla}_{\mathbf{k}} \epsilon_{\mathbf{k}} \to v_F^* \hat{\mathbf{k}} \quad \text{for} \quad |\mathbf{k}| \to k_F. \tag{2}$$

At the Fermi level, the density of quasi-particle states (per spin) $N^*(\xi) = (2\pi)^{-3} \int d^3k \delta(\xi - \xi_{\mathbf{k}})$ takes the value

$$N^*(0) = \frac{m^* k_F}{2\pi^2}. \tag{3}$$

The only difference with the case of the ideal Fermi gas is that the mass is replaced by the effective mass.

A generic low-lying excited state of the ideal Fermi gas, defined by its momentum distribution function $n_{\mathbf{k}\sigma} \neq n_{\mathbf{k}}^0$ (with $n_{\mathbf{k}}^0 = \Theta(k_F - |\mathbf{k}|)$ the momentum distribution in the ground state), can be constructed by combining particle and hole excitations. As the interaction is switched on, it evolves into a quasi-eigenstate of the interacting system characterized by the distribution function $n_{\mathbf{k}\sigma}$.[2] Because of the one-to-one correspondence between particle (or hole) excitations in the ideal Fermi gas and quasi-particle excitations in the Fermi liquid, quasi-particles follow the Fermi-Dirac statistics. Since the concept of quasi-particles refers only to low-lying excited states, $\delta n_{\mathbf{k}\sigma} = n_{\mathbf{k}\sigma} - n_{\mathbf{k}}^0$ should be appreciable only in the vicinity of the Fermi surface.

[1] In a charged Fermi liquid, the quasi-particles (holes) carry charge e ($-e$) where e is the charge of the bare particles.
[2] Note that the term "quasi-particle" often refers to both quasi-particles and quasi-holes.

1) Thermodynamics

In the interacting Fermi liquid, the change in energy due a change $\delta n = n - n^0$ in the quasi-particle distribution function reads

$$\delta E[\delta n] = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \delta n_{\mathbf{k}\sigma} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}. \tag{4}$$

The quadratic term in (4) is due to the interactions between quasi-particles and the function f is referred to as the Landau function. Spin-rotation invariance implies that it can be written in terms of a spin symmetric and a spin antisymmetric part,

$$f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f^s(\mathbf{k},\mathbf{k}') + \sigma\sigma' f^a(\mathbf{k},\mathbf{k}'). \tag{5}$$

Furthermore, for states near the Fermi surface one can set $|\mathbf{k}| = |\mathbf{k}'| = k_F$ so that $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')$ depends only on the angle θ between $\mathbf{k}_F = k_F\hat{\mathbf{k}}$ and $\mathbf{k}'_F = k_F\hat{\mathbf{k}}'$,

$$f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F) = f^s(\theta) + \sigma\sigma' f^a(\theta), \tag{6}$$

where the functions $f^s(\theta)$ and $f^a(\theta)$ can be expanded in Legendre polynomials,

$$f^{s,a}(\theta) = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos\theta), \qquad f_l^{s,a} = (2l+1)\int \frac{d\Omega}{4\pi} f^{s,a}(\theta) P_l(\cos\theta) \tag{7}$$

($d\Omega = d\varphi d\theta \sin\theta$ denotes the elementary solid angle in the direction (φ,θ)). It is convenient to introduce dimensionless parameters – the Landau parameters – by multiplying $f_l^{s,a}$ by the density of states at the Fermi level,

$$F_l^{s,a} = 2N^*(0) f_l^{s,a} \tag{8}$$

(recall that $N^*(\xi)$ is the quasi-particle density of states per spin and $2N^*(\xi)$ the total density of states).

The energy $\tilde{\epsilon}_{\mathbf{k}}$ of a quasi-particle is defined as the variation of the total energy of the system due to the introduction of this quasi-particle, i.e.

$$\tilde{\epsilon}_{\mathbf{k}} = \frac{\delta E[n]}{\delta n_{\mathbf{k}\sigma}} = \epsilon_{\mathbf{k}} + \int_{\mathbf{k}'} \sum_{\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}'\sigma'}, \tag{9}$$

where $\int_{\mathbf{k}} = \int d^3k/(2\pi)^3$. Thus the quasi-particle energy $\tilde{\epsilon}_{\mathbf{k}} \equiv \tilde{\epsilon}_{\mathbf{k}}[\delta n]$ depends on the distribution $\delta n_{\mathbf{k}\sigma} = n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}}$ of quasi-particles present in the system; it coincides with $\epsilon_{\mathbf{k}}$ only when $\delta n = 0$. In an isotropic liquid, spin-rotation invariance ensures that $\tilde{\epsilon}_{\mathbf{k}}$ is independent of σ.

1.1) *Since the states of the interacting Fermi liquid are in a one-to-one correspondence with the states of the free Fermi gas, the entropy takes the same form as in a free Fermi gas,*

$$S[n] = -\sum_{\mathbf{k},\sigma} \left[n_{\mathbf{k}\sigma} \ln n_{\mathbf{k}\sigma} + (1 - n_{\mathbf{k}\sigma}) \ln(1 - n_{\mathbf{k}\sigma}) \right]. \tag{10}$$

In the grand canonical ensemble, the relevant thermodynamic potential is the grand potential $\Omega[n] = E[n] - \mu N[n] - TS[n]$, where $N[n] = \sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}$ is the total particle number. Using the stationarity condition $\delta\Omega[n]/\delta n_{\mathbf{k}\sigma} = 0$, derive the self-consistent equation satisfied by the equilibrium distribution function $\bar{n} \equiv \{\bar{n}_{\mathbf{k}\sigma}\}$ and show that it can be written in terms of the Fermi-Dirac distribution function $n_F(x) = 1/(e^x + 1)$.

Using

$$\left.\frac{\delta N[n]}{\delta n_{\mathbf{k}\sigma}}\right|_{n=\bar{n}} = 1, \tag{11}$$

$$\left.\frac{\delta E[n]}{\delta n_{\mathbf{k}\sigma}}\right|_{n=\bar{n}} = \epsilon_{\mathbf{k}} + \int_{\mathbf{k}'} \sum_{\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')(\bar{n}_{\mathbf{k}'\sigma'} - n^0_{\mathbf{k}'}) \equiv \tilde{\epsilon}_{\mathbf{k}}[\bar{n}], \tag{12}$$

$$\left.\frac{\delta S[n]}{\delta n_{\mathbf{k}\sigma}}\right|_{n=\bar{n}} = -\ln\left(\frac{\bar{n}_{\mathbf{k}\sigma}}{1 - \bar{n}_{\mathbf{k}\sigma}}\right) \tag{13}$$

one obtains

$$\bar{n}_{\mathbf{k}\sigma} = n_F(\tilde{\epsilon}_{\mathbf{k}}[\bar{n}] - \mu), \tag{14}$$

where $n_F(x) = (e^{\beta x} + 1)^{-1}$ is the Fermi-Dirac distribution function.

1.2) *Using Eq. (4) compute the specific heat $C_V = \partial E/\partial T|_N$ to lowest order in T. Consider first non-interacting quasi-particles and then argue that the interactions do not contribute to $\mathcal{O}(T)$ using $\int d|\mathbf{k}|\,\mathbf{k}^2(\bar{n}_{\mathbf{k}\sigma} - n^0_{\mathbf{k}}) = \mathcal{O}(T^2)$. The $\mathcal{O}(T^2)$ shift of the chemical potential, which is necessary to keep the total number of particles fixed, is also irrelevant.*

Neglecting interactions between quasi-particles,

$$E = E(T=0) + \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}}[n_F(\xi_{\mathbf{k}}) - n^0_{\mathbf{k}}] \tag{15}$$

and

$$\begin{aligned}
\frac{dE}{dT} &= \frac{2}{T^2}\sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}} e^{\beta\xi_{\mathbf{k}}}\xi_{\mathbf{k}}}{(e^{\beta\xi_{\mathbf{k}}}+1)^2} \\
&\simeq 2\frac{V}{T^2}N^*(0)\int_{-\mu}^{\infty} d\xi \frac{\xi(\xi+\mu)e^{\beta\xi}}{(e^{\beta\xi}+1)^2} \qquad \text{(integral dominated by } |\xi| \lesssim T\text{: } N(\xi)\simeq N^*(0)\text{)} \\
&= 2TVN^*(0)\int_{-\mu/T}^{\infty} dx\, x(x+\mu/T)\frac{e^x}{(e^x+1)^2} \qquad (x=\beta\xi) \\
&\simeq 2TVN^*(0)\int_{-\infty}^{\infty} dx\, x(x+\mu/T)\frac{e^x}{(e^x+1)^2} \qquad (T \ll \mu) \\
&= 2TVN^*(0)\int_{-\infty}^{\infty} dx\, x^2\frac{e^x}{(e^x+1)^2} \qquad \text{(term odd in } x \text{ does not contribute to integral)} \\
&= V\frac{2\pi^2}{3}N^*(0)T,
\end{aligned} \tag{16}$$

which is the usual result for free electrons except for the effective mass m^* instead of the (bare) mass m. Since $\int d|\mathbf{k}|\,\mathbf{k}^2(\bar{n}_{\mathbf{k}\sigma} - n^0_{\mathbf{k}}) = \mathcal{O}(T^2)$, $\tilde{\epsilon}_{\mathbf{k}}[\bar{n}] = \epsilon_{\mathbf{k}} + \mathcal{O}(T^2)$, so that interactions do not modify the linear T dependence of the specific heat. The $\mathcal{O}(T^2)$ shift of the chemical potential, which is necessary to keep the total number of particles fixed, cannot alter the result (16) to order $\mathcal{O}(T)$ either.

1.3) *Show that the isothermal compressibility*

$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_{T,N} = \frac{1}{n^2}\frac{\partial n}{\partial \mu}\bigg|_T \tag{17}$$

is given by

$$\kappa = \frac{1}{n^2}\frac{2N^*(0)}{1+F_0^s}. \tag{18}$$

[Hint: compute the change in $\mu = \epsilon_{k_F}$ due to a change in k_F to obtain $\partial\mu/\partial n$; use $\frac{1}{4\pi}\int d\Omega\, f^s(\theta) = f_0^s$.] What are the allowed values of F_0^s for the system to be stable? Deduce the macroscopic sound velocity c_s as a function of the sound velocity $v_F/\sqrt{3}$ in the ideal Fermi gas, using $c_s^2 = m^{-1}\partial P/\partial n = 1/nm\kappa$ and the relation between the effective mass and the bare mass, $m^*/m = 1 + F_1^s/3$, which follows from Galilean invariance.

The chemical potential is defined by $\mu = \epsilon_F$. If the Fermi momentum varies by a small amount dk_F, the chemical potential varies by $d\mu$,

$$\mu + d\mu = \epsilon_{k_F+dk_F} + \int_{\mathbf{k}'}\sum_{\sigma'} f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}')[\theta(k_F+dk_F-|\mathbf{k}'|) - \theta(k_F-|\mathbf{k}'|)], \tag{19}$$

taking into account the variation of the quasi-particle distribution $n^0_{\mathbf{k}} = \theta(k_F - |\mathbf{k}|)$ induced by the change of k_F ($\mathbf{k}_F$ is an arbitrary momentum on the Fermi surface: $|\mathbf{k}_F| = k_F$). This implies

$$\begin{aligned}
\frac{\partial \mu}{\partial k_F} &= \frac{\partial \epsilon_{k_F}}{\partial k_F} + \int_{\mathbf{k}'} \sum_{\sigma'} f_{\sigma\sigma'}(\mathbf{k}_F, \mathbf{k}') \frac{\partial}{\partial k_F} n^0_{\mathbf{k}'} \\
&= \frac{\partial \epsilon_{k_F}}{\partial k_F} + \int_{\mathbf{k}'} \sum_{\sigma'} f_{\sigma\sigma'}(\mathbf{k}_F, \mathbf{k}') \delta(k_F - |\mathbf{k}'|) \\
&= \frac{k_F}{m^*} + \frac{2}{8\pi^3} k_F^2 \int d\Omega \, f^s(\theta) \qquad \text{(integration over solid angle)} \\
&= \frac{k_F}{m^*}(1 + F_0^s). \qquad (20)
\end{aligned}$$

Using $n = k_F^3/3\pi^2$ (mean density), we finally obtained

$$n^2 \kappa = \frac{\partial n}{\partial \mu} = \frac{\partial n}{\partial k_F} \frac{\partial k_F}{\partial \mu} = \frac{2N^*(0)}{1 + F_0^s}. \qquad (21)$$

We thus obtain the free fermion result $2N^*(0)$ (with the effective mass instead of the bare mass) with a correction due to interactions between quasi-particles. Stability requires $\kappa > 0$, i.e. $F_0^s > -1$. The macroscopic sound velocity is

$$c_s = \frac{1}{\sqrt{nm\kappa}} = \frac{v_F}{\sqrt{3}} \left(\frac{1 + F_0^s}{1 + F_1^s/3} \right)^{1/2} \qquad (22)$$