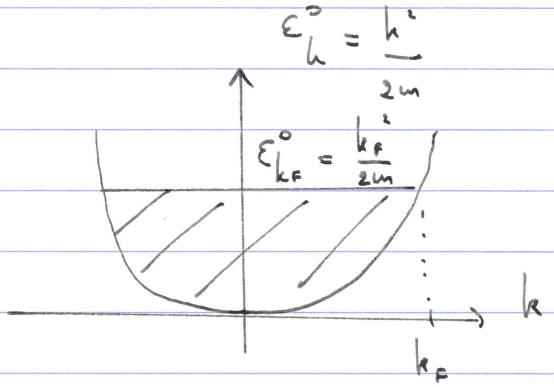
Fermi - liquid theory

Non - interacting bosons → interacting bosons
BEC depleted BEC
particle excitation $\varepsilon_k = \frac{k^2}{2m}$ sound mode $E_k = c|k|$

What about fermions? Is the non - interacting g.s. (Fermi sphere / sea) an (approximately) valid description?

1) The QP concept

1.1) Ideal Fermi gas



density: $$n = \frac{1}{V}\sum_{k,\sigma}\theta(k_F - |k|) = 2\int\frac{d^3k}{(2\pi)^3}\theta(k_F-|k|) = \frac{2}{8\pi^3}\frac{4}{3}\pi k_F^3 = \frac{k_F^3}{3\pi^2}$$

energy: $$E_0 = \sum_{k,\sigma}\varepsilon^0_k\,\theta(k_F-|k|) = \frac{3}{5}n\varepsilon^0_F V$$

momentum distribution: $$n^0_k = \langle \hat\psi^+_{k\sigma}\hat\psi_{k\sigma}\rangle = \theta(k_F-|k|)$$

low - lying excited state: $n_{k\sigma} = n^0_k + \delta n_{k\sigma}$

$$\delta E[n] = E[n] - E_0 = \sum_{k\sigma}\varepsilon^0_k\,\delta n_{k\sigma}$$ additive

particle energy: $$\varepsilon^0_k = \frac{\delta E[n]}{\delta n_{k\sigma}}$$

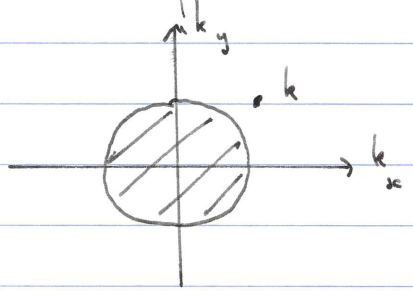
$$\varepsilon^0_k = \varepsilon^0_{k_F} + v_F (|k| - k_F) + O((|k|-k_F)^2)$$

$v_F = \frac{k_F}{m}$ Fermi velocity

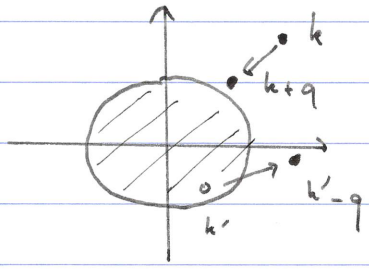
group velocity $\bar{v}_k = \nabla_k \varepsilon^0_k = \frac{k}{m}$

1.2) Interacting Fermi gas.

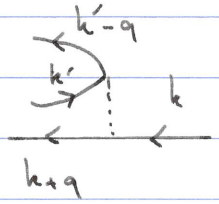
Is a particle excitation "well-defined", i.e. long-lived?



initial state



decay of initial state due to p-h excitations → finite life-time



The phase space available for p-h excitations is suppressed when $|k| \to k_F$:

constraints: $|k|, |k+q| > k_F$
$|k'| < k_F$
$|k'-q| > k_F$

+ energy conservation: $\varepsilon^0_{k+q} - \varepsilon^0_k = -(\varepsilon^0_{k'-q} - \varepsilon^0_{k'})$

→ phase space $\int_{(\text{constraints})} d^3q\, d^3k'\, \delta(\varepsilon^0_{k+q} - \varepsilon^0_k + \varepsilon^0_{k'-q} - \varepsilon^0_{k'}) \sim (|k|-k_F)^2$

→ life-time: $\frac{1}{\tau_k} \sim (|k|-k_F)^2 \ll |\xi_k| \sim ||k|-k_F|$

1.3) Microscopic justification

✱ Spectral function of the ideal Fermi gas:

$$\hat{H}_0 = \sum_{k\sigma} \xi^0_k \, \hat{\psi}^+_{k\sigma} \hat{\psi}_{k\sigma} \quad , \qquad \xi^0_k = \varepsilon^0_k - \mu$$

* g.s.: $|0\rangle = \prod_{k\sigma\ (|k|<k_F)} \hat{\psi}^+_{k\sigma} |\text{vac}\rangle$ Fermi sea

$$\hat{H}_0 |0\rangle = (E_0 - \mu N) |0\rangle$$

* particle excitation: $\hat{\psi}^+_{k\sigma} |0\rangle \,, \quad |k| > k_F$

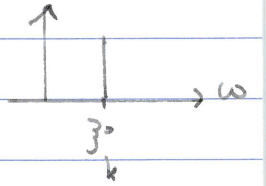
$$e^{-i\hat{H}_0 t} \hat{\psi}^+_{k\sigma} |0\rangle = e^{-i(E_0 - \mu N + \xi^0_k)t} \hat{\psi}^+_{k\sigma} |0\rangle$$

exact eigenstate with excitation energy ξ^0_k.

This also shows up in the retarded Green fct:

$$G^R_0(k,t) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega + i0^+ - \xi^0_k} = -i\,\theta(t)\, e^{-i\xi^0_k t}$$

$$A_0(k,\omega) = -\frac{1}{\pi} \operatorname{Im} G^R_0(k,\omega) = \delta(\omega - \xi^0_k)$$



✱ Spectral function of the interacting FL:

$$G^R(k,\omega) = \left[\omega + i0^+ - \xi^0_k - \Sigma^R(k,\omega)\right]^{-1}$$

pole: $\omega = \xi^0_k + \Sigma^R(k,\omega) = \xi_k - i\Gamma_k$

$$\xi_k = \xi^0_k + \operatorname{Re}\Sigma^R(k, \xi_k - i\Gamma_k) \simeq \xi^0_k + \operatorname{Re}\Sigma^R(k,\xi_k)$$

$$\Gamma_k = -\operatorname{Im}\Sigma^R(k, \xi_k - i\Gamma_k) \simeq -\operatorname{Im}\Sigma^R(k,\xi_k)$$

anticipating that $\Gamma_k \ll |\xi_k|$

* Fermi momentum: $\xi_{k_F} = \varepsilon_{k_F} - \mu = 0$

$= \xi^0_{k_F} + \mathrm{Re}\,\Sigma^R(k_F, 0) = 0$

Luttinger theorem (fixed density): k_F is not changed by interactions, but μ is.

* QP weight and life-time:

For $\omega \simeq \xi_k$: $\omega - \xi^0_k - \Sigma^R(k,\omega)$

$\simeq \underbrace{\omega - \xi^0_k - \mathrm{Re}\,\Sigma^R(k,\xi_k)}_{\omega - \xi_k} - (\omega - \xi_k)\left.\frac{\partial}{\partial\omega}\mathrm{Re}\,\Sigma^R(k,\omega)\right|_{\omega=\xi_k} - i\underbrace{\mathrm{Im}\,\Sigma^R(k,\xi_k)}_{-\Gamma_k}$

$= \frac{\omega - \xi_k}{\mathfrak{z}_k} + i\Gamma_k$ with $\frac{1}{\mathfrak{z}_k} = 1 - \left.\frac{\partial}{\partial\omega}\mathrm{Re}\,\Sigma^R(k,\omega)\right|_{\omega=\xi_k}$

$= \frac{1}{\mathfrak{z}_k}\left(\omega - \xi_k + \frac{i}{2\tau_k}\right)$ with $\frac{1}{2\tau_k} = \mathfrak{z}_k\Gamma_k = -\mathfrak{z}_k\,\mathrm{Im}\,\Sigma^R(k,\xi_k)$

Thus, we can write

$$G^R(k,\omega) = \underbrace{\frac{\mathfrak{z}_k}{\omega - \xi_k + \frac{i}{2\tau_k}}}_{\text{Quasi-particle excitation with energy } \xi_k,\ \text{life-time } \tau_k \text{ and QP weight } \mathfrak{z}_k\ (=\text{pole residue})} + \underbrace{G^R_{inc}(k,\omega)}_{\text{incoherent part (no particular structure in } \omega)}$$

$$G^R(k,t) = -i\,\mathfrak{z}_k\,\theta(t)\,e^{-i\xi_k t - t/2\tau_k} + G^R_{inc}(k,t)$$

well-defined excitation if $\frac{1}{\tau_k} \ll |\xi_k|$

$G^R_{inc}(k,t)$: decays quickly in time

microscopic calculation:

$\frac{1}{\tau_k} \sim (|k| - k_F)^2$

Z_k is a measure of the overlap between the state $\psi^+_{k\sigma}|0\rangle$ and the QP state = fraction of bare particle contained in the QP.

* **Effective mass:**

group velocity: $\vec{v}^*_k = \vec{\nabla}_k \xi_k = v^*_k \hat{k} \xrightarrow[|k|\to k_F]{} v^*_F \hat{k}$

i.e. $v^*_F = \lim_{|k|\to k_F} v^*_k \equiv \frac{k_F}{m^*}$

$$\xi_k = \xi^0_k + \mathrm{Re}\,\Sigma^R(k, \xi_k)$$

$$v^*_k = v_k + \left(\frac{\partial}{\partial k} \mathrm{Re}\,\Sigma^R(k,\omega) + \underbrace{\frac{\partial}{\partial \omega} \mathrm{Re}\,\Sigma^R(k,\omega)}_{1 - 1/Z_k} \cdot \vec{\nabla}\xi_k \right)_{\omega=\xi_k} \qquad v^*_k$$

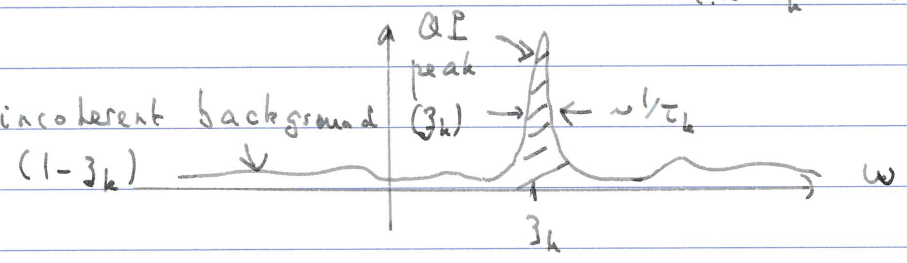
$$= Z_k \left[v_k + \frac{\partial}{\partial k} \mathrm{Re}\,\Sigma^R(k,\omega)\Big|_{\omega=\xi_k} \right]$$

$$\frac{k}{m^*} = Z_k \left[\frac{k}{m} + \hat{k} \frac{\partial}{\partial |k|} \Sigma^R(k,\omega)\Big|_{\omega=\xi_k} \right]$$

$$\frac{m}{m^*} = Z_{k_F} \left[1 + \frac{m}{k_F} \frac{\partial}{\partial |k|} \Sigma^R(k,\omega)\Big|_{\substack{\omega=\xi_k=0 \\ |k|=k_F}} \right]$$

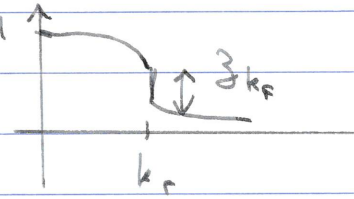
* **Spectral function:**

$$A(k,\omega) = -\frac{1}{\pi} \mathrm{Im}\, G^R(k,\omega) = \frac{1}{\pi} \frac{Z_k/2\tau_k}{(\omega - \xi_k)^2 + 1/4\tau_k^2} + A_{inc}(k,\omega)$$



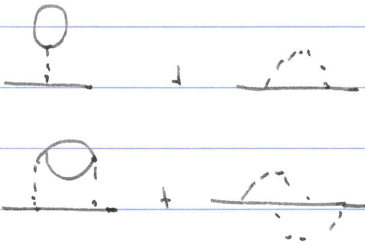
$$\int_{-\infty}^{\infty} d\omega\, A(k,\omega) = 1$$

* **Momentum distribution:** $\langle \psi^+_{k\sigma} \psi_{k\sigma} \rangle = -\langle T_\tau \psi_{k\sigma}(\tau) \psi^+_{k\sigma}(\tau^+) \rangle$



discontinuous jump.

* Microscopic calculation:

$\Sigma =$  $\begin{cases} z_k = 1 \\ \text{no life time} \quad \Sigma(k,\omega) = \Sigma(k) \text{ real} \end{cases}$

Im $\Sigma^R(k,\omega) \sim \omega^2 + \pi^2 T^2$

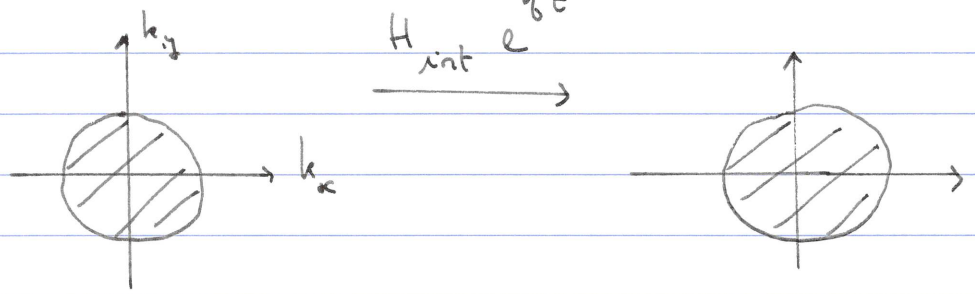
* Response functions and collective modes:

non-FLT: ex: SSB

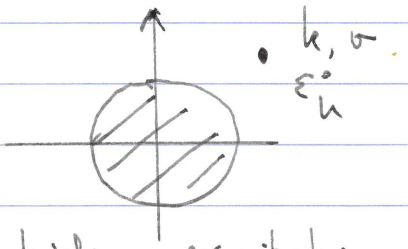
necessary for a complete description; see phenomenological FLT. [interactions between QPs cannot be neglected.]

2) Landau's (phenomenological) FLT

2.1) Adiabatic continuity assumption: any state of the ideal Fermi gas, with distribution $n_{k\sigma} = n^0_k + \delta n_{k\sigma}$, generates an eigenstate of the system as the interactions are switched on. $0^+ \equiv t$

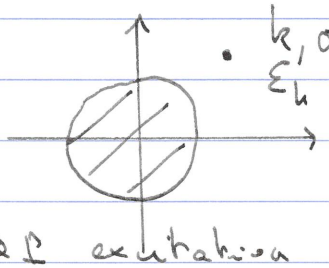


g.s.: ideal gas



particle excitation

g.s.: interacting system



QP excitation [same quantum numbers]

(similarly for hole excited)

finite life time but $\begin{cases} \tau_k \to \infty \text{ for } |k| \to k_F \\ 1/\tau_k \ll |z_k| \end{cases}$

near Fermi surface: $z_k \simeq z_{k_F} + v_F^* (|k| - k_F)$; $v_F^* = \frac{k_F}{m^*}$

density of QP states: $N^*(z) = \frac{1}{V} \sum_k \delta(z - z_k)$

$N^*(0) = \frac{m^* k_F}{2\pi^2}$

low-lying excited state: $n_{k\sigma} = n^0_k + \delta n_{k\sigma}$
(p and h excitations)

↑ QP distribution function

one-to-one correspondence: QP are fermions.

2.2) Landau's functional:

low-lying state: $E = E_0 + \delta E$

$\delta E[\delta n] = \sum_{k\sigma} \varepsilon_k \delta n_{k\sigma}$ no interactions between QPs

$\delta E[\delta n] = \sum_{k\sigma} \varepsilon_k \delta n_{k\sigma} + \frac{1}{2V} \sum_{kk'\sigma\sigma'} f_{\sigma\sigma'}(k,k') \delta n_{k\sigma} \delta n_{k'\sigma'}$

↑ Landau's function

QP energy:

$$\tilde{\varepsilon}_{k\sigma} = \frac{\delta E[\delta n]}{\delta n_{k\sigma}} = \varepsilon_k + \frac{1}{V} \sum_{k'\sigma'} f_{\sigma\sigma'}(k,k') \delta n_{k'\sigma'}$$

↳ ~~$\tilde{\varepsilon}_k$ if spin-rotation invariance~~

* Landau's parameters:

$$f_{\sigma\sigma'}(k,k') = f^s(k,k') + \sigma\sigma' f^a(k,k')$$
$$\simeq f^s(\vec{k}_F, \vec{k}'_F) + \sigma\sigma' f^a(\vec{k}_F, \vec{k}'_F)$$
$$\equiv f^s(\theta) + \sigma\sigma' f^a(\theta) \text{ with } \theta = \widehat{(\vec{k}_F, \vec{k}'_F)}$$

$f^{s,a}(\theta) = \sum_{l=0}^{\infty} f^{s,a}_l P_l(\cos\theta)$ Legendre polynomials

$F^{s,a}_l = 2N^*(0) f^{s,a}_l$ Landau's parameters.
(dimensionless).

Thermodynamics:

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{N,V} = V\,\frac{m^* k_F}{3}\,T \qquad (T \to 0)$$

$$\kappa = \frac{1}{n^2}\left.\frac{\partial n}{\partial \mu}\right|_T = \frac{1}{n^2}\,\frac{2N^*(0)}{1+F_0^s}$$

stability $\Rightarrow \quad F_0^s > -1$

more generally: $F_l^{s,a} > -2l-1$ (Pomeranchuk instabilities)

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3} \qquad \text{(Galilean invariance)}$$

Pauli susceptibility: $\chi_P = \left(\frac{g\mu_B}{2}\right)^2 \frac{2N^*(0)}{1+F_0^a}$

instability if $F_0^a < -1$: ferromagnetism.

2.3) Non-equilibrium properties: When physical properties vary on macroscopic scales $\gg k_F^{-1}$, one can adopt a semi-classical description where $\delta n_{k\sigma}(r,t)$ gives the density of QP excitations with momentum k, spin σ in the vicinity of r:

$$\delta E[\delta n] = \sum_{k\sigma} \int d^3r\, \varepsilon_k\, \delta n_{k\sigma}(r,t)$$

$$+ \frac{1}{2V} \sum_{kk'\sigma\sigma'} \int d^3r\, f_{\sigma\sigma'}(k,k')\, \delta n_{k\sigma}(r,t)\, \delta n_{k'\sigma'}(r,t)$$

f: instantaneous in time, short-range in space.

Kinetic equation:

$$\tilde{\varepsilon}_{k\sigma}(r,t) = \frac{\delta E}{\delta n_{k\sigma}(r,t)} = \varepsilon_k + \frac{1}{V}\sum_{k'\sigma'} f_{\sigma\sigma'}(k,k')\,\delta n_{k'\sigma'}(r,t)$$

$\equiv$ quasi-classical Hamiltonian: $\partial_t r = \nabla_k \tilde{\varepsilon}_{k\sigma}(r,t)$

$$\partial_t k = -\nabla_r \tilde{\varepsilon}_{k\sigma}(r,t)$$

$$\frac{d}{dt} n_{k\sigma}(r,t) = I[n_{k\sigma}] \equiv \partial_t n_{k\sigma}(r,t)\Big|_{\text{collision}}$$

↳ collision integral

$$= \partial_t n_{k\sigma}(r,t) + \nabla_r n_{k\sigma}(r,t)\cdot \dot{r} + \nabla_k n_{k\sigma}(r,t)\,\dot{k}$$

$$= \partial_t n_{k\sigma}(r,t) + \nabla_r n_{k\sigma}(r,t)\cdot\nabla_k \tilde{\varepsilon}_{k\sigma}(r,t) - \nabla_k n_{k\sigma}(r,t)\cdot\nabla_r \tilde{\varepsilon}_{k\sigma}(r,t)$$

To first order in $\delta n_{k\sigma}(r,t) = n_{k\sigma}(r,t) - n^0_k$:

$$= \partial_t \delta n_{k\sigma}(r,t) + \nabla_r \delta n_{k\sigma}(r,t)\cdot\nabla_k \varepsilon_k$$

$$- \nabla_k n^0_k \cdot \int_{k'} \sum_{\sigma'} f_{\sigma\sigma'}(k,k')\,\nabla_r \delta n_{k'\sigma'}(r,t)$$

with $\nabla_k n^0_k = \nabla_k \theta(k_F - |k|)$

$$= \nabla_k \theta(-\xi_k) = -\delta(\xi_k)\,\bar{v}^*_k \qquad \ddagger$$

Hence the linearized kinetic equation

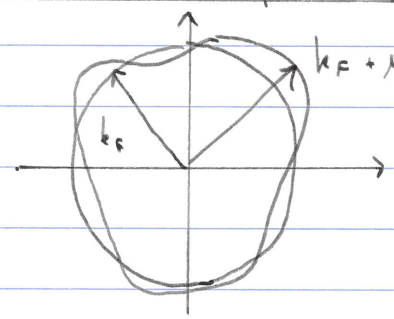
$$\partial_t \delta n_{k\sigma}(r,t) + \nabla_r \delta n_{k\sigma}(r,t)\cdot \bar{v}^*_k$$

$$+ \bar{v}^*_k\,\delta(\xi_k)\cdot\int_{k'}\sum_{\sigma'} f_{\sigma\sigma'}(k,k')\,\nabla_r \delta n_{k'\sigma'}(r,t)$$

$$= I[n_{k\sigma}]$$

* **Collective modes:** a collective mode with momentum q and frequency ω is a coherent superposition of QP-QH pair excitations. When $q \to 0$, the QP/QH excitations have vanishing energies and the collective mode can be seen as a time-dependent displacement of the Fermi surface.

* **static FS displacement:** $k_F \hat{k} \to (k_F + \mu_\sigma(\hat{k}))\,\hat{k}$



$$\xi_k = \nu_F^* (|k| - k_F)$$

$$\to \nu_F^* (|k| - k_F - \mu_\sigma(\hat{k})) = \xi_k - \nu_F^* \mu_\sigma(\hat{k})$$

$$\delta n_{k\sigma} = \theta(-\xi_k + \nu_F^* \mu_\sigma(\hat{k})) - \theta(-\xi_k)$$

$$= \nu_F^* \mu_\sigma(\hat{k})\, \delta(\xi_k)$$

* **dynamic FS displacement:** (and also space dependent).

$$\mu_\sigma(\hat{k})\, e^{i(q\cdot r - \omega t)} + \text{c.c.}$$

$\mu_\sigma(\hat{k}) \in \mathbb{C}$

$$\delta n_{k\sigma}(r,t) = \nu_F^* \delta(\xi_k) \left[\mu_\sigma(\hat{k})\, e^{i(q\cdot r - \omega t)} + \text{c.c.} \right]$$

$$\mu_\sigma(\hat{k}) = \mu^s(\hat{k}) + \sigma\, \mu^a(\hat{k})$$

μ^s: density (sound) mode; μ^a: spin-density mode

Relaxation-time approximation: $I[n] \sim -\dfrac{\delta n_{k\sigma}(r,t)}{\tau}$

* <u>first sound</u>: $\omega\tau \ll 1$ (hydrodynamic mode).

Fermi surface relaxes quickly towards equilibrium except for μ^s_{00} and μ^s_{10} component whose fluctuations (and return to equilibrium) are constrained by the conservation of particle number).

[we use: $\mu^{s,a}(\hat{k}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mu^{s,a}_{lm} Y^m_l(\hat{k})$ ← spherical harmonics.]

$$\frac{\omega^2}{(v_F^* q)^2} = \frac{1}{3}\left[(1+F_0^s)\left(1+\frac{F_1^s}{3}\right)\right]$$

i.e. $c_s = \frac{\omega}{|q|} = \frac{1}{\sqrt{kn m}}$ FS keeps its spherical shape.

* <u>zero sound</u>: sound mode in collisionless regime $\omega\tau \gg 1$.

cf. tutorial class.