Coherent-state functional integral (fermions)

1) Grassmann variables

assume $\hat{\psi}_\alpha |\psi\rangle = \psi_\alpha |\psi\rangle \quad \forall \alpha$

then $\hat{\psi}_\alpha \hat{\psi}_\beta |\psi\rangle = \hat{\psi}_\alpha \psi_\beta |\psi\rangle \underset{(*)}{=} \psi_\alpha \psi_\beta |\psi\rangle$

$= -\hat{\psi}_\beta \hat{\psi}_\alpha |\psi\rangle = -\hat{\psi}_\beta \psi_\alpha |\psi\rangle \underset{(*)}{=} -\psi_\beta \psi_\alpha |\psi\rangle$

[(*) true for $[\psi_\alpha, \hat{\psi}_\beta]_- = 0$ and $[\psi_\alpha, \hat{\psi}_\beta]_+ = 0$]

i.e. $[\psi_\alpha, \psi_\beta]_+ = -[\psi_\beta, \psi_\alpha]_+ = 0$

the ψ's anticommute : Grassmann variables.

\# Grassmann algebra : [vector space endowed with multiplication rule]

set of generators $\{\psi_1, \dots, \psi_n\}$: $[\psi_i, \psi_j]_+ = 0$; $\psi_i^2 = 0$

element of the algebra = polynomial of first order in the ψ_i's

$$f(\psi_1 \dots \psi_n) = \underbrace{c_0}_{\in \mathbb{C}} + \sum_{k=1}^{n} \sum_{\substack{i_1 \dots i_k = 1 \\ i_1 < i_2 \dots < i_k}} \underbrace{c_{i_1 \dots i_k}}_{\in \mathbb{C}} \psi_{i_1} \dots \psi_{i_k}$$

example: 1 generator ψ : $f(\psi) = c_0 + c_1 \psi$

2 generators ψ_1, ψ_2 : $f(\psi_1, \psi_2) = c_0 + c_1 \psi_1 + c_2 \psi_2 + c_{12} \psi_1 \psi_2$

* **Differentiation:** $\frac{\partial}{\partial \psi_i} \psi_j = \delta_{ij}$

$$[\psi_i, \psi_j]_+ = 0 \quad \Rightarrow \quad [\partial_{\psi_i}, \partial_{\psi_j}]_+ = 0 \,, \quad \partial_{\psi_i}^2 = 0$$

$$\partial_{\psi_i} \psi_j \psi_i = - \partial_{\psi_i} \psi_i \psi_j = - \psi_j$$

* **Integration:** identical to differentiation

$$\int d\psi_i \, f(\psi_1 \dots \psi_n) = \partial_{\psi_i} f(\psi_1 \dots \psi_n)$$

ensures that

i) $\int d\psi_i \, \partial_{\psi_i} f = \partial_{\psi_i}^2 f = 0$ as for ordinary functions vanishing at ∞

ii) $\partial_{\psi_i} \underbrace{\int d\psi_i \, f(\psi_1 \dots \psi_n)}_{\text{independent of } \psi_i} = \partial_{\psi_i}^2 f = 0$

* **Grassmann algebra with $2n$ generators:** $\{\psi_1 \dots \psi_n, \psi_1^* \dots \psi_n^*\}$

complex conjugation:

$$(\psi_i)^* = \psi_i^*$$
$$(\psi_i^*)^* = \psi_i$$
$$(c \psi_i)^* = c^* \psi_i^* \qquad (c \in \mathbb{C})$$
$$(\psi_i \psi_j)^* = \psi_j^* \psi_i^*$$

example: $\{\psi, \psi^*\}$: $f(\psi, \psi^*) = c_0 + c_1 \psi^* + c_2 \psi + c_{12} \psi^* \psi$

$$\partial_\psi f = c_2 - c_{12} \psi^*$$
$$\partial_{\psi^*} f = c_1 + c_{12} \psi$$
$$\partial_{\psi^*} \partial_\psi f = - c_{12}$$

Gaussian integrals:

$$\int d\psi^* d\psi\, e^{-a\psi^*\psi} = \int d\psi^* d\psi\, (1 - a\psi^*\psi) = a$$

$$\int \prod_i d\psi_i^* d\psi_i \exp\Big\{ - \sum_{ij} \psi_i^* A_{ij} \psi_j \Big\} = \det A$$

$$Z[J^*, J] = \int \prod_i d\psi_i^* d\psi_i \exp\Big\{ - \psi^\dagger A \psi + \sum_i (J_i^* \psi_i + \text{c.c.}) \Big\}$$

$$= (\det A) \exp\Big\{ \sum_{ij} J_i^* A^{-1}_{ij} J_j \Big\} \quad \text{with } J_\nu^{(*)} \text{ Grassmann variable.}$$

$$Z[J^*, J] = \int \mathcal{D}[\psi^*, \psi] \exp\Big\{ - \int_{x,y} \psi^*(x) A(x,y) \psi(y) + \int_x (J^* \psi + \text{c.c.}) \Big\}$$

$$= (\det A) \exp\Big\{ \int_{x,y} J^*(x) A^{-1}(x,y) J(y) \Big\}$$

2) Coherent states:

$\{ \hat\psi_\alpha^\dagger, \hat\psi_\alpha \} \longrightarrow$ generators $\{ \psi_\alpha^*, \psi_\alpha \}$

$$\big[\psi_\alpha^{(*)}, \hat\psi_\beta^{(\dagger)} \big]_+ = 0$$

$$(\psi_\alpha \hat\psi_\beta)^\dagger = \hat\psi_\beta^\dagger \psi_\alpha^* , \quad (\psi_\alpha \hat\psi_\beta^\dagger)^\dagger = \hat\psi_\beta \psi_\alpha^* , \text{ etc.}$$

NB: $\notin$ Fock space.

$$|\psi\rangle = \exp\Big\{ - \sum_\alpha \psi_\alpha \hat\psi_\alpha^\dagger \Big\} |\text{vac}\rangle = \prod_\alpha (1 - \psi_\alpha \hat\psi_\alpha^\dagger) |\text{vac}\rangle$$

$\langle \hat N \rangle = \sum_\alpha \psi_\alpha^* \psi_\alpha \notin \mathbb{R}$.

$$\langle \psi | = \langle \text{vac} | \exp\Big\{ \sum_\alpha \psi_\alpha^* \hat\psi_\alpha \Big\} = \langle \text{vac} | \prod_\alpha (1 + \psi_\alpha^* \hat\psi_\alpha)$$

Properties:

$$\hat\psi_\alpha |\psi\rangle = \psi_\alpha |\psi\rangle$$
$$\hat\psi_\alpha^\dagger |\psi\rangle = - \partial_{\psi_\alpha} |\psi\rangle$$
$$\langle \psi | \hat\psi_\alpha^\dagger = \langle \psi | \psi_\alpha^*$$
$$\langle \psi | \hat\psi_\alpha = \partial_{\psi_\alpha^*} \langle \psi |$$

overlap: $\langle \psi | \psi' \rangle = \exp\Big\{ \sum_\alpha \psi_\alpha^* \psi'_\alpha \Big\}$

closure relation: $\int \prod_\alpha d\psi_\alpha^* d\psi_\alpha \, e^{-\sum_\alpha \psi_\alpha^* \psi_\alpha} |\psi\rangle\langle\psi| = \hat{1}$

normal-ordered operator: $\langle\psi| A(\hat{\psi}_\alpha^\dagger, \hat{\psi}_\alpha) |\psi'\rangle = A(\psi_\alpha^*, \psi'_\alpha) \langle\psi|\psi'\rangle$

3) Functional integral

Trace:

$$Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_n \langle n| e^{-\beta\hat{H}} |n\rangle$$

$$= \sum_n \int \underbrace{\frac{d(\psi^*,\psi)}{\prod_\alpha d\psi_\alpha^* d\psi_\alpha}} \, e^{-\sum_\alpha \psi_\alpha^* \psi_\alpha} \underbrace{\langle n|\psi\rangle\langle\psi| e^{-\beta\hat{H}} |n\rangle}$$

do not commute!

$$= \langle -\psi| e^{-\beta\hat{H}} |n\rangle\langle n|\psi\rangle$$

$$= \int d(\psi^*,\psi)\, e^{-\sum_\alpha \psi_\alpha^* \psi_\alpha} \langle -\psi| e^{-\beta\hat{H}} |\psi\rangle$$

Partition function: FI

$$Z = \int d(\psi^*,\psi)\, e^{-\sum_\alpha \psi_\alpha^* \psi_\alpha} \underbrace{\langle -\psi|}_{\psi_N} e^{-\varepsilon\hat{H}} \cdots e^{-\varepsilon\hat{H}} \underbrace{|\psi\rangle}_{\psi_0}$$

(↑ closure relations)

$$\langle\psi_k| e^{-\varepsilon\hat{H}} |\psi_{k-1}\rangle = \langle\psi_k| 1 - \varepsilon\hat{H} |\psi_{k-1}\rangle + O(\varepsilon^2)$$

$$= \langle\psi_k|\psi_{k-1}\rangle \left[1 - \varepsilon H(\psi_k^*, \psi_{k-1})\right]$$

$$= \exp\left\{ \sum_\alpha \psi_{k\alpha}^* \psi_{k-1,\alpha} - \varepsilon H(\psi_{k\alpha}^*, \psi_{k-1,\alpha}) \right\}$$

$$Z = \int \prod_{k=1}^N d(\psi_k^*, d\psi_k) \exp\left\{ -\varepsilon \sum_{k=1}^N \left[\sum_\alpha \psi_{k\alpha}^* \frac{\psi_{k\alpha} - \psi_{k-1,\alpha}}{\varepsilon} + H(\psi_k^*, \psi_{k-1}) \right] \right\}$$

same as for bosons, but with anti-periodic boundary conditions: $\psi_N^{(*)} = -\psi_0^{(*)}$

continuum-time limit: $\{\psi_{0,\alpha} \dots \psi_{N,\alpha}\} \to \psi_\alpha(\tau)$

$$\varepsilon \sum_{k=1}^{N} \to \int_0^\beta d\tau$$

$$\psi^*_{k,\alpha} \frac{\psi_{k,\alpha} - \psi_{k-1,\alpha}}{\varepsilon} \to \psi^*_\alpha(\tau)\, \partial_\tau\, \psi_\alpha(\tau)$$

$$H(\psi^*_{k,\alpha}, \psi_{k-1,\alpha}) \to H(\psi^*_\alpha(\tau^+), \psi_\alpha(\tau))$$

$$\prod_{k=1}^{N} \prod_\alpha d\psi^*_{k,\alpha}\, d\psi_{k-1,\alpha} \to \mathcal{D}[\psi^*, \psi]$$

$$Z = \int_{\psi^{(*)}(\beta) = -\psi^{(*)}(0)} \mathcal{D}[\psi^*, \psi]\; e^{-S[\psi^*, \psi]}$$

with

$$S[\psi^*, \psi] = \int_0^\beta d\tau \Big[\sum_\alpha \psi^*_\alpha \partial_\tau \psi_\alpha + H(\psi^*_\alpha, \psi_\alpha) \Big]$$

e.g. $$S[\psi^*, \psi] = \int_0^\beta d\tau \int d^d r \sum_\sigma \psi^*_\sigma \left(-\frac{\nabla^2}{2m} - \mu + \partial_\tau \right) \psi_{r\sigma}$$

$$+ \frac{1}{2} \int_0^\beta d\tau \int d^d r\, d^d r' \sum_{\sigma\sigma'} v(r - r') \psi^*_\sigma(r\tau)\, \psi^*_{\sigma'}(r'\tau)\, \psi_{\sigma'}(r'\tau)\, \psi_\sigma(r\tau)$$

for spin-σ fermions interacting with potential $v(r-r')$.

* Green function:

$$G(\alpha\tau, \alpha'\tau') = -\langle T_\tau\, \hat\psi_\alpha(\tau)\, \hat\psi^+_{\alpha'}(\tau') \rangle$$

$$= -\frac{1}{Z} \mathrm{Tr} \left\{ e^{-\beta \hat H}\, T_\tau\, \hat\psi_\alpha(\tau)\, \hat\psi^+_{\alpha'}(\tau') \right\}$$

$$\equiv -\frac{1}{Z} \int_{\psi^{(*)}(\beta) = -\psi^{(*)}(0)} \mathcal{D}[\psi^*, \psi]\; \psi_\alpha(\tau)\, \psi^*_{\alpha'}(\tau')\; e^{-S[\psi^*, \psi]}$$

4) Non-interacting particles

$$\hat{H}_0 = \sum_\alpha \xi_\alpha \hat{\psi}^+_\alpha \hat{\psi}_\alpha \quad , \quad \xi_\alpha = \varepsilon_\alpha - \mu$$

$$S_0 = \int_0^\beta d\tau \sum_\alpha \psi^*_\alpha (\partial_\tau + \xi_\alpha) \psi_\alpha \quad \text{with} \quad \psi^{(*)}_\alpha(\beta) = -\psi^{(*)}_\alpha(0)$$

$$\rightarrow \quad \psi_\alpha(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} e^{-i\omega_n \tau} \psi_\alpha(i\omega_n)$$

$$\psi^*_\alpha(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} e^{i\omega_n \tau} \psi^*_\alpha(i\omega_n)$$

with $\omega_n = \frac{2\pi}{\beta}\left(n + \frac{1}{2}\right)$ fermionic Matsubara frequency

$$S_0 = \sum_{\alpha, \omega_n} \psi^*_\alpha(i\omega_n)(-i\omega_n + \xi_\alpha)\psi_\alpha(i\omega_n)$$

$$Z_0 = \int \prod_{\alpha, \omega_n} d\psi^*_\alpha(i\omega_n)\, d\psi_\alpha(i\omega_n)\, e^{-S_0}$$

$$= \prod_{\alpha, \omega_n} (-i\omega_n + \xi_\alpha)$$

$$\Omega_0 = -\frac{1}{\beta} \sum_{\alpha, \omega_n} \text{Log}(-i\omega_n + \xi_\alpha) \times \text{<V factor } e^{i\omega_n 0^+}$$

$$= -\frac{1}{\beta} \sum_\alpha \text{Log}(1 + e^{-\beta \xi_\alpha})$$

$$G_0(\alpha, i\omega_n) = -\langle \psi_\alpha(i\omega_n) \psi^*_\alpha(i\omega_n) \rangle$$

$$= -\frac{1}{Z} \int \mathcal{D}[\psi^*, \psi]\, \psi_\alpha(i\omega_n) \psi^*_\alpha(i\omega_n)\, e^{-S_0}$$

$$= \frac{1}{i\omega_n - \xi_\alpha}$$

5) Perturbation theory

5.1) Wick's theorem

$$Z_0[J^*, J] = \int \mathcal{D}[\psi^*, \psi] \exp\left\{ - \underbrace{\int d\tau d\tau' \sum_{\alpha\alpha'} \psi^*_\alpha(\tau) \left[-G_0^{-1}(\alpha\tau, \alpha'\tau')\right] \psi_{\alpha'}(\tau')}_{S_0} + \int d\tau \sum_\alpha \left(J^*_\alpha(\tau)\, \psi_\alpha(\tau) + \psi^*_\alpha(\tau)\, J_\alpha(\tau) \right) \right\}$$

(J^*: Grassmann variable)

$$= \underbrace{\det(-G_0^{-1})}_{Z_0[0,0] \equiv Z_0} \exp\left\{ - \int d\tau d\tau' \sum_{\alpha\alpha'} J^*_\alpha(\tau)\, G_0(\alpha\tau, \alpha'\tau')\, J_{\alpha'}(\tau') \right\}$$

$$\langle \psi_{\alpha_1}(\tau_1)\, \psi^*_{\alpha_1'}(\tau_1') \rangle_0 = -\frac{1}{Z_0} \left. \frac{\delta^2 Z_0[J^*, J]}{\delta J^*_{\alpha_1}(\tau_1)\, \delta J_{\alpha_1'}(\tau_1')} \right|_{J^* = J = 0}$$

$$= -G_0(\alpha_1 \tau_1, \alpha_1' \tau_1')$$

$$\equiv \overbrace{\psi_{\alpha_1}(\tau_1)\, \psi^*_{\alpha_1'}(\tau_1')}$$

$$\langle \psi(1)\, \psi(2)\, \psi^*(2')\, \psi^*(1') \rangle_0 = \frac{1}{Z_0} \left. \frac{\delta^4 \; e^{-\int J^* G_0 J} \, Z_0[J^*, J]}{\delta J^*(1)\, \delta J^*(2)\, \delta J(2')\, \delta J(1')} \right|_{J^* = J = 0}$$

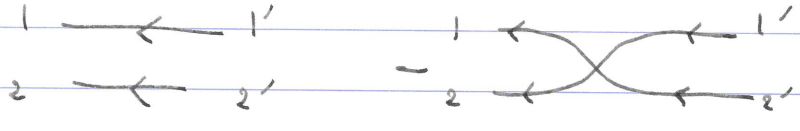
$$= \left. \frac{\delta^4 \; e^{-J^* G_0 J}}{\delta J^*(1)\, \delta J^*(2)\, \delta J(2')\, \delta J(1')} \right|_{J^* = J = 0}$$

$$= G_0(1,1')\, G_0(2,2') - G_0(1,2')\, G_0(2,1')$$

$$\equiv \overbrace{\psi(1)\psi^*(1')}\; \overbrace{\psi(2)\psi^*(2')} - \overbrace{\psi(1)\psi^*(2')}\; \overbrace{\psi(2)\psi^*(1')}$$

$$\equiv \underbrace{\psi(1)\, \psi(2)\, \psi^*(2')\, \psi^*(1')} + \underbrace{\psi(1)\, \psi(2)\, \psi^*(2')\, \psi^*(1')}$$

$$\equiv$$



$$= G_0^{(4)}(12, 2'1')$$

two-particle Green function

More generally (Wick's th):

$$\langle \psi_{\alpha_1}(\tau_1)\cdots\psi_{\alpha_n}(\tau_n)\,\psi^*_{\alpha'_1}(\tau'_1)\cdots\psi^*_{\alpha'_n}(\tau'_n)\rangle_0$$

$$= (-1)^\eta \sum_{p\in\mathcal{S}_n} (-1)^p G_0(\alpha_1\tau_1,\alpha'_{p(1)}\tau'_{p(1)})\cdots G_0(\alpha_n\tau_n,\alpha'_{p(n)}\tau'_{p(n)})$$

$$= \sum \text{all } \underbrace{\text{complete contractions}}$$

each ψ contracted with a ψ^*: $\overbrace{\psi\psi^*} = -G_0$, $\overbrace{\psi^*\psi} = G_0$

overall sign $(-1)^p$.

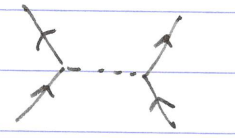
5.2) Thermodynamic potential

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int \mathcal{D}[\psi^*,\psi]\, e^{-S_0 - S_{int}} = \langle e^{-S_{int}}\rangle_0 = \langle e^{-}\rangle_0$$

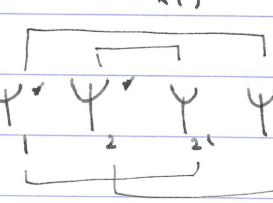
with $S_{int} = \frac{1}{2}\sum_{\{\alpha\}}\int d\tau \underbrace{(\alpha_1\alpha_2|\hat{v}|\alpha'_1\alpha'_2)}_{v_{\alpha_1\alpha_2,\alpha'_1\alpha'_2}}\psi^*_{\alpha_1}\psi^*_{\alpha_2}\psi_{\alpha'_2}\psi_{\alpha'_1}$

* first order:

$$\left(\frac{Z}{Z_0}\right)^{(1)} = \langle -\frac{1}{2}$$



$$\rangle_0$$



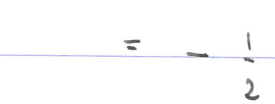
$$= -\frac{1}{2}\int d\tau \sum_{\{\alpha\}} v_{12,1'2'} \langle \psi^*_1\psi^*_2\psi_{2'}\psi_{1'}\rangle_0$$

$$= -\frac{1}{2}\int d\tau \sum_{\{\alpha\}} v_{12,1'2'}\Big[\overbrace{\psi^*_1\psi_{1'}}\,\overbrace{\psi^*_2\psi_{2'}} - \overbrace{\psi^*_1\psi_{2'}}\,\overbrace{\psi^*_2\psi_{1'}}\Big]$$

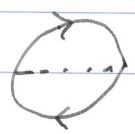
$$(-\overbrace{\psi_{1'}\psi^*_1})(-\overbrace{\psi_{2'}\psi^*_2}) - \cdots$$

$$= -\frac{1}{2}\int d\tau \sum_{\{\alpha\}} v_{12,1'2'}\Big[G_0(1'1)G_0(2'2) - G_0(2'1)G_0(1'2)\Big]$$

$$= -\frac{1}{2}$$



$$+\frac{1}{2}$$



↓ − for bosons.

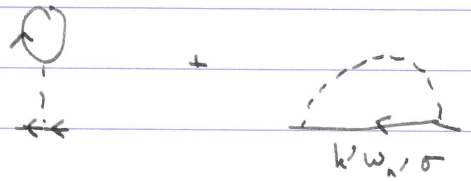
sign of a diagram (order n): $(-1)^n \zeta^{n_L}$
where n_L is the number of closed loops and
$\zeta = 1/-1$ for B / F.

* Linked cluster theorem: $\log \frac{Z}{Z_0} = \sum$ connected graphs

5.3) Green function and self-energy:

Dyson's equation $G = G_0 + G_0 \Sigma G$

$$\text{i.e.} \quad G_\sigma(k, i\omega_n) = \frac{1}{i\omega_n - \xi_k - \Sigma_\sigma(k, i\omega_n)}$$

$\Sigma_\sigma^{(1)}(k, i\omega_n) =$ 

$$= \frac{1}{\beta V} \sum_{k'\omega'} \nu(0)\, G_{0,\sigma'}(k', i\omega_{n'})\, e^{i\omega_{n'} 0^+}$$

$$- \frac{1}{\beta V} \sum_{k'\omega_{n'}} \nu(k-k')\, G_{0,\sigma}(k', i\omega_{n'})\, e^{i\omega_{n'} 0^+}$$

sign of a diagram: $(-1)^n \zeta^{n_L}$ with $\zeta = -1$ for fermions.