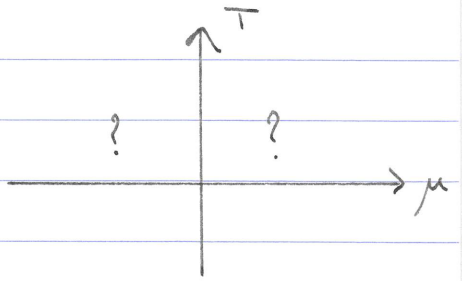
Interacting boson systems



$$S = \sum_{\tilde{k}} \psi^*(\tilde{k})(-i\omega_n + \xi_k)\psi(\tilde{k}) + \frac{g}{2\beta V}\sum_{\tilde{k}\tilde{k}'\tilde{q}} \psi^*\psi^*\psi\psi$$

1) Dilute classical gas: $\mu < 0$, $T \ll |\mu|$

✻ $g = 0$:

$$\Omega_0 = \frac{1}{\beta}\sum_k \log(1 - e^{-\beta\xi_k}) \quad \text{with} \quad e^{-\beta\xi_k} = z\, e^{-\beta\varepsilon_k}, \quad z = e^{\beta\mu} \ll 1$$

$$\simeq -\frac{V}{\beta}\int_k z\, e^{-\beta\varepsilon_k}$$

$$= -\frac{V}{\beta} z \int_0^\infty d\varepsilon\, \rho(\varepsilon)\, e^{-\beta\varepsilon} \quad \text{with} \quad \rho(\varepsilon) = \int_k \delta(\varepsilon - \varepsilon_k) = \frac{(2m)^{3/2}}{4\pi^2}\sqrt{\varepsilon}$$

$$= -\frac{V}{\beta} z \frac{(2m)^{3/2}}{4\pi^2}\underbrace{\int_0^\infty d\varepsilon \sqrt{\varepsilon}\, e^{-\beta\varepsilon}}_{T^{3/2}\int_0^\infty dx \sqrt{x}\, e^{-x} = T^{3/2}\frac{\sqrt{\pi}}{2}}$$

$$= -\frac{V z}{\beta}\left(\frac{mT}{2\pi}\right)^{3/2} \quad \text{with} \quad \lambda = \left(\frac{2\pi}{mT}\right)^{1/2} \text{ thermal de Broglie wavelength}$$

$$= -\frac{V z}{\beta}\frac{1}{\lambda^3}$$

$$P = -\frac{\Omega_0}{V} = \frac{zT}{\lambda^{3/2}} \quad [+ O(z^2)]$$

$$n = \frac{\partial P}{\partial \mu} = \frac{z}{\lambda^3} \Rightarrow P = nT = n k_B T \quad \text{classical gas with Boltzmann distribution}$$

$$\ll \frac{1}{\lambda^3} \quad (\text{i.e. } \lambda \ll n^{-1/3})$$

$$n(\varepsilon_k) = \frac{1}{e^{\beta\xi_k} - 1} \simeq z\, e^{-\beta\varepsilon_k}$$

\# $g \neq 0$: $\Omega^{(1)} = -\frac{1}{\beta}(\log Z)^{(1)}$

$= -\frac{1}{\beta}\left[\bigcirc\cdots\bigcirc + \textcircled{!} \right]$

$\sim T \frac{g}{\beta V} \sum_{\tilde{k}\tilde{k}'} G_0(\tilde{k}) e^{i\omega_n 0^+} G_0(\tilde{k}') e^{i\omega_{n'} 0^+}$

$\sim T g n^2 \beta V$ using $\frac{1}{\beta V}\sum_{\tilde{k}} G_0(\tilde{k}) e^{i\omega_n 0^+} = -n$

$\sim g V n^2$

$P^{(1)} \sim g n^2$

$\frac{P^{(1)}}{P^{(0)}} \sim \frac{g n^2}{nT} \sim \frac{g n}{T} \sim \frac{g}{T} \frac{e^{\beta\mu}}{\lambda^3} \ll 1$

for a very dilute gas, (short-range) interactions are not important: $P = nT + O(z^2)$; i.e. $\frac{P}{nT} = O(z)$

From the point of view of perturbation theory:

$G_0(k, i\omega_n) = \frac{1}{i\omega_n - \varepsilon_k - |\mu|}$ no pole, even for $\omega_n = 0$

$G_0^R(k, \omega) = \frac{1}{\omega + i0^+ - \varepsilon_k - |\mu|}$

$\rightarrow$ pole: $\omega = \varepsilon_k + |\mu| \geq |\mu|$

excitations are gapped / "massive"

i.e. no dangerous excitations.

2) Superfluid transition

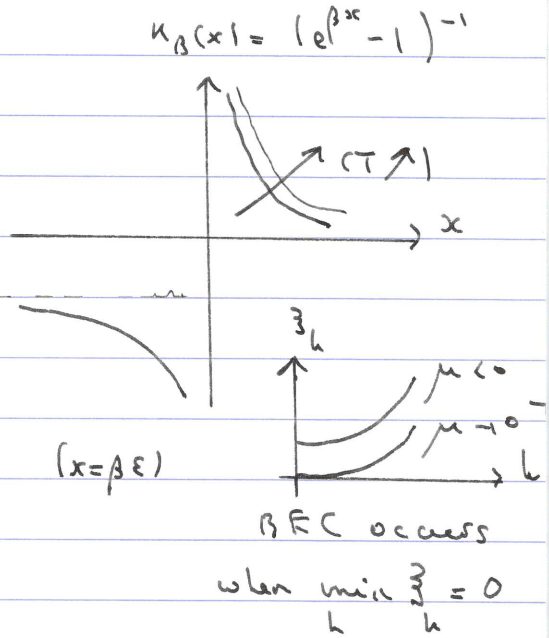
2.1) BEC of free bosons

$N = \langle \hat{N} \rangle$ fixed

$$N = \sum_k n_B(\varepsilon_k - \mu) \qquad (\mu < 0)$$

$$= V \int \frac{d^3k}{(2\pi)^3} n_B(\varepsilon_k - \mu)$$

$$= V \int_0^\infty d\varepsilon\, \rho(\varepsilon)\, n_B(\varepsilon - \mu)$$



if $T \downarrow$, $\mu \uparrow$ since N fixed

$\mu \to 0^-$:

$$n = \frac{(2m)^{3/2}}{4\pi^2} \int_0^\infty d\varepsilon\, \frac{\sqrt{\varepsilon}}{e^{\beta\varepsilon} - 1}$$

$$= \frac{(2mT)^{3/2}}{4\pi^2} \underbrace{\int_0^\infty dx\, \frac{\sqrt{x}}{e^{x} - 1}}_{\frac{\sqrt{\pi}}{2}\,\zeta(\frac{3}{2})}$$

$$= \left(\frac{2mT}{4\pi}\right)^{3/2} \zeta\left(\frac{3}{2}\right)$$

$$T_{BEC} = \frac{2\pi}{m} \left(\frac{n}{\zeta(3/2)}\right)^{2/3}$$

$T < T_{BEC}$: macroscopic number of particles in state $k = 0$

$$N = N_0 + \sum_{k \neq 0} n_B(\varepsilon_k) \qquad (\mu = 0)$$

$$= N_0 + V \int \frac{d^3k}{(2\pi)^3} n_B(\varepsilon_k)$$

$$= N_0 + V \left(\frac{mT}{2\pi}\right)^{3/2} \zeta\left(\frac{3}{2}\right) = N_0 + N \left(\frac{T}{T_{BEC}}\right)^{3/2}$$

$T = 0$: $N = N_0$ all particles in $k = 0$ state.

2-2) Interacting bosons: $N = \sum_k \langle \hat\psi^+(k)\, \hat\psi(k) \rangle$

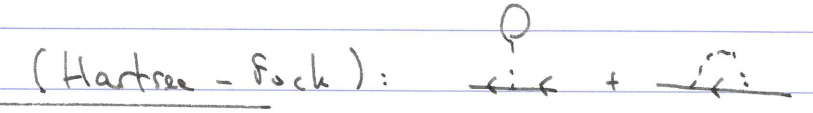
$$\langle \hat\psi^+(k)\, \hat\psi(k) \rangle = -\frac{1}{\beta} \sum_{\omega_n} e^{i\omega_n 0^+} G(k, i\omega_n)$$

$$= -\frac{1}{\beta} \sum_{\omega_n} \frac{e^{i\omega_n 0^+}}{i\omega_n - \xi_k - \Sigma(k, i\omega_n)}$$

$$= n_B(\xi_k) \quad \text{if self-energy } \Sigma = 0$$

not defined if $\mu \geq 0$.

For $\mu \geq 0$, we must compute Σ.

* Lowest order (Hartree-Fock): 

$$\Sigma(k, i\omega_n) = -\frac{2g}{\beta V} \sum_{k', \omega_{n'}} e^{i\omega_{n'} 0^+} G_0(k', i\omega_{n'})$$

$$= \frac{2g}{V} \sum_{k'} n_B(\varepsilon_{k'} - \mu)$$

still not defined for $\mu \geq 0$.

* Self-consistent Hartree-Fock: $G_0 \to G$

$$\Sigma(k, i\omega) = -\frac{2g}{\beta V} \sum_{k', \omega'} e^{i\omega' 0^+} G(k', i\omega')$$

$$= -\frac{2g}{\beta V} \sum_{k', \omega'} \frac{e^{i\omega' 0^+}}{i\omega' - \xi_{k'} - \Sigma(k', i\omega')}$$

$$= 2gn$$

with $n = n(\mu, T) = -\frac{1}{\beta V} \sum_{k, \omega} e^{i\omega 0^+} G(k, i\omega)$

$$= -\frac{1}{\beta V} \sum_{k, \omega} \frac{e^{i\omega 0^+}}{i\omega - \xi_k - 2gn}$$

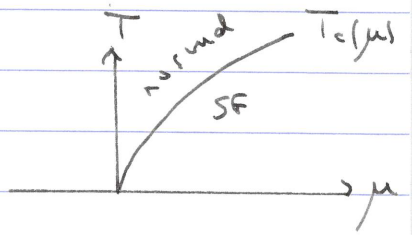
$$= \frac{1}{V} \sum_k n_B(\xi_k + 2gn)$$

solution possible if $\mu - 2gn < 0$

→ phase transition: $T_c(\mu)$: $\mu - 2g\,n(\mu, T_c) \to 0^-$

$$n(\mu, T_c) \cdot \frac{\mu}{2g} = \int_k n_B(\varepsilon_k) = \left(\frac{mT_c}{2\pi}\right)^{3/2} 3\left(\frac{3}{2}\right)$$

i.e. $T_c(\mu) = \frac{2\pi}{m}\left(\frac{\mu}{2g\,3(3/2)}\right)^{\frac{2}{3}}$



since $n = \frac{\mu}{2g}$: $T_c = \frac{2\pi}{m}\left(\frac{n}{3(3/2)}\right)^{\frac{2}{3}}$

$$\equiv T_{BEC}(n)$$

$G(k, i\omega_n)\big|_{k=\omega_n=0} = \dfrac{1}{\mu - 2gn}$ diverges when $T \to T_c^+$.

The correlation of the field Ψ become infinite range. This suggests that $\langle \Psi(r,\tau) \rangle$ is the order parameter of the normal-superfluid transition. [Ising model: $\chi_{ij} = \langle \sigma_i \sigma_j \rangle$, $\chi(q=0) \to \infty$ for $T \to T_c^+$, $\langle \sigma_i \rangle = 0$. p.]

For $T < T_c$: $\langle \Psi(r,\tau) \rangle \neq 0$. $\equiv \Psi_0$ SSB U(1)

$$\Psi(r,\tau) = \Psi_0 + \Psi'(r,\tau).$$

$$\langle \Psi^*(s,\tau') \Psi(r,\tau) \rangle = |\Psi_0|^2 + \dots$$

↳ $N_0 = V|\Psi_0|^2$ particles in condensate.

* Propagator:

$$G(k, i\omega_n) = \frac{1}{i\omega_n - \frac{k^2}{2m} + \mu - 2g\,n(\mu, T)}$$

$$\equiv \left[i\omega_n - \frac{k^2}{2m} - \frac{\xi^{-2}}{2m} \right]^{-1}$$

with $\xi^{-2}/2m = -\mu + 2g\,n(\mu, T)$

$$= -\mu + 2g\left[n(\mu, T_c) + (T - T_c)\,\partial_T n\big|_{T_c} \right]$$

$$= 2g\,(T - T_c)\,\partial_T n\big|_{T_c}$$

$\xi \sim (T - T_c)^{-\nu}$ correlation length

$\nu = \frac{1}{2}$ (mean-field value).

* quantum-classical crossover:

$$G(r, \tau = 0) = \frac{1}{\beta} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot r}}{i\omega_n - \frac{1}{2m}(k^2 + \xi^{-2})}$$

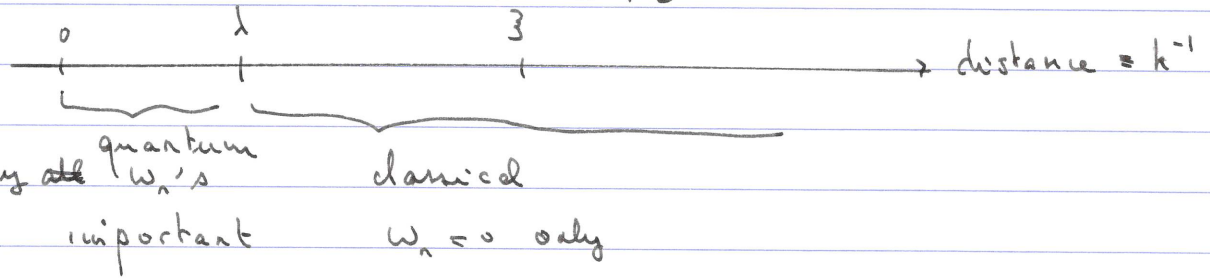
with $\xi^{-2} \to 0$ for $T \to T_c$.

NB: 2 characteristic lengths: ξ, λ

if $\frac{\xi^{-2}}{2m} \ll \omega_{n=1} = 2\pi T$, i.e. $\xi \gg \lambda = \sqrt{\frac{2\pi}{mT}}$,

the long-distance behavior $|r| \gg \lambda$ is determined by $\omega_{n=0} = 0$, i.e. by the thermal (classical) fluctuations.

$$G(r, \tau = 0) = -2mT \int_k \frac{e^{ik \cdot r}}{k^2 + \xi^{-2}}$$



Effective classical description at long distances:

$$\psi(r, \tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} e^{-i\omega_n \tau} \psi(r, i\omega_n)$$

→ classical field $\psi_{cl}(r) = \frac{1}{\sqrt{\beta}} \psi(r, i\omega_n = 0)$

$$e^{-S_{eff}[\psi_{cl}^*, \psi_{cl}]} = \int \mathcal{D}[\psi^*_{\omega_n \neq 0}, \psi_{\omega_n \neq 0}] \, e^{-S[\psi^*, \psi]}$$

↳ integrate out quantum fluctuations

effective long-distance description.

$$S_{\text{eff}}[\psi_d^*, \psi_{cl}] \sim \int d^3r \left\{ \psi_d^* (r_0 - \nabla^2) \psi_d + u_0 |\psi_d|^4 + \dots \right\}$$

no time (τ) dependence but quantum fluctuations determine coefficients r_0, u_0.

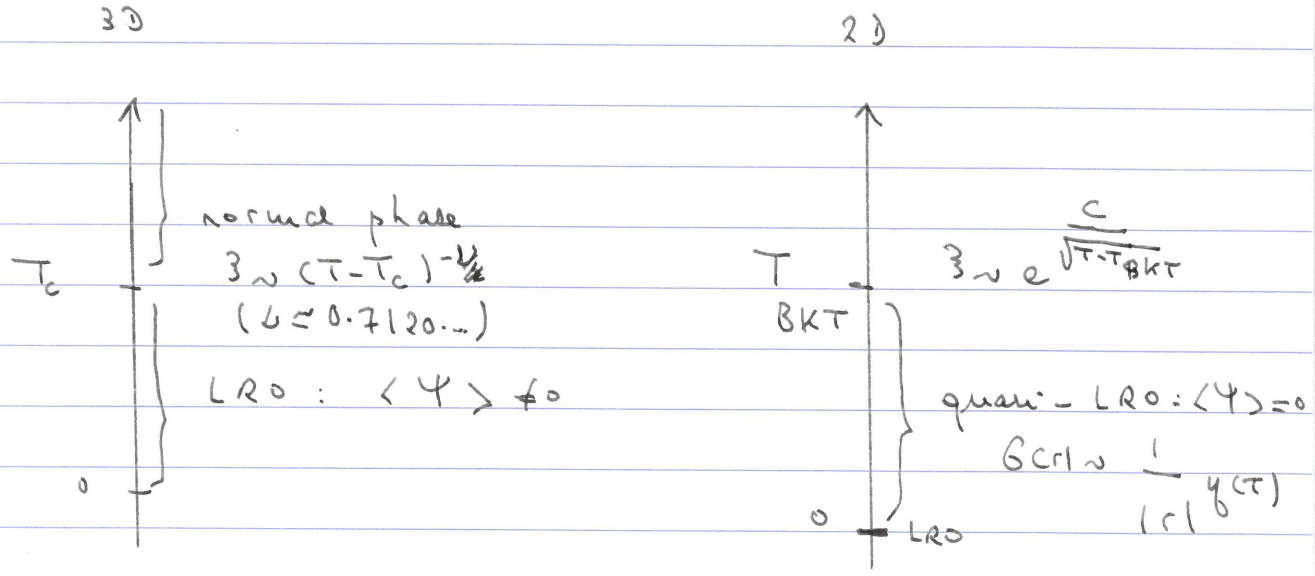
UV momentum cutoff $\Lambda \sim \lambda^{-1}$.

$$\psi_d = \frac{1}{\sqrt{2}} (\psi_1 + i \psi_2) \; ; \quad \vec{\psi} = (\psi_1, \psi_2)$$

$$S_{\text{eff}}[\vec{\psi}] \sim \int d^3r \left\{ (\nabla \vec{\psi})^2 + r_0 \vec{\psi}^2 + u_0 (\vec{\psi}^2)^2 + \dots \right\}$$

ψ^4 - theory or $O(2)$ model.

→ universality class of the XY model (same critical behavior).



Berezinskii - Kosterlitz - Thouless phase transition driven by unbinding of vortices.

3) Gross-Pitaevskii theory

3.1) GP equation:

$T \ll T_c$ $\langle \Psi \rangle$ o.p.

simplest approach: neglect all (quantum and thermal) fluctuations

$$Z = \int \mathcal{D}[\Psi^*, \Psi]\, e^{-S[\Psi^*, \Psi]} \rightarrow e^{-S[\Psi^*, \Psi]}$$

$$\frac{\delta S}{\delta \Psi^*(r,t)} = \frac{\delta S}{\delta \Psi(r,t)} = 0$$

$$\left(\partial_t - \mu - \frac{\nabla^2}{2m} + V_{ext}(r) + g\,|\Psi(r,t)|^2 \right) \Psi(r,t) = 0$$

↳ $-i\partial_t$ in real time [GP equation = non-linear Schrödinger eq.]

$$\Psi(r,t) = \sqrt{n(r,t)}\; e^{i\theta(r,t)}$$

with $n(r,t) = |\Psi(r,t)|^2$ (density) (all particles in the condensate)

$$\rightarrow \begin{cases} \partial_t n + \vec{\nabla} \cdot \left(n \dfrac{\vec{\nabla}\theta}{m} \right) = 0 & (1) \\ \dot{\theta} - \mu + V_{ext} + gn + \dfrac{(\nabla n)^2}{8mn} - \dfrac{\nabla^2 n}{4mn} + \dfrac{(\nabla\theta)^2}{2m} = 0 & (2) \end{cases}$$

(1) continuity eq: $\vec{j} = n \vec{v}_s$

$$\vec{v}_s = \frac{\vec{\nabla}\theta}{m}$$ velocity of the condensate

2) take $\vec{\nabla}$: $m \dfrac{d}{dt} \vec{v}_s = m \dfrac{\partial}{\partial t} \vec{v}_s + \vec{\nabla}\left(\frac{1}{2} m \vec{v}_s^2 \right)$

$= m\, \partial_t \vec{v}_s + m (\vec{v}_s \cdot \vec{\nabla}) \vec{v}_s$

and use $\vec{\nabla}\left(\frac{1}{2}\vec{v}^2\right) = (\vec{v}\cdot\vec{\nabla})\vec{v}$ when $\vec{\nabla} \times \vec{v} = 0$

$$= -\vec{\nabla}\left(\underbrace{gn + V_{ext}}_{\mu(r,t)\ \text{chemical potential}} \right) + \frac{1}{2m} \vec{\nabla}\left(\frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right)$$

local potential; quantum pressure term $O(\hbar^2)$

≡ Euler equation (hydrodynamics)

3.2) Dynamics of a uniform condensate:

$V_{ext} = 0$: equilibrium $\Psi = \sqrt{n}$ choose real

$\mu = gn$

small fluctuations about eq.: $\delta n(r,t) = n(r,t) - n$

$\dot{\theta}$, $\nabla\theta$

linearized equations (from (1.21)):

$$\begin{cases} \delta\dot{n} + \frac{n}{m}\nabla^2\theta = 0 \\ \dot{\theta} + g\,\delta n - \frac{\nabla^2\delta n}{4mn} = 0 \end{cases}$$

i.e.

$$\begin{cases} -i\omega\,\delta n - \frac{n}{m}q^2\theta = 0 \\ -i\omega\,\theta + \left(g + \frac{q^2}{4mn}\right)\delta n = 0 \end{cases} \quad ; \quad \theta \equiv \theta(q,\omega),\ \delta n \equiv \delta n(q,\omega)$$

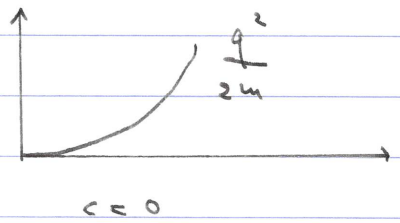
$$\begin{vmatrix} +i\omega & +\frac{n}{m}q^2 \\ g + \frac{q^2}{4mn} & -i\omega \end{vmatrix} = \omega^2 - \frac{n}{m}q^2\left(g + \frac{q^2}{4mn}\right) = 0$$

$$\omega = \left(g\frac{n}{m}q^2 + \frac{|q|^4}{4m^2}\right)^{1/2} \simeq \begin{cases} c|q| & \text{if } |q| \ll \xi^{-1} = \sqrt{gnm} \\ \frac{q^2}{2m} + gn & \text{if } |q| \gg \xi^{-1} \end{cases}$$

NB: a single mode (θ and n are conjugate variables)

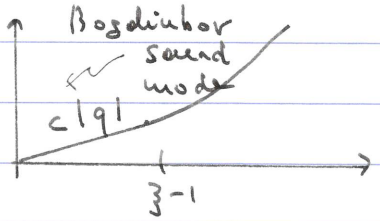
ξ = healing length.

free boson vs interacting bosons.



$c = 0$

$\kappa \sim \frac{\partial n}{\partial \mu} = \infty$ infinite compressibility



$c > 0$

$\kappa < \infty$

Superfluidity follows from Landau's criterion.