Coherent - state functional integral

1) Coherent states :

$$\hat{\psi}_\alpha |\psi\rangle = \psi_\alpha |\psi\rangle \quad \forall \alpha, \quad \psi_\alpha \in \mathbb{C}$$

sol^n : $$|\psi\rangle = \exp\left\{\sum_\alpha \psi_\alpha \hat{\psi}^+_\alpha\right\} |vac\rangle$$

$$= \sum_{n_{\alpha_i}=0}^{\infty} \frac{\psi_{\alpha_1}^{n_{\alpha_1}} \cdots \psi_{\alpha_p}^{n_{\alpha_p}} \cdots}{(n_{\alpha_1}! \cdots n_{\alpha_p}! \cdots)^{\frac{1}{2}}} |n_{\alpha_1} \cdots n_{\alpha_p} \cdots\rangle$$

[use $[\hat{\psi}_\alpha, (\psi^+_\beta)^n] = \delta_{\alpha\beta}\, n\, (\psi^+_\alpha)^{n-1}$]

$$\hat{\psi}^+_\alpha |\psi\rangle = \frac{\partial}{\partial \psi_\alpha} |\psi\rangle$$

$$\langle\psi| \hat{\psi}^+_\alpha = \langle\psi| \psi^*_\alpha$$

$$\langle\psi| \hat{\psi}_\alpha = \frac{\partial}{\partial \psi^*_\alpha} \langle\psi|$$

where $\langle\psi| = \langle vac| \exp\left\{\sum_\alpha \psi^*_\alpha \hat{\psi}_\alpha\right\}$ bra

overlap : $$\langle\psi|\psi'\rangle = \exp\left\{\sum_\alpha \psi^*_\alpha \psi'_\alpha\right\}$$

closure relation : $$\int \prod_\alpha \underbrace{\frac{d\psi^*_\alpha\, d\psi_\alpha}{2i\pi}}_{\frac{1}{\pi} d\mathrm{Re}(\psi_\alpha)\, d\mathrm{Im}(\psi_\alpha)} e^{-\sum_\alpha |\psi_\alpha|^2} |\psi\rangle\langle\psi| = \hat{1}$$

the basis $\{|\psi_\alpha\rangle\}$ is overcomplete

Normal - ordered operator : $A(\hat{\psi}^+_\alpha, \hat{\psi}_\alpha)$: all $\hat{\psi}^+$'s on the left, $\hat{\psi}$'s on the right

$$\langle \Psi | A(\hat{\psi}^\dagger_\alpha, \hat{\psi}_\alpha) | \Psi' \rangle = \langle \Psi | \Psi' \rangle \, A(\Psi^*_\alpha, \Psi'_\alpha)$$

$$= e^{\sum_\alpha \Psi^*_\alpha \Psi'_\alpha} \, A(\Psi^*_\alpha, \Psi'_\alpha)$$

Special properties of coherent states: cf. harmonic oscillator
quantum optics

2) Functional integral:

2.1) Partition function:

$$Z = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle$$

$$= \sum_n \langle n | \int d(\Psi^*, \Psi) \, e^{-\sum_\alpha |\Psi_\alpha|^2} | \Psi \rangle \langle \Psi | e^{-\beta \hat{H}} | n \rangle$$

$$= \int d(\Psi^*, \Psi) \, e^{-\sum_\alpha |\Psi_\alpha|^2} \langle \Psi | e^{-\beta \hat{H}} | \Psi \rangle$$

where $d(\Psi^*, \Psi) = \prod_\alpha \dfrac{d\Psi^*_\alpha \, d\Psi_\alpha}{2i\pi}$; $\Psi^{(*)} \equiv \Psi^{(*)}_N$

$$= \int d(\Psi^*_N, \Psi_N) \, e^{-\sum_\alpha |\Psi_{N\alpha}|^2} \langle \Psi_N | e^{-\varepsilon \hat{H}} \cdots e^{-\varepsilon \hat{H}} | \Psi_N \rangle$$

$$= \int \prod_{k=1}^N d(\Psi^*_k, \Psi_k) \, e^{-\sum_{k=1}^N \sum_\alpha |\Psi_{k\alpha}|^2} \prod_{k=1}^N \langle \Psi_k | e^{-\varepsilon \hat{H}} | \Psi_{k-1} \rangle$$

with $\Psi^{(*)}_0 = \Psi^{(*)}_N$

now

$$\langle \Psi_k | e^{-\varepsilon \hat{H}} | \Psi_{k-1} \rangle = \langle \Psi_k | 1 - \varepsilon \hat{H} | \Psi_{k-1} \rangle + O(\varepsilon^2)$$

$$= \langle \Psi_k | \Psi_{k-1} \rangle \left(1 - \varepsilon H(\Psi^*_{k\alpha}, \Psi_{k-1,\alpha})\right)$$

if $H(\hat{\psi}^\dagger_\alpha, \hat{\psi}_\alpha)$ normal ordered

$$\simeq \exp\left\{ \sum_\alpha \Psi^*_{k\alpha} \Psi_{k-1\alpha} - \varepsilon H(\Psi^*_{k\alpha}, \Psi_{k-1\alpha}) \right\}$$

$$Z = \int \prod_{k=1}^{N} d(\psi^*_k, \psi_k) \exp\Big\{ - \sum_{k=1}^{N} \sum_\alpha \psi^*_{k\alpha} (\psi_{k\alpha} - \psi_{k-1\,\alpha})$$
$$- \varepsilon \sum_{k=1}^{N} H(\psi^*_{k\alpha}, \psi_{k-1\,\alpha}) \Big\}$$

<u>continuous trajectory</u>: $\psi_{0\alpha} \dots \psi_{N\alpha} \to \psi_\alpha(\tau)$ with $\tau \in [0, \beta]$

$$\varepsilon \sum_{k=1}^{N} \to \int_0^\beta d\tau$$

$$\psi^*_{k\alpha} \frac{\psi_{k\alpha} - \psi_{k-1\,\alpha}}{\varepsilon} \to \psi^*_\alpha(\tau)\, \partial_\tau \psi_\alpha(\tau)$$

$$H(\psi^*_{k\alpha}, \psi_{k-1\,\alpha}) \to H(\psi^*_\alpha(\tau^+), \psi_\alpha(\tau))$$

Hence

$$Z = \int_{\psi^{(*)}(\beta) = \psi^{(*)}(0)} \mathcal{D}[\psi^*, \psi]\; e^{-S[\psi^*, \psi]}$$

with

$$\mathcal{D}[\psi^*, \psi] = \lim_{N \to \infty} \prod_{k=1}^{N} \prod_\alpha \frac{d\psi^*_{k\alpha}\, d\psi_{k\alpha}}{2i\pi}$$

and $$S[\psi^*, \psi] = \int_0^\beta d\tau \Big\{ \sum_\alpha \psi^*_\alpha \partial_\tau \psi_\alpha + H(\psi^*_\alpha(\tau^+), \psi_\alpha(\tau)) \Big\}$$

<u>Euclidean action</u>

<u>Example</u>: particles interacting with 2-body ~~potential~~ interaction

$$\hat{H} = \int d^d r\, \frac{\nabla \hat\psi^\dagger \cdot \nabla \hat\psi}{2m} + \frac{1}{2} \int d^d r\, d^d r'\, v(r - r')\, \hat\psi^\dagger(r)\, \hat\psi^\dagger(r')\, \hat\psi(r')\, \hat\psi(r)$$

$$S = \int_0^\beta d\tau \Big\{ \int d^d r\, \psi^*(r\tau) \Big[\partial_\tau - \mu - \frac{\nabla^2}{2m} \Big] \psi(r\tau)$$
$$+ \frac{1}{2} \int d^d r\, d^d r'\, v(r - r')\, \psi^*(r\tau)\, \psi^*(r'\tau)\, \psi(r'\tau)\, \psi(r\tau) \Big\}$$

2.2) Green function:

$$G(\alpha\tau,\alpha'\tau') = -\langle T_\tau \hat{\psi}_\alpha(\tau)\, \hat{\psi}^+_{\alpha'}(\tau')\rangle$$

$$= -\frac{1}{Z} \mathrm{Tr}\left\{ e^{-\beta\hat{H}}\, e^{\tau\hat{H}}\, \hat{\psi}_\alpha\, e^{-(\tau-\tau')\hat{H}}\, \hat{\psi}^+_{\alpha'}\, e^{-\tau'\hat{H}} \right\} \qquad (\tau>\tau')$$

$$= -\frac{1}{Z}\int \prod_{k=1}^{N} d(\psi^*_\alpha, d\psi_\alpha)\, e^{-\sum_{k=1}^{N}\sum_\gamma |\psi_{k\gamma}|^2} \langle\psi_N| e^{-\beta\hat{H}}|\psi_{N-1}\rangle$$

$$\dots \langle \psi_{k_1+1}| e^{-\varepsilon\hat{H}}\hat{\psi}_\alpha|\psi_{k_1}\rangle \dots \langle\psi_{k_2}|\hat{\psi}^+_{\alpha'} e^{-\varepsilon\hat{H}}|\psi_{k_2-1}\rangle \dots$$

$$\dots \langle\psi_1|e^{-\varepsilon\hat{H}}|\psi_0\rangle$$

if $\tau \equiv k_1\varepsilon$
$\quad\;\tau' = k_2\varepsilon$

$$= -\frac{1}{Z}\int \prod_{k=1}^{N} d(\psi^*_k, \psi_k) \dots \psi_{k_1\alpha} \dots \psi^*_{k_2\alpha'} \dots$$

$$= -\frac{1}{Z}\int \mathcal{D}[\psi^*,\psi]\, \psi_\alpha(\tau)\,\psi^*_{\alpha'}(\tau')\, e^{-S[\psi^*,\psi]}$$

For $\tau<\tau'$: $G(\alpha\tau,\alpha'\tau') = -\langle \hat{\psi}^+_{\alpha'}(\tau')\,\hat{\psi}_\alpha(\tau)\rangle$

$$= -\frac{1}{Z}\mathrm{Tr}\left\{ e^{-(\beta-\tau')\hat{H}}\,\hat{\psi}^+_{\alpha'}\, e^{-(\tau'-\tau)\hat{H}}\,\hat{\psi}_\alpha\, e^{-\tau\hat{H}}\right\}$$

$$\equiv -\frac{1}{Z}\int \mathcal{D}[\psi^*,\psi]\,\psi^*_{\alpha'}(\tau')\,\psi_\alpha(\tau)\, e^{-S'[\psi^*,\psi]}$$

same expression: time-ordered correlation functions are given by functional integrals.

3) Non-interacting bosons:

$\hat{H}_0 = \sum_\alpha \xi_\alpha \hat{\psi}^+_\alpha \hat{\psi}_\alpha$ in the diagonal basis $(\xi_\alpha = \varepsilon_\alpha - \mu)$

$$S_0 = \int_0^\beta d\tau \sum_\alpha \psi^*_\alpha (\partial_\tau + \xi_\alpha) \psi_\alpha$$

$$= - \sum_{\omega_n, \alpha} \psi^*_\alpha (i\omega_n) (i\omega_n - \xi_\alpha) \psi_\alpha (i\omega_n)$$

$$\equiv - \sum_{\omega_n, \alpha} \psi^*_\alpha (i\omega_n) G_0^{-1} (\alpha, i\omega_n) \psi_\alpha (i\omega_n)$$

where $\psi_\alpha(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} e^{-i\omega_n \tau} \psi_\alpha(i\omega_n)$

$$\psi_\alpha(i\omega_n) = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau \, e^{i\omega_n\tau} \psi_\alpha(\tau)$$

$\omega_n = \frac{2\pi}{\beta} n$ Matsubara frequency.

NB: thermodynamics $Z \rightarrow G_0 \rightarrow$ spectrum $\omega = \xi_\alpha$ (dynamics).

$$Z_0 = \int \mathcal{D}[\psi^*, \psi] \exp\left\{ \sum_{\omega_n, \alpha} \psi^*_\alpha (i\omega) G_0^{-1}(\alpha, i\omega_n) \psi_\alpha (i\omega_n) \right\}$$

$= \mathcal{N} \prod_{\alpha, \omega_n} \left[- G_0(\alpha, i\omega_n) \right]$ with $\mathcal{N} = 1$ if we treat carefully the integration measure $\mathcal{D}[\psi^*, \psi]$.

$$= \prod_{\alpha, \omega_n} (-i\omega_n + \xi_\alpha)^{-1}$$

$$\Omega_0 = -\frac{1}{\beta} \operatorname{Log} Z_0 = \frac{1}{\beta} \sum_{\omega_n, \alpha} \operatorname{Log}(-i\omega_n + \xi_\alpha)$$

divergent Matsubara sum! [pb with continuum-time limit]

correct expression: $\Omega_0 = \frac{1}{\beta} \sum_{\omega_n, \alpha} \operatorname{Log}(-i\omega_n + \xi_\alpha) \, e^{i\omega_n 0^+}$

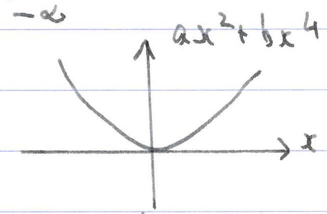
see tutorial class.

<u>Remark:</u> The computation of Z reduces to the computation of a functional integral.

How can we compute the integral

$$I = \int_{-\infty}^{\infty} dx \; e^{-ax^2 - bx^4} \qquad (b>0).$$

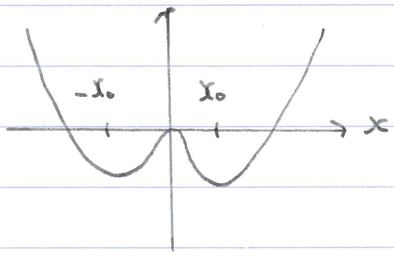
i) $a>0$:



$$I = \int dx \; e^{-ax^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (bx^4)^n \qquad \equiv \text{Perturbation Theory}$$

$$\frac{I}{I_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} b^n \underbrace{\langle x^{4n} \rangle_0}_{\text{depends on } \langle x^2\rangle_0 = \frac{1}{2a} \text{ (Wick's th.)}} \quad \text{with } \langle \dots \rangle_0 = \frac{1}{I_0} \int dx \dots e^{-ax^2}$$

ii) $a<0$



$$a + 2bx_0^2 = 0$$

$$x_0 = \left(\frac{-a}{2b}\right)^{\frac{1}{2}}$$

$$f(x) = ax^2 + bx^4 = f(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0) + \dots$$

expansion about x_0.

$$I \simeq e^{-f(x_0)} \int dx \; e^{-\frac{1}{2}(x-x_0)^2 f''(x_0) + \dots}$$

saddle-pt (mean-field theory)

fluctuations around mean field.

4) Perturbation theory:

4.1) Wick's theorem:

$$Z_0[J^*, J] = \int \mathcal{D}[\psi^*, \psi] \exp\left\{ -S_0[\psi^*, \psi] + \int d\tau \sum_\alpha (J^*_\alpha \psi_\alpha + c.c.) \right\}$$

$$= \int \mathcal{D}[\psi^*, \psi] \exp\left\{ \int d\tau d\tau' \sum_{\alpha\alpha'} \psi^*_\alpha(\tau) G_0^{-1}(\alpha\tau, \alpha'\tau') \psi_{\alpha'}(\tau') + \int d\tau \sum_\alpha (\dots) \right\}$$

$$= Z_0[0,0] \exp\left\{ -\int d\tau d\tau' \sum_{\alpha\alpha'} J^*_\alpha(\tau) G_0(\alpha\tau, \alpha'\tau') J_{\alpha'}(\tau') \right\}$$

using the rules of Gaussian integration.

Consider now

$$\langle \psi_{\alpha_1}(\tau_1) \dots \psi_{\alpha_n}(\tau_n) \psi^*_{\alpha'_n}(\tau'_n) \dots \psi^*_{\alpha'_1}(\tau'_1) \rangle_0$$

$$= \frac{1}{Z_0} \int \mathcal{D}[\psi^*, \psi] \dots e^{-S_0}$$

$$= \frac{1}{Z_0} \left. \frac{\delta^{2n} Z_0[J^*, J]}{\delta J^*(1) \dots \delta J^*(n) \delta J(n') \dots \delta J(1')} \right|_{J^* = J = 0}$$

\# $\underline{n = 1:}$ $\langle \psi_{\alpha_1}(\tau_1) \psi^*_{\alpha'_1}(\tau'_1) \rangle_0 = \frac{1}{Z_0} \left. \frac{\delta^2 Z_0[J^*, J]}{\delta J^*(1) \delta J(1')} \right|_{J^* = J = 0}$

$$= \frac{1}{Z_0} \frac{\delta}{\delta J(1')} \cancel{\int \mathcal{D}[\psi^*, \psi]} \left(-\sum_{\alpha'} \int d\tau' G_0(\alpha_1\tau_1, \alpha'\tau') J_{\alpha'}(\tau') \; Z_0 \exp\{\dots\} \right)$$

$$= -G_0(\alpha_1\tau_1, \alpha'_1\tau'_1).$$

$$\equiv -\langle T_\tau \hat{\psi}_{\alpha_1}(\tau_1) \hat{\psi}^\dagger_{\alpha'_1}(\tau'_1) \rangle_0$$

i.e. $G_0(\alpha_1\tau_1, \alpha'_1\tau'_1) = -\langle \psi_{\alpha_1}(\tau_1) \psi^*_{\alpha'_1}(\tau'_1) \rangle$ propagator graphically represented by a directed line

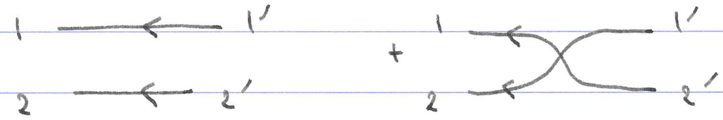
$$\alpha_1\tau_1 \longleftarrow \alpha'_1\tau'_1$$

annihilation — particle creation

\# $\underline{n=2}$: $\langle \Psi(1)\Psi(2)\Psi^*(2')\Psi^*(1')\rangle_0$

$$= G_0(1,1')\,G_0(2,2') + G_0(1,2')\,G_0(2,1')$$

$\equiv$



can be obtained by considering all possible contractions $\Psi\Psi^*$ and using the rule $\Psi(1)\Psi^*(2) = -G_0(1,2)$.

$$\langle \Psi(1)\Psi(2)\Psi^*(2')\Psi^*(1')\rangle_0$$

$$= \Psi(1)\Psi(2)\Psi^*(2')\Psi^*(1') + \Psi(1)\Psi(2)\Psi^*(2')\Psi^*(1')$$

Wick's theorem:

$$\langle \Psi_{\alpha_1}(\tau_1)\dots\Psi_{\alpha_n}(\tau_n)\,\Psi^*_{\alpha'_n}(\tau'_n)\dots\Psi^*_{\alpha'_1}(\tau'_1)\rangle_0$$

$$= \sum \text{all complete contractions}$$

$$= (-1)^n \sum_{p\in S_n} G_0(\alpha_1\tau_1,\alpha'_{p(1)}\tau'_{p(1)})\dots\,G_0(\alpha_n\tau_n,\alpha'_{p(n)}\tau'_{p(n)})$$

4.2) Thermodynamic potential:

$$Z = \int \mathcal{D}[\Psi^*,\Psi]\, e^{-S_0 - S_{int}}$$

$$S_{int} = \frac{1}{2}\int d\tau \sum_{\alpha_1\dots\alpha'_2} \underbrace{(\alpha_1\alpha_2|\hat{v}|\alpha'_1\alpha'_2)}_{v_{\alpha_1\alpha_2,\alpha'_1\alpha'_2}}\,\Psi^*_{\alpha_1}\Psi^*_{\alpha_2}\Psi_{\alpha'_2}\Psi_{\alpha'_1}$$

$$\frac{Z}{Z_0} = \langle e^{-S_{int}} \rangle_0 \qquad \text{with} \qquad \langle \cdots \rangle_0 = \frac{1}{Z_0} \int \mathcal{D}[\psi^*, \psi] \cdots e^{-S_0}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0$$

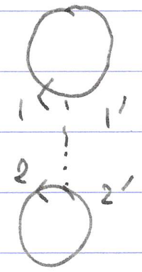
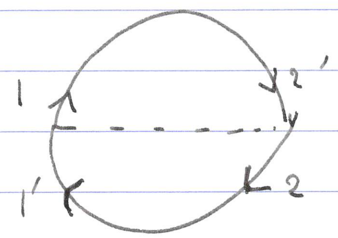
$$= 1 - \langle S_{int} \rangle_0 + \frac{1}{2!} \langle S_{int}^2 \rangle_0 + \cdots$$

\# $\underline{O(v):}$

$$-\langle S_{int} \rangle_0 = -\frac{1}{2} \int d\tau \sum_{\alpha_1 \ldots \alpha_2'} v_{\alpha_1 \alpha_2, \alpha_1' \alpha_2'} \underbrace{\langle \psi^*(1)\, \psi^*(2)\, \psi(2')\, \psi(1') \rangle_0}$$

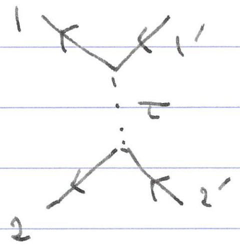
$/\!/$ [Wick's th.]

$$G_0(1', 1^+)\, G_0(2', 2^+) + G_0(1', 2^+)\, G_0(2', 1^+)$$

$\equiv$  $+$ 

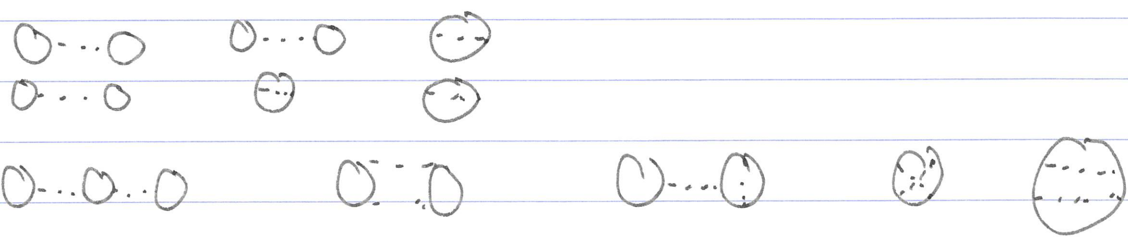
⚠ $G(1', 1)$
$= G_0(\alpha_1' \tau, \alpha_1 \tau)$
$\equiv G_0(\alpha_1' \tau, \alpha_1 \tau^+)$

if we graphically represent the interaction by the vertex

 $\equiv v_{\alpha_1 \alpha_2, \alpha_1' \alpha_2'}$

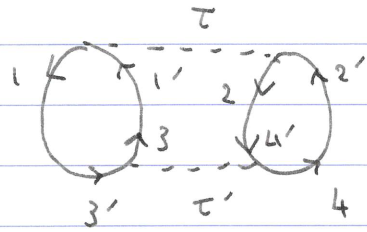
\# $\underline{O(v^2):}$ $\quad \frac{1}{2!} \langle S_{int}^2 \rangle \equiv \frac{1}{2!} \int d\tau\, d\tau' \langle \psi^* \psi^* \psi \psi \;\; \psi^* \psi^* \psi \psi \rangle_0$

4! contractions but only 8 diagrams



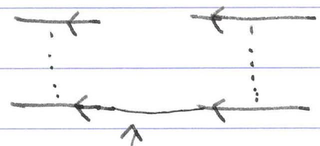
Perturbation theory: write all diagrams to a given order and compute each diagram.

e.g.



sign and factor

$$\left\langle e^{-\frac{1}{2}\bar{\psi}\bar{\psi}\psi\psi}\right\rangle_0 \to \frac{1}{2!}\left(-\frac{1}{2}\right)^2 \Big\langle$$



$$\Big\rangle_0$$

2 choices for this contraction.

then only one choice

$$= \frac{1}{2!}\left(\frac{1}{2}\right)^2 \times 2 = \frac{1}{4}$$

Hence the expression of the diagram:

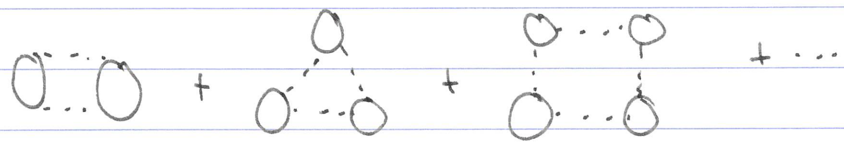
$$\frac{1}{4}\int d\tau\, d\tau' \sum_{\alpha_1 \dots \alpha_4'} V_{\alpha_1\alpha_2,\alpha_1'\alpha_2'}\, V_{\alpha_3\alpha_4,\alpha_3'\alpha_4'}$$

$$\times\; G_0(\alpha_3'\tau', \alpha_1\tau)\, G_0(\alpha_1'\tau, \alpha_3\tau')$$

$$\times\; G_0(\alpha_4'\tau', \alpha_2\tau)\, G_0(\alpha_2'\tau, \alpha_4\tau')$$

Resummation of a subclass of diagrams:

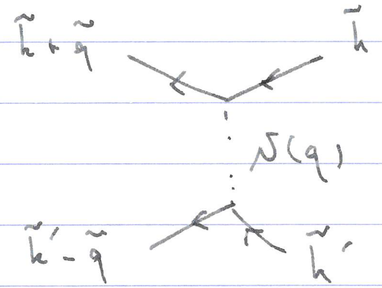
e.g.



Diagrams in Fourier space:

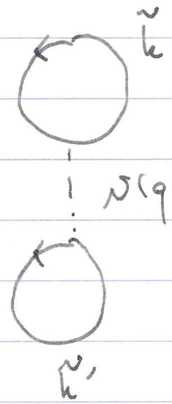
$$S_{int} = \frac{1}{2\beta V} \sum_{\tilde{k}\tilde{k}'\tilde{q}} v(q)\, \psi^*(\tilde{k}+\tilde{q})\, \psi^*(\tilde{k}'-\tilde{q})\, \psi(\tilde{k}')\, \psi(\tilde{k})$$

$\equiv$



with
$$\tilde{k} \equiv (k, i\omega_n)$$
$$\tilde{k}' \equiv (k', i\omega_{n'})$$
$$\tilde{q} \equiv (q, i\omega_\nu)$$

example:



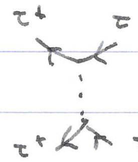
$$-\frac{1}{2\beta V} \sum v(q) \langle \psi^*_{\tilde{k}+\tilde{q}}\, \psi^*_{\tilde{k}'-\tilde{q}}\, \psi_{\tilde{k}'}\, \psi_{\tilde{k}} \rangle$$

$$= -\frac{1}{2\beta V} \sum v(q)\, G_0(\tilde{k}, \tilde{k}+\tilde{q})\, G_0(\tilde{k}', \tilde{k}'-\tilde{q})$$

$$= -\frac{1}{2\beta V} v(q=0) \sum_{\tilde{k}\tilde{k}'} G_0(\tilde{k})\, G_0(\tilde{k}')$$

NB: CV factors missing.

time space $G_0(\tau, \tau^+)\, G_0(\tau', \tau'^+)$



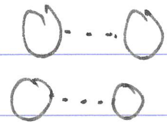
$$\to \sum_{\omega_n, \omega_{n'}} G_0(i\omega_n)\, e^{i\omega_n 0^+}\, G_0(i\omega_{n'})\, e^{i\omega_{n'} 0^+}$$

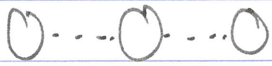
⇝ rules for CV factors:

+ necessary to compute Ω_0 (see tutorial)
+ add $e^{i\omega_n 0^+}$ for each closed loop in a diagram.

Linked cluster theorem:

$$\log \frac{Z}{Z_0} = \sum \text{connected diagrams.}$$

e.g.  is not connected

 is connected.

$O(v)$: 2 connected graphs (2 contractions)
$O(v^2)$: 5 " " " " (8 diagrams, 24 contractions).

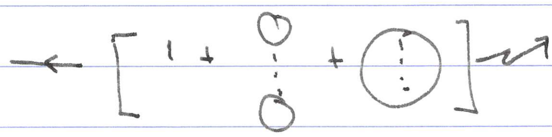
4.3) Green function:

$$G(1,2) = -\langle \psi(1)\, \psi^*(2) \rangle$$

$$= -\frac{Z_0}{Z} \langle \psi(1)\, \psi^*(2)\, e^{-S_{int}} \rangle_0$$

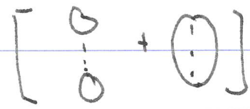
$$= -\frac{Z_0}{Z_0 + Z_1} \langle \psi(1)\, \psi^*(2)\, (1 - S_{int}) \rangle_0 \quad \text{to } O(v)$$

$$= \underbrace{G_0(1,2)\left[1 - \frac{Z_1}{Z_0}\right]} + \langle \psi_*(1)\, \psi^*(2)\, S_{int} \rangle_0 + O(v^2)$$



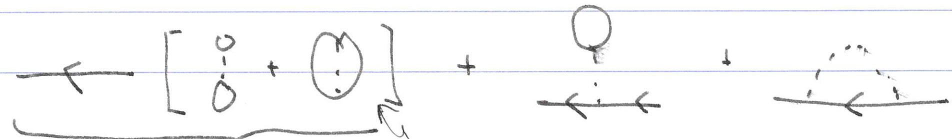
signs ignored!

$$= G_0(1,2) + \underbrace{G_0(1,2)\Big[\quad\Big]}_{(\mathrm{I})} + \langle \psi(1)\, \psi^*(2)\, S_{int} \rangle_0$$



now

$$\langle \psi(1)\, \psi^*(2)\, S_{int} \rangle_0 = \frac{1}{2} \int \langle \psi(1)\, \psi^*(2)\; v\, \psi^* \psi^* \psi \psi \rangle_0$$

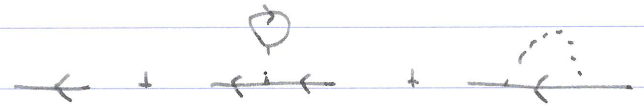
$=$ 

(II)

signs ignored

(I) and (II) cancel (when signs and symmetry factors properly taken into account).

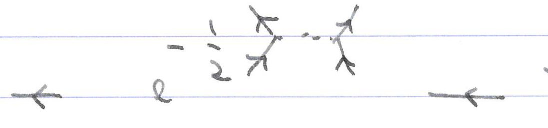
Only the connected diagrams survive:

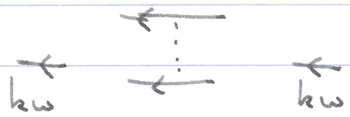
$G(1,2) =$  $+ O(v^2)$

More generally: $G(1,2) = -\frac{1}{Z_0}\int \mathcal{D}[\Psi^*, \Psi]\, \Psi(1)\, \Psi^*(2)\, e^{-S}\Big|_{\text{connected diagrams}}$

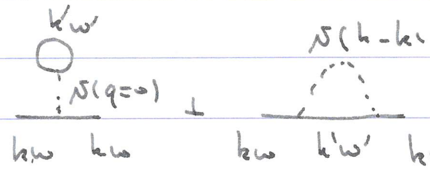
✱ <u>$O(v)$ in Fourier space</u>: $G(k, i\omega_n) = -\langle \Psi(k, i\omega)\, \Psi^*(k, i\omega)\rangle$

$= -\langle \Psi(k, i\omega)\, \Psi^*(k, i\omega)\, e^{-S_{int}}\rangle_0 \Big|_{\text{con. graphs}}$

$\Rightarrow -\Big\langle$  $\Big\rangle_0 \Big|_{\text{c.gr.}}$

$= \frac{1}{2}\Big\langle$  $\Big\rangle_0 \Big|_{\text{c.gr.}} + O(v^2)$

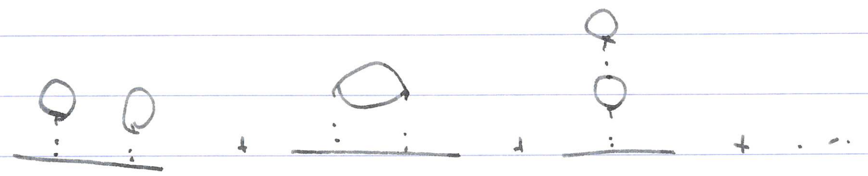
Wick's theorem

$= -\Big\{$  $\Big\}$

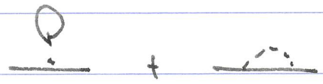
NB: $\overline{\Psi\Psi^*} = -G_0$

$= -\frac{1}{\beta V}\sum_{k'\omega'} v(q=0)\, G_0(k', i\omega')\, e^{i\omega' 0^+}$

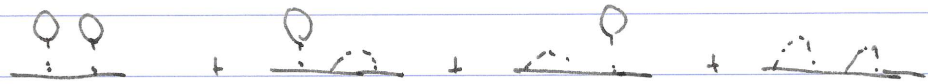
$\quad -\frac{1}{\beta V}\sum_{k'\omega'} v(k-k')\, G_0(k', i\omega')\, e^{i\omega 0^+}$

✱ <u>$O(v^2)$</u>: 

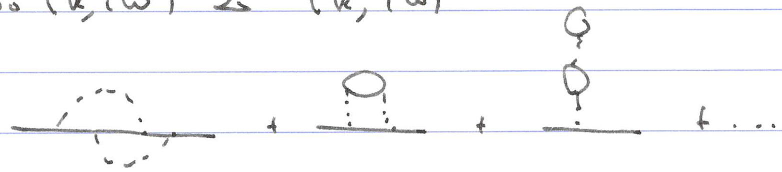
4.4) Self-energy and Dyson's equation:

$O(\nu)$: $G =$  $= G_0(k,i\omega)^2\, \Sigma^{(1)}(k,i\omega)$

$- \; G_0$

$O(\nu^2)$: 

$$\equiv G_0^3(k,i\omega)\, \Sigma^{(1)}(k,i\omega)^2$$

+ other terms : 

$$\equiv G_0(k,i\omega)^2\, \Sigma^{(2)}(k,i\omega)$$

One can write G in terms of the self-energy $\Sigma = \Sigma^{(1)} + \Sigma^{(2)} + \dots$:

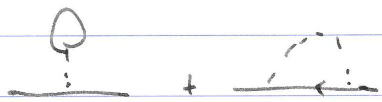
$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \dots$$

$$= G_0 + G_0 \Sigma G \qquad \text{Dyson's equation}$$

$$G^{-1} = G_0^{-1} - \Sigma$$

i.e. $G_\sigma(k, i\omega_n) = \left[i\omega_n - \xi_k - \Sigma_\sigma(k, i\omega_n) \right]^{-1}$

The self-energy is given by the sum of all amputated (no external legs) 1-particle-irreducible (cannot be separated into 2 disconnected pieces by cutting 1 line) diagrams.

$\Sigma^{(1)}_\sigma(k, i\omega_n) =$ 

$$= -\frac{1}{\beta V} \sum_{k'\omega'\sigma'} \nu(q=0)\, G_{0\sigma'}(k', i\omega')\, e^{i\omega' 0^+}$$

$$- \frac{1}{\beta V} \sum_{k'\omega'} \nu(k-k')\, G_{0\sigma}(k', i\omega')\, e^{i\omega' 0^+}$$

4.5) Spectral representation of the one-particle Green function:

$$G_\sigma(k,\tau) = -\langle T_\tau \hat{\psi}_\sigma(k,\tau)\, \hat{\psi}^\dagger_\sigma(k,0)\rangle$$

$$= -\frac{1}{Z}\,\mathrm{Tr}\left\{ e^{-(\beta-\tau)\hat{H}}\,\hat{\psi}_\sigma(k)\, e^{-\tau\hat{H}}\,\hat{\psi}^\dagger_\sigma(k)\right\} \quad \text{for } \tau>0$$

$$= -\frac{1}{Z}\sum_{n,m} e^{-(\beta-\tau)\varepsilon_n}\langle n|\hat{\psi}_\sigma(k)|m\rangle\, e^{-\tau\varepsilon_m}\langle m|\hat{\psi}^\dagger_\sigma(k)|n\rangle$$

$$= -\frac{1}{Z}\sum_{n,m} e^{-(\beta-\tau)\varepsilon_n-\tau\varepsilon_m}\,|\langle m|\hat{\psi}^\dagger_\sigma(k)|n\rangle|^2$$

$$G_\sigma(k,i\omega_n) = -\frac{1}{Z}\sum_{n,m}|\langle\ \rangle|^2 e^{-\beta\varepsilon_n}\underbrace{\int_0^\beta d\tau\, e^{\tau(i\omega_n+\varepsilon_n-\varepsilon_m)}}_{\frac{e^{\beta(\varepsilon_n-\varepsilon_m)}-1}{i\omega_n+\varepsilon_n-\varepsilon_m}}$$

$$= -\frac{1}{Z}\sum_{n,m}|\langle m|\hat{\psi}^\dagger_\sigma(k)|n\rangle|^2\,\frac{e^{-\beta\varepsilon_m}-e^{-\beta\varepsilon_n}}{i\omega_n+\varepsilon_n-\varepsilon_m}$$

$$= \int d\omega\,\frac{A_\sigma(k,\omega)}{i\omega_n-\omega} \qquad \text{spectral representation}$$

with

$$A_\sigma(k,\omega) = \frac{1}{Z}\sum_{n,m}|\langle m|\hat{\psi}^\dagger_\sigma(k)|n\rangle|^2\left(e^{-\beta\varepsilon_n}-e^{-\beta\varepsilon_m}\right)\delta(\omega+\varepsilon_n-\varepsilon_m)$$

spectral function: provides information on one-particle excitation spectrum $\omega=\varepsilon_n-\varepsilon_m$

$$T=0:\quad A_\sigma(k,\omega) = \sum_m |\langle m|\hat{\psi}^\dagger_\sigma(k)|0\rangle|^2\,\cancel{(1-e^{-\beta\varepsilon_m})}\,\delta(\omega+\varepsilon_0-\varepsilon_m)$$

$$-\sum_n |\langle 0|\hat{\psi}^\dagger_\sigma(k)|n\rangle|^2\,\delta(\omega+\varepsilon_n-\varepsilon_0)$$

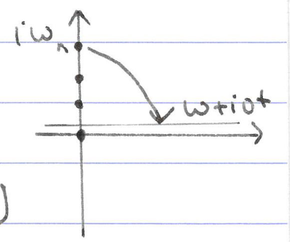
$\omega = \varepsilon_m-\varepsilon_0$ particle excitation

$\omega = \varepsilon_0-\varepsilon_n$ hole excitation.

Retarded Green function:

$$G^R_\sigma(k,t) = -i\Theta(t)\langle[\Psi_\sigma(k,t), \Psi^+_\sigma(k,0)]\rangle$$

$$G^R_\sigma(k,\omega) = \int d\omega' \frac{A_\sigma(k,\omega')}{\omega + i0^+ - \omega'}$$



i.e.

$$G^R_\sigma(k,\omega) = G_\sigma(k, i\omega_n \to \omega + i0^+)$$

and $$A_\sigma(k,\omega) = -\frac{1}{\pi}\,\mathrm{Im}\, G^R_\sigma(k,\omega)$$

Strategy: compute $G_\sigma(k, i\omega_n)$ (e.g. from perturbation theory)

$\to G^R_\sigma(k,\omega)$

$\to A_\sigma(k,\omega)$

Summary

① $\hat{H} = H(\hat{\psi}^{+}_{\alpha}, \hat{\psi}_{\alpha})$ normal ordered

$$Z = \int_{\psi^{(*)}(\beta) = \psi^{*}(0)} \mathcal{D}[\psi^{*}, \psi] \, e^{-S[\psi^{*}, \psi]}$$

with $S = \int_0^\beta d\tau \left(\sum_\alpha \psi^{*}_\alpha \partial_\tau \psi_\alpha + H(\psi^{*}_\alpha(\tau^{+}), \psi_\alpha(\tau)) \right)$ Eucl. action.

$$G(\alpha\tau, \alpha'\tau') = - \langle T_\tau \hat{\psi}_\alpha(\tau) \hat{\psi}^{+}_{\alpha'}(\tau') \rangle$$ time-ordered corr. f^{n}

$$= - \frac{1}{Z} \int \mathcal{D}[\psi^{*}, \psi] \, \psi_\alpha(\tau) \psi^{*}_{\alpha'}(\tau') \, e^{-S}$$

e.g.: $n(r) = \langle \hat{\psi}^{+}(r) \hat{\psi}(r) \rangle$

$$= \langle T_\tau \hat{\psi}(r\tau) \hat{\psi}^{+}(r\tau^{+}) \rangle$$

$$= - G(r\tau, r\tau^{+})$$

$$= - \frac{1}{\beta} \sum_\omega G(r, r; i\omega) \, e^{i\omega 0^{+}}$$

② Non-interacting systems: $\hat{H}_0 = \sum_{\alpha,\alpha'} h_{\alpha\alpha'} \hat{\psi}^{+}_\alpha \hat{\psi}_{\alpha'}$

$$S_0 = \int_0^\beta d\tau \left(\sum_\alpha \psi^{*}_\alpha \partial_\tau \psi_\alpha + H_0(\psi^{*}_\alpha, \psi_\alpha) \right)$$

$$\equiv \int_0^\beta d\tau \, d\tau' \sum_{\alpha\alpha'} \psi^{*}_\alpha(\tau) \underbrace{M(\alpha\tau, \alpha'\tau')}_{-G_0^{-1}(\alpha\tau, \alpha'\tau').} \psi_{\alpha'}(\tau')$$

$$Z_0 = \left[\det(-G_0) \right]^{-1}$$

$$= \prod_{\alpha, \omega_n} (-i\omega_n + \xi_\alpha)^{-1}$$ diagonal basis

$$\Omega_0 = - \frac{1}{\beta} \operatorname{Log} Z_0 = \frac{1}{\beta} \sum_{\omega_n, \alpha} \operatorname{Log}(-i\omega_n + \xi_\alpha) \, e^{i\omega_n 0^{+}}$$

③ PT: $$Z = \int \mathcal{D}[\psi^*, \psi]\, e^{-S_0 - S_{int}}$$

$$= \int \mathcal{D}[\psi^*, \psi]\, e^{-S_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_{int}^n$$

$$\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0$$

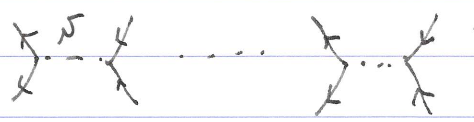
$$\downarrow$$

$$\langle \psi^* \psi^* \psi \psi \cdots\cdots \psi^* \psi^* \psi \psi \rangle_0$$

$= \sum$ all complete contraction

$$\overbrace{\psi^*_{\alpha_1}(\tau_1)\, \psi_{\alpha_2}}(\tau_2) = -G_0(2,1)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle \cdots \rangle_0$$



→ Feynman diagrams.