Quantum Field Theory I
Master ICFP 2026-2027
Problem Set 6

October 19, 2026

1 Feynman diagrams in a scalar theory

The aim of this exercise is to get draw Feynman diagrams in a simple theory, the scalar theory with ϕ^4 interactions. Our aim is to compute the S matrix in the $2 \to 2$ sector, within perturbation theory.

The lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

1. Recall the expression for the Feynman propagator in direct space and in Fourier space. Compute also the interaction hamiltonian in the interaction picture.

2. Draw the Feynman diagrams for $\langle O|T\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)e^{-i\int d^4x\mathcal{H}_I}|0\rangle$ at order λ^0. Are these connected?

3. Draw the Feynman diagrams at order λ^1. Which ones involve vacuum bubbles? Which ones are connected?

4. Compute $\langle O|Te^{-i\int d^4x\mathcal{H}_I}|0\rangle$ at order λ^1 and deduce the correlation function $\mathcal{G}(x_1, x_2, x_3, x_4)$. Pay particular attention to the diagrams with bubble. As proven in the lecture, the same cancellation occurs at higher orders in perturbation theory. For this reason, from now on, we will only consider Feynman diagrams with no vacuum bubble.

5. We now want to illustrate the fact that the disconnected Feynman diagrams can be interpreted in terms of the exact propagator. We now consider the 2-point correlation function. Draw the Feynman diagrams of this object at order λ^2 and find the symmetry factor of each of these diagrams.

6. Draw the disconnected Feynman diagrams for the 4-point correlation function at order λ^2 (with no vacuum bubble) and compute the symmetry factor for each of them. Reexpress these in terms of the propagators obtained in the previous question. It is enough to consider one set of diagrams, for instance those which connect x_1/x_2 and x_3/x_4. This relation between the disconnected diagrams and 2-point correlation functions holds true at all orders.

7. Draw now the connected Feynman diagrams for the 4-point correlation function at order λ^2 (with no vacuum bubble) and compute the symmetry factor for each of them. Interpret these diagrams in terms of amputated diagrams and 2-point correlation functions.

8. Write the integral $f(p)$ for one of the nontrivial diagrams in direct space and in Fourier space. Express the amputated diagram (that is when the propagators on the external legs have been factorized) as an integral over momenta.

9. We now study the integral in some detail. We will later on develop more systematic methods, but for now, consider the case where the external momenta are 0, $f(0)$ and change the variable of integration $q^0 \to iq_{\mathrm{E}}$ (this corresponds to changing the axis on which the integral is performed in the complex plane of q^0). Rewrite the integral as an integral over euclidean momenta. You can then perform the angular integrals. What can you say about the remaining integral?

10. There are several ways or regularizing the theory. One consists in introducing an ultraviolet cutoff. The integral over momenta is limited to $q < \Lambda$. Perform the integral in this case and find the leading behavior in the limit $\Lambda \to \infty$.

11. Another way of regularizing the integral consists in considering it in an arbitrary real dimension d (or even complex if we wish). Under what condition on d does the integral converge? Compute the integral in this case. You can use the fact that, for $a > -1$ and $a - b < -1$,

$$\int_0^\infty \frac{x^a}{(x+1)^b} = \frac{\Gamma(1+a)\Gamma(-1-a+b)}{\Gamma(b)}$$

    Compute the divergent contribution in an expansion in small ϵ, where $d = 4 - 2\epsilon$. Compare with the result of the previous question.

12. Observe that $f(p) - f(0)$ is now finite.