

Quantum Field Theory I

Master ICFP 2026-2027

Problem Set 4

October 7, 2026

1 Energy momentum tensor of the Dirac field

Consider a Dirac field whose Lagrange density is $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$.

1. Show that the theory is invariant under a global change of phase. At the infinitesimal level, this corresponds to a transformation $\psi \rightarrow \psi + \delta\psi$ with $\delta\psi = i\theta\psi$. Find the Noether current and check that it is conserved.
2. In the case of a massless particle, show that the theory is also invariant under an infinitesimal transformation $\delta\psi = i\theta\gamma^5\psi$. Find the associated conserved current. Check that it is indeed conserved in the massless theory but is not in the massive one. This symmetry is called chiral symmetry and plays a key role in QCD for understanding the mass of the particles.
3. Find the energy momentum tensor. Check that it is indeed conserved. Observe that it is not symmetric under permutation of the indices.
4. ★ Under an infinitesimal boost, the Dirac bispinor transforms as: $\psi \rightarrow \psi + \delta\psi$ with $\delta\psi = \omega_{\mu\nu}(x^\mu\partial^\nu - \frac{i}{2}S^{\mu\nu})\psi$. Find the associated Noether current.

2 Propagators

The aim of this exercise is to get more familiar with the notion of propagators which are key in the rest of the lecture. As we now show, propagators can be introduced independently of quantum field theory.

Let us call $D(x, y)$ a solution of the partial differential equation $(\square + m^2)D(x - y) = -i\delta^{(4)}(x - y)$. We look for solutions that can be Fourier transformed.

1. Show that the propagator can be used to find a solution $u(x)$ of the partial differential equation $(\square + m^2)u(x) = J(x)$ where J is some given function (often called a source).

2. Since the problem is invariant under translations, we are inclined to go to Fourier space. Write the equation defining D in the reciprocal space and solve for $\hat{D}(p)$.
3. We could formally make the inverse Fourier transform and obtain an expression for the propagator. There are however two caveats.
 - the propagator is not uniquely defined: we can always add a solution of the homogeneous equation and get another propagator;
 - The inverse Fourier transform is ill-defined because we hit poles, which lie in $p^0 = \pm\sqrt{\vec{p}^2 + m^2}$.

As seen in the lecture, we can solve the second problem by slightly deforming the integration contour in p^0 so that it avoids the poles, passing either above or below. We define the retarded propagator as the result of the integral when we avoid the poles by above (or, equivalently, by moving the poles in the lower half plane). The advanced propagator is obtained by avoiding the poles by passing below. Finally, the Feynman propagator is obtained by avoiding the pole at $p^0 < 0$ by below and the pole at $p^0 > 0$ by above.

Use the theorem of residues to perform the p^0 integrals for the three propagators. You will have to differentiate the cases where $x^0 - y^0$ is positive or negative. Express your results in terms of

$$\Delta(x) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-iE_p t + i\vec{p}\vec{x}}}{2E_p}$$

4. In the previous question, we found different propagators. As discussed above, this is not a surprise because we can always add a solution of the homogeneous equation to a propagator and obtain a new propagator. Otherwise stated, the difference of two propagators should be a solution of the homogeneous equation. Check that this is true.
5. The Feynman propagator is of utmost importance in QFT because it appears in perturbative calculations of scattering amplitudes. It is basically the mismatch between the time ordering $T\Phi_I(x)\Phi_I(y)$ and the normal ordering $:\Phi_I(x)\Phi_I(y):$. Compute the difference of these two quantities and check that this it can be expressed in terms of the Feynman propagator.