

Quantum Field Theory I

Master ICFP 2026-2027

Problem Set 3

October 7, 2026

1 Diracology

You will exercise with the γ matrices and related aspects of Dirac bispinors. This will prove useful later on when we will perform actual calculations.

1. Using Clifford algebra, show that $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = 1$.
2. Compute the traces:

$$\text{Tr } \gamma^\mu \quad \text{Tr } \gamma^\mu \gamma^\nu \quad \text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \quad \text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$

The general strategy is to use the anticommutator of two gamma matrices and use the cyclicity of the trace. For odd powers of γ matrices, it is convenient to introduce the identity in the form $(\gamma^5)^2$.

3. Show that the following quantities are hermitian:

$$\bar{\psi}\psi \quad \bar{\psi}\gamma^\mu\psi \quad \bar{\psi}\gamma^\mu\gamma^5\psi$$

4. Show that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$ are Lorentz scalars while $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^5\psi$ transform as vectors under Lorentz transformations.
5. The plane wave solutions of the Dirac equation are

$$u(\vec{p})e^{-ip \cdot x} \quad v(\vec{p})e^{ip \cdot x}$$

Check that

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta \\ \sqrt{p \cdot \bar{\sigma}} \eta \end{pmatrix} \quad v(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta \\ -\sqrt{p \cdot \bar{\sigma}} \eta \end{pmatrix}$$

are indeed solutions.

6. Show that

$$\bar{u}_s(\vec{p})u_{s'}(\vec{p}) = -\bar{v}_s(\vec{p})v_{s'}(\vec{p}) = 2m\delta_{ss'} \quad \bar{v}_s(\vec{p})u_{s'}(\vec{p}) = \bar{u}_s(\vec{p})v_{s'}(\vec{p}) = 0$$

2 Noether charges as generators of transformations

We consider the complex Klein-Gordon field.

1. Find the variables Π_ϕ and Π_{ϕ^*} which are conjugate to ϕ and ϕ^* . Write the equal time commutation relations.
2. Write the Noether charge Q associated with the change of phase of ϕ in terms of the fields and their conjugate variables. Compute the commutators $[Q, \phi]$, $[Q, \phi^*]$, $[Q, \Pi_\phi]$ and $[Q, \Pi_{\phi^*}]$ and comment.
3. Write the Noether charge H associated with the invariance under time translation in terms of the fields and their conjugate variables. Compare with the Legendre transform of the lagrangian. Compute the commutators $[H, \phi(x)]$, $[H, \phi^*(x)]$, $[H, \Pi_\phi(x)]$ and $[H, \Pi_{\phi^*}(x)]$. Comment.
4. Write the Noether charge P^i associated with the invariance under time translation in terms of the fields and their conjugate variables. Compute the commutators $[P^i, \phi(x)]$, $[P^i, \phi^*(x)]$, $[P^i, \Pi_\phi(x)]$ and $[P^i, \Pi_{\phi^*}(x)]$. Comment.
5. Compute the Noether charge for a boost. Show that this operator is hermitian.
6. We now focus on the free theory, $V(\rho) = m^2\rho$. The plane wave solutions write:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ip \cdot x} + b_{\vec{p}}^\dagger e^{ip \cdot x} \right)$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_{\vec{p}} e^{ip \cdot x} + a_{\vec{p}}^\dagger e^{-ip \cdot x} \right)$$

Write the operators $a_{\vec{p}}$ $b_{\vec{p}}$ $a_{\vec{p}}^\dagger$ $b_{\vec{p}}^\dagger$ in terms of the fields and their conjugate variables at time $t = 0$. Deduce their commutation relations.

7. Express the Noether charges Q , H and P^i in terms of the creation and annihilation operators.
8. The ground state $|0\rangle$ is defined as the state which is annihilated by both $a_{\vec{p}}$ and $b_{\vec{p}}$. Show that this state is an eigenstate of Q , H and P^i and give the associated eigenvalues.
9. We now consider the 1-particle state $a_{\vec{p}}^\dagger|0\rangle$. Show that it is also an eigenstate of the previous operators and find the associated eigenvalues.
10. Same questions for the 1-particle state $b_{\vec{p}}^\dagger|0\rangle$. What physical observable differentiate between these two kinds of states?