

Quantum Field Theory I

Master ICFP 2026-2027

Second Problem Set

September 15, 2026

1 Representations of a Lie group

Consider a Lie group G . The vicinity of the identity is characterized by a set of generators: $g \simeq 1 + i \delta\theta^a T^a$. We can generate elements of the group which are farther apart by using the exponentiation map:

$$g(\theta) = e^{i\theta^a T^a}$$

The commutator of two generators gives a linear combination of generators: $[T^a, T^b] = i f^{abc} T^c$, where f^{abc} are the structure constants.

1. The structure constants can be used to characterize the multiplication law of the group, even far from the identity. This can be understood by using the Baker-Campbell-Hausdorff formula, which states that $e^X e^Y = e^Z$, with:

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots$$

where the dots represent terms that involve nested commutators of X and Y . The multiplication of two elements of the group is still an element of the group: $g(\theta).g(\phi) = g(m(\theta, \phi))$. By using the Baker-Campbell-Hausdorff formula, find the first few terms of the expansion of m . Convince yourself that the multiplicative law m is completely determined by the structure constants.

2. Assume that we found another set of matrices t^a with the same commutation relations as the original generators: $[t^a, t^b] = i f^{abc} t^c$. We now build an exponential map with the new generators: $g'(\theta) = e^{i\theta^a t^a}$. Show that $g'(\theta).g'(\phi) = g'(m(\theta, \phi))$. This shows that the multiplication law is preserved.
3. Show that the matrices $j_a = \frac{1}{2}\sigma^a$, $k_a = \frac{i}{2}\sigma^a$ have the same commutation relations as the generators of the Lorentz group. Same question for $j_a = \frac{1}{2}\sigma^a$, $k_a = -\frac{i}{2}\sigma^a$. These are the generators of rotations and boosts in the representations $(1/2, 0)$ and $(0, 1/2)$ of the Lorentz group.

2 Noether currents in the complex Klein-Gordon model

Consider a complex field ϕ whose dynamics is characterized by the action:

$$S = \int d^d x \partial_\mu \phi^\star \partial^\mu \phi - V(\phi^\star \phi)$$

where the function V is called the potential.

1. Find the equations of motion of the theory.
2. Show that the theory is invariant under a global change of phase: $\phi \rightarrow e^{i\theta} \phi$. Find the associated Noether current, Check that it is conserved when we impose the equations of motion.
3. Same questions for the infinitesimal translations $\phi \rightarrow \phi + \epsilon^\mu \partial_\mu \phi$.
4. Same questions for the infinitesimal Lorentz transformations $\phi \rightarrow \phi - \omega^{\mu\nu} x_\mu \partial_\nu \phi$ where ω is antisymmetric.

3 Spontaneous Symmetry Breaking

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + m^2 \frac{\phi^2}{2} - \frac{\lambda}{4} \phi^4. \quad (3.1)$$

with $m^2, \lambda > 0$.

1. How many solutions with $\phi(x) = c$ can you find? Which ones have the lowest energy?
2. Do the lowest energy solutions respect the internal (i.e. excluding Poincaré) symmetries of this Lagrangian? Hint: what simple operation can you perform on the scalar without changing the Lagrangian?
3. Write $\phi(x) = c + \pi(x)$ and substitute it into the Lagrangian. Show that $\pi(x) = 0$ is a solution of the new equations of motion.