TD 7

1) $D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$

$(i\partial\!\!\!/_x + m)\, D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i\,(\not{p}+m)}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$

$= D_F(x-y)$

2) $(i\partial\!\!\!/ - m)\, D_F(x-y) = (i\partial\!\!\!/_x + m)(i\partial\!\!\!/_x - m)\, D(x-y)$

$= (-\Box - m^2) D(x-y)$

$= i\,\delta^{(4)}(x-y)$

3) $D(x-y)\,(-i\overleftarrow{\partial\!\!\!/}_y + m) = \int \frac{d^4p}{(2\pi)^4} \frac{i\,(\not{p}+m)}{p^2 - m^2 + i\varepsilon} e^{-ip(x-y)}$

$= D_F(x-y)$

4) $D_F(x-y)\,(-i\overleftarrow{\partial\!\!\!/}_y - m) = D(x-y)\,(-i\overleftarrow{\partial\!\!\!/}_y + m)(-i\overleftarrow{\partial\!\!\!/}_y - m)$

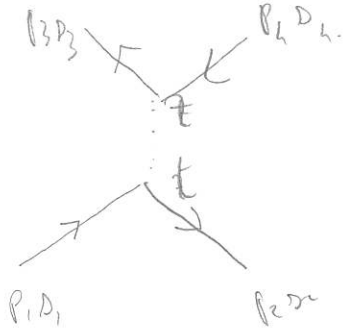
$= -(\Box_y + m^2) D(x-y)$

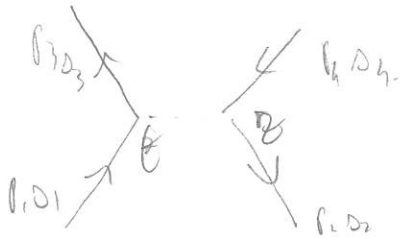
$= i\,\delta^{(4)}(x-y)$

$$ {}_{out}\langle \{p_3,s_3\},\{p_4,s_4\} \mid \{p_1,s_1\},\{p_2,s_2\}\rangle_{in} $$

$$ = \int_{x_1\cdots x_4} \bar{v}_{s_2}(i\not\partial_2 - m)\,\bar{u}_{s_3}(i\not\partial_3 - m)\,\langle 0|T\psi_{s_3}(x_3)\bar{\psi}_{s_4}(x_4)\, e^{-i\int H_I}\, \bar{\psi}_{s_1}(x_1)\psi_{s_2}(x_2)|0\rangle\,(i\overleftarrow{\not\partial}_1 + m)u_{s_1}\,(i\overleftarrow{\not\partial}_4+m)u_{s_4} $$
$$ e^{i(-p_1x_1 - p_2x_2 + p_3x_3 + p_4x_4)} $$

2) order 2





3) lot of diags:

$$ (-ig)^2 \int_{x_1\cdots x_4} \bar{v}_{s_2}(i\not\partial_2 - m)\,\bar{u}_{s_3}(i\not\partial_3 - m)\,\langle 0|T\psi_3\bar{\psi}_4\,\phi_2\bar{\psi}_2\psi_2\,\phi_1\bar{\psi}_1\psi_1\,\bar{\psi}_1\psi_2|0\rangle\,(i\overleftarrow{\not\partial}_1+m)u_{s_1}\,(i\overleftarrow{\not\partial}_4+m)v_{s_4} $$
$$ \int_{z,t} e^{i(-p_1x_1 - p_2x_2 + p_3x_3 + p_4x_4)} $$

$$ = -(-ig)^2 \int_{x_1\cdots x_4} \left(\bar{v}_{s_2}(i\not\partial_2 - m)D_F(x_2-t)D_F(t-x_4)(i\overleftarrow{\not\partial}_1+m)u_{s_1}\right)\left(\bar{u}_{s_3}(i\not\partial_3-m)D_F(x_3-z)D_F(z-x_4)(i\overleftarrow{\not\partial}+m)v_{s_4}\right) $$
$$ \int_{z,t} D(z-t) \quad e^{i(-p_1x_1 - p_2x_2 + p_3x_3 + p_4x_4)} $$

↑ because of wick th.

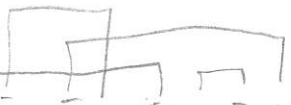
$$ = -(-ig)^2 \int_{x_1\cdots x_4} \left(-\bar{v}_{s_2}(\vec{p}_2)u_{s_1}(\vec{p}_1)\right) S(t-x_2)\,S(t-x_1)\left(-\bar{u}_{s_3}(p_3)v_{s_4}(p_4)\right) $$
$$ S(x_3-z)\,S(x_4-z)\; e^{i(-p_1x_1 - p_2x_2 + p_3x_3 + p_4x_4)} \int D(z-t) $$

$$ = -(-ig)^2 \int_{z,t} e^{i(-t(p_1+p_2)+z(p_3+p_4))}\, D(z-t) \quad (\bar{v}_{s_2}u_{s_1})(\bar{u}_{s_3}v_{s_4}) $$

$$= -(-ig)^2 \int_{z,t} e^{i\left((z-t)(p_3+p_4) - t(p_1+p_2-p_3-p_4)\right)} D(z-t)\,(\bar{v}_{D_2} v_{D_1})(\bar{u}_{D_3} u_{D_4})$$

$$= (-ig)^2 (2\pi)^4 \delta(p_1+p_2-p_3-p_4)\; D(p_1+p_2)\,(\bar{v}_2 v_1)(\bar{u}_3 u_4)$$

Secondo diagramma



$$(-ig)^2 \int_{\substack{x_1,\dots,x_4 \\ z,t}} \bar{v}_2 (i\overleftarrow{\partial}_{x_2} - m)\, \bar{u}_3 (i\overleftarrow{\partial}_{x_3} - m) \langle 0|T\psi_3 \bar{\psi}_4 \phi_z \bar{\psi}_2 \psi_z \phi_t \bar{\psi}_t \psi_t \bar{\psi}_1 \psi_2 |0\rangle (i\overrightarrow{\partial}_{x_1} - m) u_1 (i\overrightarrow{\partial}_{x_4} + m) v_4$$
$$D(z-t)\; e^{i(-p_1 x_1 - p_2 x_2 + p_3 x_3 + p_4 x_4)}$$

$$= + (-ig)^2 \int_{x_1,\dots,x_4,z,t} \left(\bar{v}_2 (i\overleftarrow{\partial}_{x_2} - m) D_F(x_2 - z) D_F(z - x_4) (i\overrightarrow{\partial}_{x_4} + m) v_4\right)$$
$$\left(\bar{u}_3 (i\overleftarrow{\partial}_{x_3} - m) D_F(x_3 - t) D_F(t - x_1) (i\overrightarrow{\partial}_{x_1} + m) u_1\right)$$
$$D(z-t)\; e^{i(-p_1 x_1 - p_2 x_2 + p_3 x_3 + p_4 x_4)}$$

$$= (ig)^2 (\bar{v}_2 v_4)(\bar{u}_3 u_1) \int_{z,t} e^{i(t(-p_1+p_3) + z(-p_2+p_4))} D(z-t)$$

$$= (ig)^2 (\bar{v}_2 v_4)(\bar{u}_3 u_1) (2\pi)^4 \delta(p_1+p_2-p_3-p_4)\; D(p_1-p_3)$$

$$\mathcal{M} = g^2 \left(\frac{\bar{v}_2 u_1\; \bar{u}_3 v_4}{\underbrace{(p_1+p_2)^2}_{s} - M^2 + i\varepsilon} - \frac{\bar{v}_2 v_4\; \bar{u}_3 u_1}{\underbrace{(p_1-p_3)^2}_{t} - M^2 + i\varepsilon} \right)$$

$$\frac{1}{4}\sum_{D_1D_2D_3D_4}|\mathcal{M}|^2 = \frac{1}{4}\sum_{D_1D_2D_3D_4}\left(|\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 - 2\,\mathrm{Re}\,\mathcal{M}_1^*\mathcal{M}_2\right)$$

$$\frac{1}{4}\sum_{D_1D_2D_3D_4}|\mathcal{M}_1|^2 = \frac{1}{4}\frac{g^4}{(D-\mu^2+i\epsilon)^2}\left(\sum_{D_1D_2}\bar{v}_2 u_1 \bar{u}_1 v_2\right)\left(\sum_{D_3D_4}\bar{u}_3 v_4 \bar{v}_4 u_3\right)$$

$$= \frac{1}{4}\frac{g^4}{D-M^2}\left(\mathrm{Tr}\,(\not{p}_1+m)(\not{p}_2-m)\;\mathrm{Tr}\,(\not{p}_4-m)(\not{p}_3+m)\right)$$

$$= 4\frac{g^4}{(D-M^2)^2}\left(p_1p_2-m^2\right)\left(p_3p_4-m^2\right) = \frac{4g^4}{(D-M^2)^2}\,\frac{D^2}{4} = \frac{g^4D^2}{(D-M^2)^2}$$

$D^2 = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1p_2 = 2p_1p_2 + 4m^2$

$$\frac{1}{4}\sum_{D_1D_2D_3D_4}|\mathcal{M}_2|^2 = \frac{g^4}{4(t-\mu^2+i\epsilon)^2}\,\mathrm{Tr}\,(\not{p}_2-m)(\not{p}_4-m)\;\mathrm{Tr}\,(\not{p}_1+m)(\not{p}_3+m)$$

$$= \frac{4g^4}{(t-\mu^2+i\epsilon)^2}\left(p_2p_4+m^2\right)\left(p_1p_3+m^2\right)$$

$$= \frac{g^2t^2}{(t-\mu^2)^2}$$

$$\frac{1}{4}\sum_{D_1D_2D_3D_4}\mathcal{M}_1^*\mathcal{M}_2 = -\frac{g^2}{4(D-\mu^2+i\epsilon)(t-\mu^2+i\epsilon)}\left(\sum_{D_i}\bar{u}_1 v_2 \bar{u}_2 v_4 \bar{v}_4 u_3 \bar{u}_3 u_1\right)$$

$$= \frac{-g^2}{4(D-\mu^2+i\epsilon)(t-\mu^2+i\epsilon)}\,\mathrm{Tr}\,(\not{p}_2-m)(\not{p}_4-m)(\not{p}_3+m)(\not{p}_1+m)$$

$$= \frac{-g^2\,4}{4(D-\mu^2+i\epsilon)(t-\mu^2+i\epsilon)}\left((p_1p_3)(p_2p_4) - p_1p_4)(p_2p_3) + (p_1p_3)(p_2p_4)\right)$$

$$= -\frac{4g^2}{(s-m^2+i\varepsilon)(t-m^2+i\varepsilon)} \qquad \left(s^2-t^2+t^2\right)$$

BUT $s+t+u = (p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2$

$= p_1^2+p_2^2+2p_1p_2+p_1^2+p_3^2-2p_1p_3+p_1^2+p_4^2-2p_1p_4$

$= m_1^2+m_2^2+m_1^2+m_3^2+m_1^2+m_4^2+2p_1(\underbrace{p_2-p_3-p_4}_{-p_1})$

$= m_1^2+m_2^2+m_3^2+m_4^2 = 0$ in the mass shell case

$u = -(s+t)$

$u^2 = s^2+t^2+2st \qquad s^2+t^2 = -2st$

$$\frac{1}{4}\sum |\mathcal{M}_1+\mathcal{M}_2|^2 = \frac{2g^2 2st}{(s-m^2+i\varepsilon)(t-m^2+i\varepsilon)}$$

All in all

$$\frac{1}{4}\sum |\mathcal{M}|^2 = g^4\left[\frac{s^2}{(s-m^2)^2}+\frac{t^2}{(t-m^2)^2}+\frac{4st}{(s-m^2)(t-m^2)}\right]$$

5) $p_1 = (|\vec{p}|, \vec{p})$ $\quad p_3 = (|\vec{p}'|, \vec{p}')$

$p_2 = (|\vec{p}|, -\vec{p})$ $\quad p_4 = (|\vec{p}'|, -\vec{p}')$.

conservation of energy $|\vec{p}| = |\vec{p}'|$

$s = (p_1 + p_2)^2 = 4\vec{p}^2$

$t = (p_1 - p_3)^2 = (0, \vec{p} - \vec{p}')^2 = -(\vec{p}^2 + \vec{p}'^2 - 2\vec{p}\vec{p}')$

$= -p^2(2 - 2\cos\theta) = -2p^2(1 - \cos\theta)$

s is positive the first reaction can give a pole

t is negative $\rightarrow$ no pole