

$$\int d^3x \phi(n) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{+i\vec{p}\vec{n}} + b_{\vec{p}}^\dagger e^{-i\vec{p}\vec{n}} \right) \quad (6)$$

$$\phi^\dagger(n) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(b_{\vec{p}} e^{+i\vec{p}\vec{n}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\vec{n}} \right)$$

$$\dot{\phi} = \Pi_{\phi^*}(n) = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(a_{\vec{p}} e^{+i\vec{p}\vec{n}} + b_{\vec{p}}^\dagger e^{-i\vec{p}\vec{n}} \right)$$

$$\dot{\phi}^\dagger = \Pi_{\phi}(n) = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left(-b_{\vec{p}} e^{+i\vec{p}\vec{n}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\vec{n}} \right)$$

$$\int d^3x e^{+i\vec{q}\vec{x}} \phi(n) = \frac{1}{\sqrt{2E_q}} \left(a_{-\vec{q}} + b_{\vec{q}}^\dagger \right)$$

$$\int d^3x e^{+i\vec{q}\vec{x}} \Pi^*(n) = i \sqrt{\frac{E_q}{2}} \left(-a_{-\vec{q}} + b_{\vec{q}}^\dagger \right)$$

$$\int d^3x e^{+i\vec{q}\vec{n}} \left(\sqrt{\frac{E_q}{2}} \phi(n) - i \frac{1}{\sqrt{2E_q}} \Pi^*(n) \right) = b_{\vec{q}}^\dagger$$

$$\int d^3x e^{+i\vec{q}\vec{n}} \left(\sqrt{\frac{E_q}{2}} \phi^\dagger(n) - i \frac{1}{\sqrt{2E_q}} \Pi_{\phi}(n) \right) = a_{-\vec{q}}$$

$$b_{\vec{q}} = \int d^3x e^{+i\vec{q}\vec{n}} \left(\sqrt{\frac{E_q}{2}} \phi(n) + i \frac{1}{\sqrt{2E_q}} \Pi^*(n) \right) = a_{-\vec{q}}$$

$$\int d^3x e^{+i\vec{q}\vec{n}} \left(\sqrt{\frac{E_q}{2}} \phi^\dagger(n) + i \frac{1}{\sqrt{2E_q}} \Pi_{\phi}(n) \right) = b_{-\vec{q}}$$

$$[a_q^\dagger, a_{q'}] = \int d^3x d^3y e^{i(\vec{q}\cdot\vec{x} - \vec{q}'\cdot\vec{y})} \left[\sqrt{\frac{E_q}{2}} \phi(\vec{x}) - \frac{i}{\sqrt{E_q}} \pi(\vec{x}), \sqrt{\frac{E_{q'}}{2}} \phi(\vec{y}) + \frac{i}{\sqrt{E_{q'}}} \pi(\vec{y}) \right] \quad (7)$$

$$= \int d^3x d^3y \delta_{\vec{x}-\vec{y}} \left(+i \sqrt{\frac{E_q}{4E_{q'}}} \delta h(\vec{y}) - i \sqrt{\frac{E_{q'}}{4E_q}} (-i) \delta h(\vec{y}) \right)$$

$$= - \int d^3x \frac{e^{i(q'-q)\cdot\vec{x}}}{2} \left(\sqrt{\frac{E_q}{E_{q'}}} + \sqrt{\frac{E_{q'}}{E_q}} \right)$$

$$= - (2\pi)^3 \delta_{\vec{q}-\vec{q}'}$$

See calculation for $[a_q^\dagger, p_{q'}]$

$[a, l^\dagger] = 0$ because all operators commute
same thing for $[a^\dagger, l]$

$$[a_q^\dagger, l_{q'}^\dagger] = \int d^3x d^3y e^{i(\vec{q}\cdot\vec{x} + \vec{q}'\cdot\vec{y})} \left[\sqrt{\frac{E_q}{2}} \phi(\vec{x}) - \frac{i}{\sqrt{E_q}} \pi(\vec{x}), \sqrt{\frac{E_{q'}}{2}} \phi(\vec{y}) - \frac{i}{\sqrt{E_{q'}}} \pi(\vec{y}) \right]$$

$$= \int d^3x d^3y \delta_{\vec{x}-\vec{y}} e^{i(\vec{q}+\vec{q}')\cdot\vec{x}} \left(-i \sqrt{\frac{E_q}{4E_{q'}}} + -i \sqrt{\frac{E_{q'}}{4E_q}} (-i) \right)$$

$$= 0$$

Same thing for $[a_q, l]$

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$$7) \quad Q = \int d^3x : (\pi_\phi \phi - \pi_{\phi^*} \phi^*) :$$

$$= \int d^3x : i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} (-b_{\vec{p}} e^{i\vec{p}\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\vec{x}}) \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} (a_{\vec{q}} e^{i\vec{q}\vec{x}} + b_{\vec{q}}^\dagger e^{-i\vec{q}\vec{x}})$$

- a ↔ b

$$= \int \frac{d^3p d^3q}{(2\pi)^6} \sqrt{\frac{E_p}{4E_q}} \left[b_{\vec{p}} a_{\vec{q}} \delta(\vec{p}-\vec{q}) + b_{\vec{p}} b_{\vec{q}}^\dagger \delta(\vec{p}-\vec{q}) - a_{\vec{p}}^\dagger a_{\vec{q}} \delta(\vec{p}-\vec{q}) - a_{\vec{p}}^\dagger b_{\vec{q}}^\dagger \delta(\vec{p}+\vec{q}) \right]$$

- a ↔ b

$$= \int \frac{d^3p}{(2\pi)^3} (b_{\vec{p}}^\dagger b_{\vec{p}} - a_{\vec{p}}^\dagger a_{\vec{p}})$$

$$H = \int d^3x \pi_{\phi^*} \pi_{\phi} + \partial_i \phi^* \partial_i \phi + m^2 \phi^* \phi$$

$$= - \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{\sqrt{E_p E_q}}{2} (-a_{\vec{p}} e^{i\vec{p}\vec{x}} + b_{\vec{p}}^\dagger e^{-i\vec{p}\vec{x}}) (-b_{\vec{q}} e^{i\vec{q}\vec{x}} + a_{\vec{q}}^\dagger e^{-i\vec{q}\vec{x}})$$

$$= \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} p q (a_{\vec{p}} e^{i\vec{p}\vec{x}} - b_{\vec{p}}^\dagger e^{-i\vec{p}\vec{x}}) (b_{\vec{q}} e^{i\vec{q}\vec{x}} - a_{\vec{q}}^\dagger e^{-i\vec{q}\vec{x}})$$

$$+ \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_q}} m^2 (a_{\vec{p}} e^{i\vec{p}\vec{x}} + b_{\vec{p}}^\dagger e^{-i\vec{p}\vec{x}}) (b_{\vec{q}} e^{i\vec{q}\vec{x}} + a_{\vec{q}}^\dagger e^{-i\vec{q}\vec{x}})$$

Tens in $a^\dagger$ the integral $\delta^3(p+q) \rightarrow$
 perf the integral in $q \rightarrow$

$$\int \frac{d^3p}{(2\pi)^3} \left(-\frac{E_p}{2} + \frac{1}{2E_p} (p^2 + m^2) \right) a_p b_{-p} = 0$$

Same calculation for tens in $a^\dagger b^\dagger$.

Tens in $b^\dagger b$: integral in $x \rightarrow \delta(p-q)$. Integrate over $q \rightarrow$

$$\int \frac{d^3p}{(2\pi)^3} \left(\frac{E_p}{2} + \frac{1}{2E_p} (p^2 + m^2) \right) b_p^\dagger b_p$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p b_p^\dagger b_p$$

Same calc for the tens in $a a^\dagger$:

$\hookrightarrow$

$$H = \int \frac{d^3p}{(2\pi)^3} \cdot E_p (a_p a_p^\dagger + b_p^\dagger b_p)$$

We can rewrite this in a normal ordered form:

$$H = \int \frac{d^3p}{(2\pi)^3} E_p (a_p^\dagger a_p + b_p^\dagger b_p + \underbrace{(2\pi)^3 \delta^3(0)}_{\infty})$$

The ~~zero~~ fluctuation of the vacuum gives an infinite zero point energy. To regularize, put the system in a box $\delta^3(0) \sim V$

We can shift the energies s.t. $\tilde{H} = \int \frac{d^3p}{(2\pi)^3} E_p (a_p^\dagger a_p + b_p^\dagger b_p)$

$$P^i = - \int d^3x \quad \partial_i \phi \Pi^i + \partial_i \phi^* \Pi^i$$

$$= - \int d^3x \frac{d^3p d^3q}{(2\pi)^6} \sqrt{\frac{E_p}{E_q}} i p^i \left[(a_{\vec{p}} e^{i p x} - b_{\vec{p}} e^{-i p x}) (-b_{\vec{q}} e^{i q x} + a_{\vec{q}} e^{-i q x}) \right. \\ \left. + (b_{\vec{p}} e^{i p x} - a_{\vec{p}} e^{-i p x}) (-a_{\vec{q}} e^{i q x} + b_{\vec{q}} e^{-i q x}) \right]$$

term in a b.c. integrated in $x \rightarrow (2\pi)^3 \delta(p+q)$. integrated in $q \rightarrow$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{p^i}{2} (a_p b_{-p} + b_p a_{-p}) = 0 \text{ under the integral.} \\ (\text{odd function of } p.)$$

Something for term in $a^\dagger b^\dagger$.

term in $b^\dagger b$: integrated in $x \rightarrow (2\pi)^3 \delta(p-q)$ integrated in $q \rightarrow$

$$\int \frac{d^3p}{(2\pi)^3} \frac{p^i}{2} (b_p^\dagger b_p + b_p b_p^\dagger) \\ = \int \frac{d^3p}{(2\pi)^3} p^i (b_p^\dagger b_p + \underbrace{(2\pi)^3 \delta(0)}) \\ \text{0 under the integral}$$

The calc also for the ϕ give a similar result.

$$P^i = \int \frac{d^3p}{(2\pi)^3} p^i (a_p^\dagger a_p + b_p^\dagger b_p)$$

8] $Q|0\rangle = \tilde{H}|0\rangle = P'|0\rangle = 0$

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The vacuum is an eigenket of these operators with eigenvals 0.

9] Note that $a_L^\dagger a_L a_P^\dagger |0\rangle = (a_L^\dagger [a_L a_P^\dagger] + a_L^\dagger a_P^\dagger a_L) |0\rangle$
 $= (2\pi)^3 \delta(\mathbf{r}-\mathbf{L}) a_P^\dagger |0\rangle$

$$L^\dagger L a_P^\dagger |0\rangle = L^\dagger a_P^\dagger L |0\rangle = 0$$

From this we conclude that

$$Q a_P^\dagger |0\rangle = - \int \frac{d^3 L}{(2\pi)^3} \delta(\mathbf{L}-\mathbf{P}) (2\pi)^3 a_P^\dagger |0\rangle$$

$$= - a_P^\dagger |0\rangle$$

Similarly $H a_P^\dagger |0\rangle = E_P a_P^\dagger |0\rangle$

$$\mathbf{P} \cdot a_P^\dagger |0\rangle = \mathbf{P} \cdot a_P^\dagger |0\rangle$$

$a_P^\dagger |0\rangle$ is a 1 particle state with charge -1
 Energy $\sqrt{p^2 + m^2}$
 momenta $\mathbf{p}$

Similar calculation for $P_p^{\dagger} |0\rangle$ show that it is an eigenstate of Q w/ P^i with eigenvalue

$$\Delta 1, \quad E_p, P^i$$

The diff between these 2 states is the electric charge.