

①

$$1) \quad g(\sigma) g(\varphi) = e^{i\sigma^a \tau^a} e^{i\varphi^b \tau^b}$$

$$= \exp \left(i\sigma^a \tau^a + i\varphi^b \tau^b - \frac{i}{2} [\sigma^a \tau^a, \varphi^b \tau^b] \right)$$

$$+ \frac{i}{12} [\sigma^a \tau^a [\sigma^b \tau^b, \varphi^c \tau^c]] - \frac{i}{12} [\varphi^a \tau^a [\varphi^b \tau^b, \sigma^c \tau^c]] + \dots$$

$$= \exp \left(i\tau^a (\sigma^a + \varphi^a - f^{abc} \sigma^b \varphi^c + \frac{1}{2} f^{abc} \sigma^b f^{cde} \sigma^d \varphi^e + \frac{1}{2} f^{abc} \varphi^b f^{cde} \varphi^d \sigma^e + \dots) \right)$$

$$2) \quad g'(\sigma) g'(\varphi) = g'(u(\sigma, \varphi))$$

$$3) \quad \cancel{[j_a, j_b]} \quad \sigma^a \sigma^b = i\varepsilon^{abc} \sigma^c + \delta^{ab} \mathbb{I}$$

$$[j_a, j_b] = i\varepsilon^{abc} j_c$$

$$[\sigma^a, \sigma^b] = 2i\varepsilon^{abc} \sigma^c$$

$$[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}] = i\varepsilon^{abc} \frac{\sigma^c}{2}$$

$$[j_a, h_b] = i\varepsilon^{abc} \left(\frac{1}{2} \sigma^c \right) = i\varepsilon^{abc} h_c$$

$$[h_a, h_b] = -i\varepsilon^{abc} \frac{\sigma^c}{2} = -i\varepsilon^{abc} j_c$$

$$\textcircled{2} \quad 1) \quad \frac{\delta S}{\delta \phi} = -\square \phi^* - \phi^* V'(\phi^* \phi) = 0$$

$$\frac{\delta S}{\delta \phi^*} = -\square \phi - \phi V'(\phi^* \phi) = 0 \quad \ddot{\phi} = \Delta \phi - \phi V'$$

$$2) \quad \phi \rightarrow \phi' = \phi + i\theta \phi$$

$$\phi^* \rightarrow \phi'^* = \phi^* - i\theta \phi^*$$

$$\delta S = \int d^d x \quad -i \partial_\mu (\phi^* \partial^\mu \phi) + i \partial_\mu \phi^* \partial^\mu (\theta \phi)$$

$$= \int d^d x \quad \partial_\mu \theta (i \partial^\mu \phi^* \phi - i \phi^* \partial^\mu \phi)$$

$$J^\mu = i (\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi)$$

$$\partial_\mu J^\mu = i (\square \phi^* \phi - \phi^* \square \phi)$$

$$= i (-\phi^* \phi V' + \phi^* \phi V') = 0$$

$$2) \quad \phi' = \phi + \epsilon^\mu \partial_\mu \phi$$

$$\delta S = \int \partial_\mu (\epsilon^\nu \partial^\mu \phi^*) \partial^\nu \phi + \partial_\mu \phi^* \partial^\mu (\epsilon^\nu \partial_\nu \phi) + V'(\phi^* \phi) (\epsilon^\mu \partial_\mu (\phi^* \phi))$$

$$= \int \partial_\mu \epsilon^\nu \left(\partial^\mu \phi^* \partial^\nu \phi + \partial_\mu \phi^* \partial^\nu \phi \right) + \epsilon^\nu \partial_\nu \left(V(\phi^* \phi) \phi^* \phi + \partial_\mu \phi^* \partial^\mu \phi \right)$$

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$$= \int \partial_r \epsilon_0 \left\{ \underbrace{\partial^r \phi^* \partial^\nu \phi + \partial^\nu \phi^* \partial^r \phi - \eta^{\mu\nu} \mathcal{L}}_{T^{\mu\nu}} \right\}$$

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \Box \phi^* \partial^\nu \phi + \partial^\nu \phi^* \Box \phi + \partial^\nu (\partial_r \phi^* \partial^r \phi) \\ &\quad - \partial^\nu (\partial_r \phi^* \partial^r \phi - V(\phi^* \phi)) \\ &= \partial^\nu \phi (\Box \phi^* + V'(\phi^* \phi) \phi^*) + c.c. \\ &= 0 \end{aligned}$$

$$3) \quad \phi' = \phi - \frac{1}{2} \omega_{\mu\nu} x^\mu \partial^\nu \phi$$

$$\begin{aligned} \delta S &= \int \partial_r \left(-\frac{1}{2} \omega_{\rho\sigma} x^\rho \partial^\sigma \phi^* \right) \partial^r \phi + \partial_r \phi^* \partial^\sigma \left(-\frac{1}{2} \omega_{\rho\sigma} x^\rho \partial^\sigma \phi \right) \\ &\quad - \frac{1}{2} \omega_{\rho\sigma} x^\rho \partial^\sigma V(\phi^* \phi) \\ &= \int \partial_r \omega_{\rho\sigma} \left(-\frac{1}{2} x^\rho \partial^\sigma \phi^* \partial^r \phi - \frac{1}{2} x^\sigma \partial^r \phi^* \partial^\rho \phi \right) \\ &\quad - \frac{1}{2} \omega_{\rho\sigma} \partial^\sigma \phi^* \partial^\rho \phi - \frac{1}{2} \omega_{\rho\sigma} \partial^\rho \phi^* \partial^\sigma \phi \\ &\quad - \frac{1}{2} \omega_{\rho\sigma} x^\rho \partial^\sigma \mathcal{L} \end{aligned}$$

$$= \int -\frac{1}{2} \partial_\mu \omega_{\rho\sigma} (x^\rho \partial^\sigma \phi^\mu \partial^\mu \phi + x^\rho \partial^\mu \phi^\mu \partial^\sigma \phi - \eta^{\rho\sigma} x^\mu \mathcal{L}) \quad (3)$$

$$= \int -\frac{1}{2} \partial_\mu \omega_{\rho\sigma} (x^\rho \tau^{\mu\sigma})$$

$$\tau^{\rho\sigma} = x^\rho \tau^{\sigma\cdot} - x^\sigma \tau^{\cdot\rho}$$

$$| \partial_\mu \tau^{\rho\sigma} = \tau^{\rho\sigma} - \tau^{\sigma\rho} = 0$$