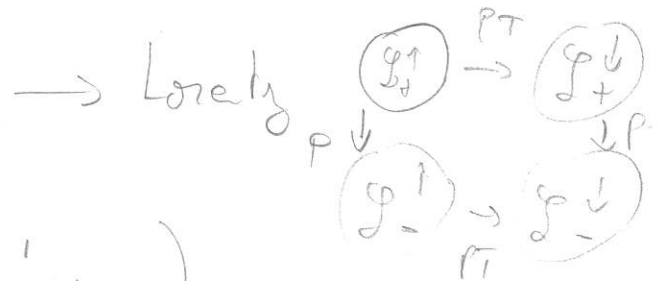


Poincaré: $x \rightarrow x' = f(x)$ such that

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu = dx'^\mu \eta_{\mu\nu} dx'^\nu$$

$$x' = \Lambda^\mu_\nu x^\nu + a^\mu$$

with $\int \Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta = \eta_{\alpha\beta}$
 Λ arbitrary



$$\Lambda_T = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \Lambda_P = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

$\mathcal{G}(\Lambda, a)$ is a representation of Poincaré if it preserves mult. rule

$$x' = \Lambda^\mu_\nu x^\nu + a^\mu$$

$$x' = \tilde{\Lambda}^\mu_\nu x^\nu + \tilde{a}^\mu$$

$$x \xrightarrow{(\tilde{\Lambda}, \tilde{a})} x' \xrightarrow{(\Lambda, a)} x''$$

$$x'' = \Lambda^\mu_\nu x'^\nu + a^\mu = \Lambda^\mu_\nu (\tilde{\Lambda}^\nu_\rho x^\rho + \tilde{a}^\nu) + a^\mu$$

$$= (\Lambda^\mu_\nu \tilde{\Lambda}^\nu_\rho) x^\rho + a^\mu + \Lambda^\mu_\nu \tilde{a}^\nu$$

$$\sqcup \quad \mathcal{G}(\Lambda, a) \mathcal{G}(\tilde{\Lambda}, \tilde{a}) = \mathcal{G}(\Lambda \tilde{\Lambda}, a + \Lambda \tilde{a})$$

inverse?

$$\mathcal{G}(\Lambda, a) \mathcal{G}(\Lambda^{-1}, -\Lambda^{-1}a) = \mathcal{G}(\Lambda \Lambda^{-1}, a - \Lambda \Lambda^{-1}a) = \mathcal{G}(1, 0)$$

$$\mathcal{G}^{-1}(\Lambda, a) = \mathcal{G}(\Lambda^{-1}, -\Lambda^{-1}a)$$

$$\begin{aligned}
 \xi_g(\Lambda a) \xi_g(1+\omega, \varepsilon) \xi_g^{-1}(\Lambda, a) &= \xi_g(\Lambda(1+\omega), a + \Lambda \varepsilon) \xi_g(\Lambda^{-1}, -\Lambda^{-1}a) \quad (2) \\
 &= \xi_g(\Lambda(1+\omega)\Lambda^{-1}, a + \Lambda \varepsilon + \Lambda(1+\omega)(-\Lambda^{-1}a)) \\
 &= \xi_g(\underbrace{1+\Lambda\omega\Lambda^{-1}}_{\text{invariant}}, \underbrace{a + \Lambda \varepsilon - a - \Lambda\omega\Lambda^{-1}a}_{\text{invariant}})
 \end{aligned}$$

$$\begin{aligned}
 \xi_g(\Lambda a) \left[1 - \frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu} - i \varepsilon_\mu p^\mu \right] \xi_g^{-1}(\Lambda a) \\
 = 1 - \frac{i}{2} \Lambda_\alpha^\beta \omega_{\beta\gamma} \Lambda^{-1\gamma}_\delta \Sigma^{\alpha\delta} - i (\Lambda_\mu^\nu \varepsilon_\nu - \Lambda_\mu^\nu \omega_{\nu\rho} \Lambda^\rho_\sigma a^\sigma) p^\mu \\
 = 1 - \frac{i}{2} \Lambda_\alpha^\mu \omega_{\mu\nu} \Lambda^{-\nu}_\beta \Sigma^{\alpha\beta} + i \Lambda_\alpha^\mu \omega_{\mu\nu} \Lambda^{-\nu}_\beta a^\beta p^\alpha \\
 - i \Lambda_\nu^\mu \varepsilon_\mu p^\nu \quad (\text{with } (2))
 \end{aligned}$$

$$(1) \quad \xi_g(\Lambda a) \frac{i}{2} \Sigma^{\mu\nu} \xi_g^{-1}(\Lambda a) = \frac{i}{2} \Lambda_\alpha^\mu \Lambda_\beta^\nu \Sigma^{\alpha\beta} - \frac{i}{2} \Lambda_\alpha^\mu \Lambda_\beta^\nu (a^\beta p^\alpha - a^\alpha p^\beta)$$

$$(2) \quad \xi_g(\Lambda a) i p^\mu \xi_g^{-1}(\Lambda a) = i \Lambda_\nu^\mu p^\nu$$

$$(2), \quad \Lambda = 1, a = \varepsilon \quad \Lambda = 1+\omega, a = \varepsilon \rightarrow \xi_g \text{ arbitrary}$$

$$\xi_g \left(-i p^\alpha, i p^\mu + i p^\mu, p^\alpha \right) = 0 = [p^\mu, p^\alpha] = 0.$$

$$-\frac{i}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta} i p^\mu + i p^\mu \left(\frac{i}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta} \right) = i (\omega_{\alpha\beta} p^\alpha \eta^{\mu\beta})$$

$$\frac{i}{2} [p^\mu \Sigma^{\alpha\beta}] = \frac{i}{2} (\eta^{\mu\alpha} p^\beta - \eta^{\mu\beta} p^\alpha)$$

$$[p^\mu \Sigma^{\alpha\beta}] = i (\eta^{\mu\alpha} p^\beta - \eta^{\mu\beta} p^\alpha)$$

$$\begin{aligned}
 & G(\Lambda a) \left(1 - \frac{i}{2} \omega_{\mu\nu} \Sigma^{\mu\nu} - i a_\mu P^\mu \right) G^{-1}(\Lambda a) \\
 &= 1 - \frac{i}{2} \Lambda_\alpha^\beta \omega_{\beta\gamma} \Lambda^{-1\gamma}_\delta \Sigma^{\alpha\delta} - i \left(\Lambda_\mu^\nu \varepsilon_\nu - \Lambda_\mu^\alpha \omega_{\alpha\beta} \Lambda^{-1\beta}_\gamma a^\gamma \right) P^\mu \\
 &= 1 - \frac{i}{2} \Lambda_\alpha^\mu \omega_{\mu\nu} \Lambda_\delta^\nu \Sigma^{\alpha\delta} - i \Lambda_\alpha^\mu \varepsilon_\mu P^\alpha + i \Lambda_\alpha^\mu \omega_{\mu\nu} \Lambda_\beta^\nu a^\beta P^\alpha.
 \end{aligned}
 \tag{20}$$

$$(1) \quad G(\Lambda a) (-i P^\mu) G^{-1}(\Lambda a) = -i \Lambda_\alpha^\mu P^\alpha$$

$$\begin{aligned}
 (2) \quad G(\Lambda a) \left(-\frac{i}{2} \Sigma^{\mu\nu} \right) G^{-1}(\Lambda a) &= -\frac{i}{2} \Lambda_\alpha^\mu \Lambda_\delta^\nu \Sigma^{\alpha\delta} + \frac{i}{2} \Lambda_\alpha^\mu \Lambda_\beta^\nu a^\beta P^\alpha \\
 &\quad - \frac{i}{2} \Lambda_\alpha^\nu \Lambda_\beta^\mu a^\beta P^\alpha \\
 &= +\frac{i}{2} \Lambda_\alpha^\mu \Lambda_\beta^\nu \left(-\Sigma^{\alpha\beta} + a^\beta P^\alpha - a^\alpha P^\beta \right)
 \end{aligned}$$

$$(1) \quad \left(1 - \frac{i}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta} - \frac{i}{2} \Sigma^{\alpha\beta} \right) \frac{i}{2} \Sigma^{\mu\nu} \left(1 + \frac{i}{2} \omega_{\alpha\beta} \Sigma^{\alpha\beta} + \frac{i}{2} \Sigma^{\alpha\beta} \right) \\ = +\frac{i}{2} (\eta_{\alpha}^{\mu} \omega_{\alpha}^{\nu} - \eta_{\beta}^{\nu} \omega_{\beta}^{\mu}) \Sigma^{\alpha\beta} - \frac{i}{4} \left(\Sigma^{\mu\nu} \rho^{\alpha} - \Sigma^{\mu\alpha} \rho^{\nu} \right)$$

$$\frac{1}{4} \omega_{\alpha\beta} [\Sigma^{\alpha\beta}, \Sigma^{\mu\nu}] + \frac{1}{4} \Sigma^{\alpha\beta} [\rho^{\alpha}, \Sigma^{\mu\nu}]$$

$$= +\frac{i}{2} \Sigma^{\mu\beta} \omega_{\beta}^{\nu} + \frac{i}{2} \omega_{\alpha}^{\mu} \Sigma^{\alpha\nu} - \frac{i}{4} \Sigma^{\alpha\beta} (\eta^{\alpha\nu} \rho^{\mu} - \eta^{\mu\nu} \rho^{\alpha}) \\ = +\frac{i}{2} (\Sigma^{\mu\beta} \omega_{\beta\alpha} \eta^{\alpha\nu} + \omega_{\alpha\beta} \eta^{\beta\mu} \Sigma^{\alpha\nu}) - \frac{i}{4} \Sigma^{\alpha\beta} (\eta^{\alpha\nu} \rho^{\mu} - \eta^{\mu\nu} \rho^{\alpha})$$

$$[\Sigma^{\alpha\beta}, \Sigma^{\mu\nu}] = i (\eta^{\beta\mu} \Sigma^{\alpha\nu} - \eta^{\alpha\mu} \Sigma^{\beta\nu} + \eta^{\beta\nu} \Sigma^{\alpha\mu} - \eta^{\alpha\nu} \Sigma^{\beta\mu})$$

$$\frac{1}{4} [\rho^{\alpha}, \Sigma^{\mu\nu}] = -\frac{i}{4} (\eta^{\alpha\nu} \rho^{\mu} - \eta^{\mu\nu} \rho^{\alpha})$$

$$[\rho^{\alpha}, \Sigma^{\mu\nu}] = i (\eta^{\alpha\mu} \rho^{\nu} - \eta^{\alpha\nu} \rho^{\mu}) \quad \text{which}$$

$$P^0 = H$$

$$P^i = p^i$$

$$P^i = (H, p^i)$$

$$\Sigma^{\mu\nu} = \begin{pmatrix} 0 & -K^x & -K^y & -K^z \\ K^x & 0 & J_z & -J_y \\ K^y & -J_z & 0 & J_x \\ K^z & J_y & -J_x & 0 \end{pmatrix}$$

$$\Sigma^{0i} = -K_i$$

$$\Sigma^{12} = +J_z$$

$$\Sigma^{23} = J_x$$

$$\Sigma^{31} = J_y$$

$$(P^{\mu})$$

$$[J_x, J_y] = [\Sigma^{23}, \Sigma^{31}] = i \Sigma^{21} \eta^{33} = i \Sigma^{12} = i J_z$$

$$[J_x, K_y] = [\Sigma^{23}, \Sigma^{20}] = -i \eta^{22} \Sigma^{30} = +i K_z$$

$$[K_x, K_y] = [\Sigma^{10}, \Sigma^{20}] = -i \eta^{00} \Sigma^{12} = -i J_z$$

$$[J_x, H] = [\Sigma^{23}, P^0] = 0$$

$$[J_x, P_y] = [\Sigma^{23}, P^2] = -i \eta^{22} P^3 = i P_z$$

$$[K_x, H] = [\Sigma^{10}, P^0] = i \eta^{00} P^1 = i P_x$$

$$[K_x, P_x] = [\Sigma^{10}, P^1] = -i \eta^{11} P^0 = -i H$$

$$[\bar{J}_i, \bar{X}_j] = i \epsilon_{ijk} \bar{J}_k$$

$$[K_i, K_j] = -i \epsilon_{ijk} \bar{J}_k$$

$$[K_i, H] = i P_i \quad [K_i, P_j] = i \delta_{ij} H$$

Gall. Via ...

⑥

$$[\tilde{K}_i, \tilde{K}_j] = \frac{1}{c^2} [K_i, K_j] = \frac{1}{c^2} \epsilon_{ijk} J_k \rightarrow 0$$

$$[\tilde{K}_i, \tilde{H}] = \frac{1}{c} [K_i, H] = i \hbar \tilde{P}_i = i \hbar \tilde{P}_i$$

$$[\tilde{K}_i, \tilde{P}_j] = \frac{1}{c} [K_i, P_j] = \frac{i}{c} \delta_{ij} H = \frac{i}{c} \delta_{ij} \frac{H}{c} = i \omega \delta_{ij}$$

Planck units

①

$$[l] = M L^2 T^{-1}$$

$$[c] = L T^{-1}$$

$$[E] = M^{-1} L^3 T^{-2}$$

$$F = G \frac{M}{r^2}$$

$$[h_B] = M L^2 T^{-2} \Theta^{-1}$$

$$[G l] = L^5 T^{-3}$$

$$\left[\frac{G l}{c^3} \right] = L^2$$

$$l_P = \sqrt{\frac{G l}{c^3}} \sim 10^{-35} \text{ m}$$

$$m_{Pl} = \frac{l}{c l_P} = \frac{l}{c} \sqrt{\frac{c^3}{G l}} = \sqrt{\frac{c l}{G}} \sim 10^{-8} \text{ kg}$$

$$\Theta_{Pl} = \frac{m_{Pl} c^2}{h_B} = \frac{1}{h_B} \sqrt{\frac{l c^5}{G}} \sim 10^{32} \text{ K}$$

$$\frac{M_c}{m_{Pl}} = \frac{2}{m_P^2} m_{Pl}$$

$$M_c = 2 \frac{m_{Pl}^3}{m_P^2} = \frac{2}{m_P^2} \left(\frac{l c}{G} \right)^{3/2}$$

$$\frac{R_S}{l_P} = 2 \frac{M}{m_{Pl}} \Rightarrow R_S = 2 M \sqrt{\frac{G l}{c^3}} \sqrt{\frac{G}{c l}} = 2 M \frac{G}{c^2}$$

$$\frac{T_{BH}}{\Theta_{Pl}} = \frac{1}{8\pi} \frac{m_{Pl}}{M} \Rightarrow T_{BH} = \frac{1}{8\pi} \sqrt{\frac{l c^5}{G h_B^2}} \sqrt{\frac{c l}{G}} \frac{1}{M}$$

$$= \frac{1}{8\pi} \sqrt{\frac{c^6 l^2}{h_B^2 G^2}} \frac{1}{M} = \frac{l c^3}{8\pi M h_B G}$$

$$[E] = M L^2 T^{-2}$$

$$\left[\frac{E}{e}\right] = T^{-1}$$

$$\left[\frac{E}{ec}\right] = L^{-1}$$

$$\cancel{[E]} \left[\frac{mc}{e}\right] = L^{-1}$$

$$[S] = [e] = 1$$

$$[\mathcal{L}] = L^{-D}$$

$$L^{2-D} [\phi]^2 = [S] = 1 \Rightarrow [\phi] = L^{\frac{D-2}{2}}$$

$$[\omega^2] = [\partial^2] = L^{-2}$$

$$L^{-D} [\lambda_R] L^{R \frac{D-2}{2}} = 1 \quad [\lambda_e] = L^{D(1-\frac{R}{2})+R} \\ = M^{D(\frac{R}{2}-1)+R}$$