Level statistics, ergodicity, and localization transition of Lévy matrices

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Beyond the Gaussian Ensemble

Real symmetric $N \times N$ Wigner Random Matrices with elements distributed independently and with a finite variance → Universality class of Gaussian Orthogonal Ensemble (GOE)

- Semicircle Law
- Wigner surmise
- ...

Several important sets of RMs fall out of the classical Gaussian paradigm

What happens if the random entries of the matrix exhibit heavy tails?

Lévy Matrices

Cizeau & Bouchaud '94
Burda & al '07
Metz, Neri & Bollé '10
Ben Arous & Guionnet '07
Bordenave & Guionnet '13

★Exciting mathematical problem in its own right

- new universality classes
- new and unexpected properties

★Extremely relevant problem in physics and in other disciplines

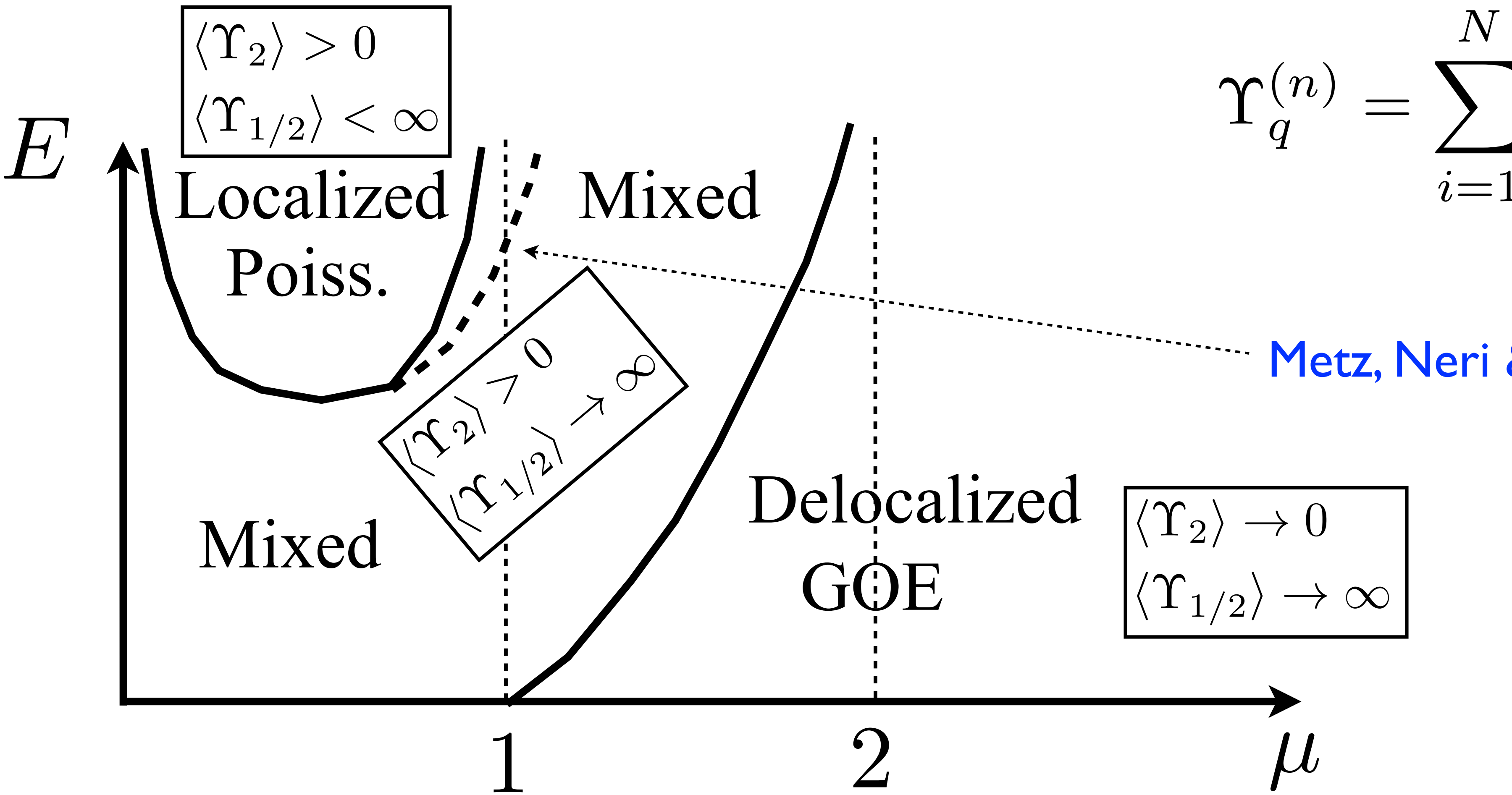
- spin-glasses with dipolar RKKY interactions
- disordered electronic systems
- portfolio optimization
- study of correlations in big data sets
- ...

The theory of Lévy Matrices is not yet well-established
Several issues are still open

The pioneering work of Cizeau & Bouchaud '94

$$w_n(i) = |\langle i|n\rangle|^2$$

$$\Upsilon_q^{(n)} = \sum_{i=1}^{N} [w_n(i)]^q$$



Metz, Neri & Bollé '10

In contradiction with recent rigorous results Bordenave & Guionnet '13

Intermediate extended but non-ergodic (non-GOE) phase ?

Biroli, Ribeiro-Texeira & Tarzia; De Luca & al '14; cfr Giulio's talk

The definition of the model

$N \times N$ real symmetric Wigner random Lévy matrices $\mathcal{H}$

h_{ij} are independent and identically distributed

$$P(h_{ij}) = \theta\left(|h_{ij}| > (N\mu)^{-1/\mu}\right) \frac{1}{2N|h_{ij}|^{1+\mu}}$$

- typical value $N^{-\frac{1}{\mu}}$
- power-law tails with exponent $1 + \mu$

$\mu > 2$ GOE universality class

$\mu < 2$ Entries have an infinite variance

Each row (or column) of $\mathcal{H}$ → $O(N)$ elements of order $N^{-\frac{1}{\mu}}$; $O(1)$ elements of $O(1)$

LM = Sparse Random Matrix + "GOE background"

The resolvent Matrix

$$G_{ij} = \left(\frac{1}{(E - i\eta)\mathcal{I} - \mathcal{H}} \right)_{ij}$$

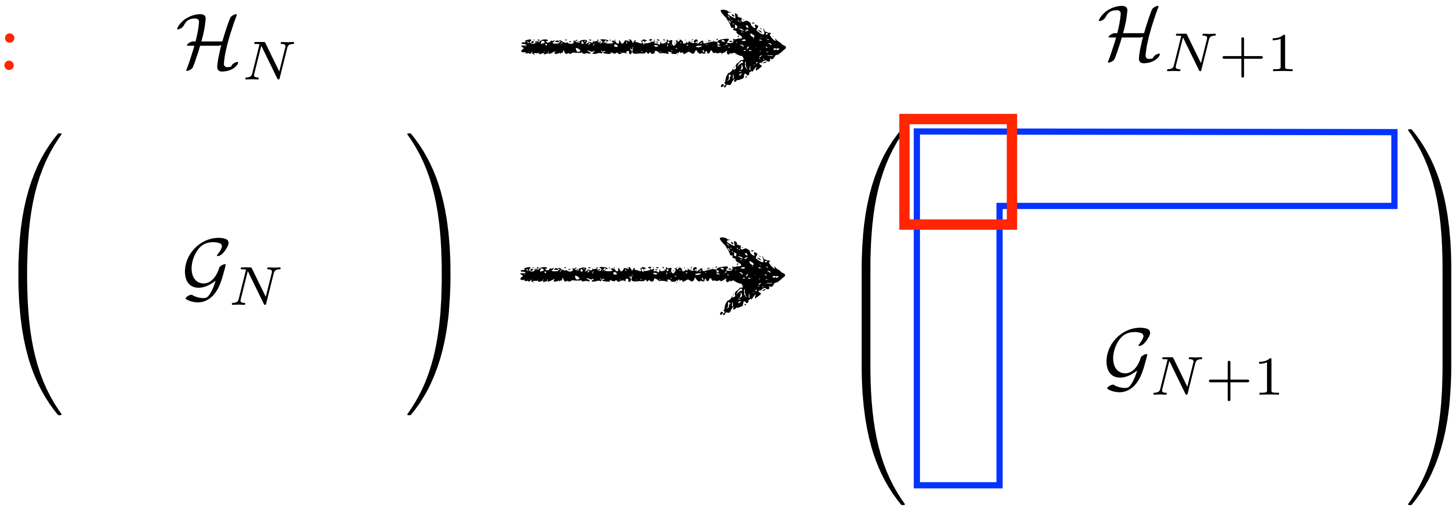
Spectral properties of $\mathcal{H}$ can be obtained through the study of $\mathcal{G}$:

DOS $\longrightarrow$ $$\rho(E) = \lim_{\eta \to 0^+} \frac{1}{N\pi} \sum_{i=1}^{N} \mathrm{Im} G_{ii}(E - i\eta)$$

IPR $\longrightarrow$ $$\langle \Upsilon_2(E) \rangle = \lim_{\eta \to 0^+} \frac{1}{N\pi} \sum_{i=1}^{N} \eta |G_{ii}(E - i\eta)|^2$$

Recursive relations for $\mathcal{G}$

Cavity method:



Exact recursive relations (in the thermodynamic limit):

$$G_{ii}^{-1} = E - i\eta - \sum_{j=1}^{N} h_{ij}^2 G_{jj}$$

Cizeau & Bouchaud '94
Ben Arous & Guionnet '07

Self-consistent equation on the probability distribution $Q(G)$

$$Q(G) = \int \prod_{i=1}^{N} \left[\mathrm{d}G_i Q(G_i) \mathrm{d}h_i P(h_i) \right] \delta\left(G^{-1} - E + i\eta + \sum_{i=0}^{N} h_i^2 G_i \right)$$

Metz, Neri & Bollé '10

Density of States

Ben Arous & Guionnet '07

$Q(G) \longrightarrow$ Complex Lévy Stable distribution

Closed exact equation for $\langle G^{\frac{\mu}{2}} \rangle$:

$$\langle G^{\frac{\mu}{2}} \rangle = -i \, \frac{e^{-i\,\frac{\pi\mu}{2}}}{\Gamma\left(\frac{\mu}{2}\right)} \int_0^\infty \mathrm{d}t \, (-it)^{\frac{\mu}{2}-1} e^{-it(E-i\eta)} \exp\left[-\frac{\Gamma\left(1-\frac{\mu}{2}\right)(-it)^{\frac{\mu}{2}}}{\mu} \langle G^{\frac{\mu}{2}} \rangle\right]$$

Exact equation for the DOS:

$$\rho(E) = \frac{1}{\pi} \lim_{\eta \to 0^+} \mathrm{Im} \left\{ i \int_0^\infty \mathrm{d}t \, e^{-it(E-i\eta)} \exp\left[-\frac{\Gamma\left(1-\frac{\mu}{2}\right)(-it)^{\frac{\mu}{2}}}{\mu} \langle G^{\frac{\mu}{2}} \rangle\right]\right\}$$

The spectrum of Lévy Matrices is a symmetric distribution with support on the whole real axis and power-law tails with exponent $1+\mu$

The mapping to Directed Polymers

Self-energy $\Sigma_i = E - i\eta - G_{ii}^{-1} = S_i + i\Delta_i$

In the limit $\eta \to 0^+, \ \Delta_i \ll S_i$ Abou-Chacra, Anderson & Thouless '73

$$S_i = \sum_{j=1}^{N} \frac{h_{ij}^2}{E - S_j}$$

$$\Delta_i = \sum_{j=1}^{N} \frac{h_{ij}^2}{(E - S_j)^2} \Delta_j$$

Δ_i satisfies the same equations as the partition function of Directed Polymers in Random Media in presence of (correlated) quenched disorder

Derrida & Spohn '07

"Free energy" (one-step RSB) of the Directed Polymers

$$\phi(m, E) = \frac{1}{Rm} \log \overline{\left(\sum_{\mathcal{P}} \prod_{(i_{n+1}, i_n)} \left| \frac{h_{i_n, i_{n+1}}}{E - S_{i_{n+1}}} \right|^{2m} \right)}$$

The mobility edge

The localization transition correspond to the freezing transition of the DP (akin to the glass transition of the Random Energy Model)

$$\begin{cases} \left.\dfrac{\partial \phi(m, E^\star)}{\partial m}\right|_{m=m^\star} = 0 \\ \phi(m^\star, E^\star) = 0 \end{cases}$$

Aizenman & Warzel '11
Abou-Chacra, Anderson & Thouless '73
Bapst '13

Exact equation for the mobility edge $(m^\star = 1/2)$

$$K_{m,\mu}^2 \left(s_\mu^2 - s_m^2\right) |\mathcal{L}(E^\star)|^2 - 2 s_\mu K_{m,\mu} \,\mathrm{Re}\mathcal{L}(E^\star) + 1 = 0$$

$$K_{m,\mu} = \frac{1}{2}\Gamma\left(m - \frac{\mu}{2}\right)\Gamma\left(1 - m - \frac{\mu}{2}\right) \qquad s_\mu = \sin\left(\frac{\pi\mu}{2}\right)$$

$$\mathcal{L}(E) = \int_0^{+\infty} \frac{\mathrm{d}k}{\pi}\, k^{\mu-1}\, \hat{L}_{\mu/2}^{C(E),\beta(E)}(k)\, e^{ikE} \qquad s_m = \sin(\pi m)$$