Fourier transforms

(Common conventions in condensed-matter physics – V = volume, $\beta = 1/T$, $k_B = \hbar = 1$)

I. Classical field theory

- (Real) fields

$$\begin{aligned} \varphi(\mathbf{r}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} \varphi(\mathbf{q}) \\ \varphi(\mathbf{q}) &= \frac{1}{\sqrt{V}} \int d^d r\, e^{-i\mathbf{q}\cdot\mathbf{r}} \varphi(\mathbf{r}) \end{aligned} \tag{1}$$

- Propagators

$$\begin{aligned} G(\mathbf{r}) &= \langle \varphi(\mathbf{r})\varphi(0)\rangle = \frac{1}{V} \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} G(\mathbf{q}) \\ G(\mathbf{q}) &= \langle \varphi(\mathbf{q})\varphi(-\mathbf{q})\rangle = \int d^d r\, e^{-i\mathbf{q}\cdot\mathbf{r}} G(\mathbf{r}) \end{aligned} \tag{2}$$

- Actions

$$\begin{aligned} S_0 &= \frac{1}{2} \int d^d r\, \varphi(\mathbf{r})(r_0 - \boldsymbol{\nabla}^2)\varphi(\mathbf{r}) \\ &= \frac{1}{2} \sum_{\mathbf{q}} \varphi(-\mathbf{q})(r_0 + \mathbf{q}^2)\varphi(\mathbf{q}) \\ &= \frac{1}{2} \sum_{\mathbf{q}} \varphi(-\mathbf{q}) G_0^{-1}(\mathbf{q})\varphi(\mathbf{q}) \end{aligned} \tag{3}$$

$$S_{\text{int}} = \frac{u_0}{4!} \int d^d r\, \varphi(\mathbf{r})^4 = \frac{u_0}{4!V} \sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \varphi(\mathbf{q}_1)\varphi(\mathbf{q}_2)\varphi(\mathbf{q}_3)\varphi(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{4}$$

II. Quantum field theory

- Fields (complex or Grassmannian variables)

$$\begin{aligned}\psi(\mathbf{r},\tau) &= \frac{1}{\sqrt{\beta V}}\sum_{\mathbf{k},\omega_n} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}\psi(\mathbf{k},i\omega_n)\\ \psi^*(\mathbf{r},\tau) &= \frac{1}{\sqrt{\beta V}}\sum_{\mathbf{k},\omega_n} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}\psi^*(\mathbf{k},i\omega_n)\end{aligned} \tag{5}$$

$$\begin{aligned}\psi(\mathbf{k},i\omega_n) &= \frac{1}{\sqrt{\beta V}}\int_0^\beta d\tau\int d^dr\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}\psi(\mathbf{r},\tau)\\ \psi^*(\mathbf{k},i\omega_n) &= \frac{1}{\sqrt{\beta V}}\int_0^\beta d\tau\int d^dr\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}\psi^*(\mathbf{r},\tau)\end{aligned} \tag{6}$$

(Matsubara frequency: $\omega_n = 2n\pi/\beta$ (bosons) or $(2n+1)\pi/\beta$ (fermions))

- Propagators

$$\begin{aligned}G(\mathbf{r},\tau) = -\langle\psi(\mathbf{r},\tau)\psi^*(0,0)\rangle &= \frac{1}{\beta V}\sum_{\mathbf{k},\omega_n} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}G(\mathbf{k},i\omega_n)\\ G(\mathbf{k},i\omega_n) = -\langle\psi(\mathbf{k},i\omega_n)\psi^*(\mathbf{k},i\omega_n)\rangle &= \int_0^\beta d\tau\int d^dr\, e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_n\tau)}G(\mathbf{r},\tau)\end{aligned} \tag{7}$$

- Actions

$$\begin{aligned}S_0 &= \int_0^\beta d\tau\int d^dr\,\psi^*(\mathbf{r},\tau)\left(\partial_\tau-\mu-\frac{\boldsymbol{\nabla}^2}{2m}\right)\psi(\mathbf{r},\tau)\\ &= \sum_{\mathbf{k},\omega_n}\psi^*(\mathbf{k},i\omega_n)\left(-i\omega_n-\mu+\frac{\mathbf{k}^2}{2m}\right)\psi(\mathbf{k},i\omega_n)\\ &= \sum_{\mathbf{k},\omega_n}\psi^*(\mathbf{k},i\omega_n)\left[-G_0^{-1}(\mathbf{k},i\omega_n)\right]\psi(\mathbf{k},i\omega_n)\end{aligned} \tag{8}$$

$$\begin{aligned}S_{\rm int} &= \frac{1}{2}\int_0^\beta d\tau\int d^dr\int d^dr'\,v(\mathbf{r}-\mathbf{r}')\psi^*(\mathbf{r},\tau)\psi^*(\mathbf{r}',\tau)\psi(\mathbf{r}',\tau)\psi(\mathbf{r},\tau)\\ &= \frac{1}{2\beta V}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}\sum_{\omega_n,\omega_{n'},\omega_\nu} v(\mathbf{q})\psi^*(\mathbf{k}+\mathbf{q},i\omega_{n+\nu})\psi^*(\mathbf{k}'-\mathbf{q},i\omega_{n'-\nu})\psi(\mathbf{k}',i\omega_{n'})\psi(\mathbf{k},i\omega_n)\end{aligned} \tag{9}$$

where $v(\mathbf{q}) = \int d^dr\, e^{-i\mathbf{q}\cdot\mathbf{r}}v(\mathbf{r})$.

III. Limits $V\to\infty$ and $T\to 0$

$$\frac{1}{V}\sum_{\mathbf{q}}\to\int\frac{d^dq}{(2\pi)^d}\qquad (V\to\infty),\qquad\qquad \frac{1}{\beta}\sum_{\omega_n}\to\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\qquad (T\to 0) \tag{10}$$