An introduction to symmetries and quantum field theory

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Updated September 28, 2021

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Foreword

These are lecture notes for the course "Symmetries and quantum field theory" given at the master 2 "concepts fondamentaux de la physique, parcours de physique quantique" in Paris. The first version was written by Jean-Noël Fuchs and was later modified by Matthieu Tissier. It is work in progress and comments are most welcome.

There are 12 or 13 sessions consisting of a 1.5h lecture and a 1.5h exercise session (under the supervision of Julien Serreau serreau@apc.in2p3.fr).

Recommended books:

• Textbooks:

The three documents that I most used to prepare these lecture notes are the following.

B. Delamotte, Un soupçon de théorie des groupes: groupe des rotations et groupe de Poincaré [1]

M. Maggiore, A modern introduction to quantum field theory [2]

L.H. Ryder, Quantum field theory [3]

• More advanced:

A. Zee, Quantum field theory in a nutshell [4]

• Less advanced (popular science reading):

A. Zee, Fearful symmetry: the search for beauty in modern physics [5]

Chapter 1

Introduction

V. Weisskopf: "There are no real one-particle systems in nature, not even few-particle systems. The existence of virtual pairs and of pair fluctuations show that the days of fixed particle number are over."

These notes serve as an introduction to relativistic quantum field theory (but they often discuss the connection to condensed matter physics as well). They start by discussing the symmetries of flat spacetime (Lorentz and Poincaré groups). Then classical field theory is reviewed in the framework of scalar fields before quantizing the non-interacting field using canonical quantization. In the last sections we construct the natural geometrical objects of spacetime, such as scalars, vectors and spinors, which are the building blocks of the standard model of particle physics. We then describe the quantization procedure for these fields. The notes end with a short presentation of spontaneous symmetry breaking.

1.1 Requirements

Quantum mechanics including representations of the rotation group and the notion of spin
Special relativity in covariant notation
Analytical mechanics (Lagrangian, action, Euler-Lagrange equations, etc.)
Notions on group theory and representations
Electromagnetism (wave propagation, gauge invariance)

1.2 Why quantum field theory?

1.2.1 Quantum + relativity $\Rightarrow$ QFT

Wedding of quantum physics and relativity imposes the language of quantum field theory (QFT). Indeed, quantum mechanics is formulated to describe a fixed number of particles. But this is only possible non-relativistically. The uncertainty relation $\Delta E \Delta t \gtrsim \hbar$ permits the violation of the energy conservation law (by an amount ΔE) for a short time (Δt) and relativity theory permits the conversion of energy into matter ($E = mc^2$ states that the rest energy is given by the mass). Therefore the number of particles (even massive) can not be fixed: (virtual) particles may appear or disappear out of nothing (vacuum). We therefore need a theory that permits to create or destroy particles. This is QFT. The change of perspective is that we now describe a field (such as the electromagnetic field or the electron field) rather than particles and that particles emerge as excitations of this field upon quantization.

Let us try to describe quantum mechanically a single relativistic massive particle (i.e. an electron, for example) and show that we run into a difficulty. Because $\Delta x \Delta p \gtrsim \hbar$ and $\Delta p = \Delta E/v$ with $v = \partial E/\partial p$ and $E = \sqrt{p^2c^2 + m^2c^4}$, then $\Delta E \gtrsim \hbar v/\Delta x$. In addition, remaining a single particle on the positive energy branch of the relativistic dispersion relation $E = \pm\sqrt{p^2c^2 + m^2c^4}$ means that the energy uncertainty ΔE has to be smaller than the mass gap $2mc^2$. Therefore $\Delta x \gtrsim \hbar v/(2mc^2) \sim \hbar/(mc)$, which means that there is a minimum localization length for a single particle, which is of the order of the Compton wavelength $\hbar/(mc)$ [1]. If we try to localize a particle better than its Compton wavelength, then its energy becomes uncertain by an amount larger than the mass gap so that other particles start being involved (those occupying the negative energy branch in the Dirac sea picture of the vacuum having a negative energy branch $E = -\sqrt{p^2c^2 + m^2c^4}$ filled with electrons). And hence, it is impossible to describe a single relativistic particle. Note that quantum + relativity also imposes the notion of antiparticles (pictured as a missing electron in the Dirac sea).

The most well-known example of a QFT is that of the electromagnetic field, the quanta of which are the photons. Photons can be created and destroyed. There is no consistent description of a single photon. The blackbody radiation is described as an ideal Bose-Einstein gas of massless particles at thermal equilibrium with a zero chemical potential $\mu = 0$: when $T \to 0$, there are less and less photons on average. Photons are not conserved, their number is not fixed.

In the present context, QFT is known as relativistic quantum field theory.

Another strong argument which indicates that standard quantum mechanics (based on Shrödinger equation) is not suited for relativistic is the following. As you well know, in standard quantum mechanics, the position of a particle is described by an operator (if you measure the position of a particle, you have a probability to find in between x and $x + dx$ with probability $|\psi(x)|^2 dx$, etc...). Time has a very different status. Time in ordinary quantum mechanics is an external parameter (not an operator!). This clashes with the basics of special relativity, which states that space and time coordinates should be treated on an equal footing. There are two ways out. Either you promote time to an operator (but this proved to be not very fruitful) or you downgrade position coordinates to be parameters (as time). In the second case, what can you quantize? The answer to this apparent paradox is to use fields. The spacetime coordinates are indeed parameters and what is quantized is the field itself.

1.2.2 Many-body problem in condensed matter physics

Another familiar example of a QFT can be found in non-relativistic condensed matter physics. Phonons appear as quanta of the displacement field of a crystal. They can also be destroyed and created. In the context of condensed matter, QFT is not required but it is frequently used as it is very convenient when dealing with quantum mechanics of a very large number of degrees of freedom. This is known as the many-body problem. QFT in the many-body problem is also known as "second quantization" description (meaning occupation number representation, Fock space, annihilation and creation operators, etc.) in contrast to using standard quantum mechanics known as "first quantization" description (meaning numbered particles, many-body wavefunction $\Psi(1, ..., N; t)$ in configuration space, symmetrization or anti-symmetrization, etc). It means describing the population of modes instead of describing the behavior of numbered particles. Actually, it is not restricted to describing non-conserved particles (such as phonons, magnons, etc.) but can be adapted to describe conserved particles as well (such as electrons in a metal or atoms in a trapped ultracold gas, which are conserved in a non-relativistic theory).

"Second quantization" is a historical name and a quite confusing one (a better name would be non-relativistic QFT or condensed matter field theory). It comes from a misinterpretation: classical physics of a single particle (Newton's equation or Lagrangian dynamics or Hamiltonian dynamics) would first be quantized into the quantum mechanical description of a single particle (Schrödinger's equation for a wavefunction $\varphi(\vec{x}, t)$), and then the wavefunction would itself be quantized again (i.e. second quantized) $\varphi(\vec{x}, t) \to \hat{\varphi}(\vec{x}, t)$ to become a quantum field. Do you see what is wrong in the preceding argument? Think about this point, it is important to grasp. Actually, there is only a single quantization procedure but accompanied by a change of perspective from a single particle to a field describing all identical particles. What is wrong is that the field that becomes quantized in a QFT is not a wavefunction (it is not a quantum state describing a single particle) but rather a field whose excitations are the particles. Once this field is quantized, it becomes an operator creating or destroying particles at position $\vec{x}$ and time t. A better notation for the quantized field

[1] For an electron, the Compton wavelength is of the order of 500 fm, i.e. 100 times smaller than the Bohr radius, the typical size of atoms and 500 times larger than the size of a nucleus.

number of degrees of freedom	classical	quantum
finite and discrete	classical mechanics	quantum mechanics
infinite and continuous	classical field theory	quantum field theory

Table 1.1: Field theory versus mechanics.

$\hat{\varphi}(\vec{x}, t)$ would be $\hat{a}(\vec{x}, t)$ to clearly identify it as a ladder operator (or annihilation operator as for the harmonic oscillator). The conjugate field $\hat{a}^\dagger(\vec{x}, t)$ is also a ladder operator (known as a creation operator). In QFT, as we will see, the Schrödinger equation for a single electron is reinterpreted as being a *classical* field equation describing the field of all electrons, i.e. the electronic field[2]. Just as Maxwell's equation is a classical wave equation describing the electromagnetic field, i.e. the photonic field.

In these lectures, we will mostly focus on relativistic QFT. For more on condensed matter field theory (i.e. the use of QFT techniques in the context of non-relativistic condensed matter physics), see other M2 courses such as "Quantum mechanics: second quantization and scattering theory" (first semester) and "Condensed matter theory" (first semester). We also strongly recommend the book by A. Altland and B. Simons, *Condensed matter field theory* [6].

In short, QFT is a convenient/efficient framework to describe a system with many particles that are not necessarily conserved (they can be created or annihilated).

1.2.3 What is a field?

A field is an object ϕ defined at each point of space-time

$$\phi(\vec{x}, t) \tag{1.1}$$

where $\vec{x}$ denotes space, t time and ϕ can be a scalar, a vector, a tensor, a spinor, a matrix or even something else. More generally, a field is a map from a base manifold[3] M (usually space-time) to a target manifold T (which depends on the nature of the field: scalar, vector, etc.):

$$\begin{aligned} \phi : M &\rightarrow T \\ x &\rightarrow \phi(x) \end{aligned} \tag{1.2}$$

A field describes an infinite number of degrees of freedom (DoF). For example, if the field is a real scalar ($\phi \in \mathbb{R}$) and if space-time is continuous $1+1$ dimensional, at each point of this 2d manifold, there is a single degree of freedom. QFT should be clearly contrasted with quantum mechanics, which is restricted to a finite countable number of degrees of freedom, see Table 1.1.

Fields are also needed to avoid action at a distance, i.e. to mediate forces at finite velocity and respect locality. For example, the electric interaction between charges is not instantaneous but propagates at the speed of light. Although, it is often approximated as an instantaneous (non-retarded) Coulomb potential.

1.3 Symmetries as a leitmotiv and guiding principle

Symmetries are all important in physics (P.W. Anderson even wrote that "it is only slightly overstating the case to say that physics is the study of symmetry" in "More is different", Science 1972). They constrain the form of theories (symmetry dictates design). We will use space-time symmetries to construct relativistic field theories. Symmetries also imply conservation laws (for example, the conservation of energy is a consequence of invariance under time translation). Even more so in quantum then in classical physics. In addition to space-time symmetries, there are also less obvious internal symmetries. For example: gauge symmetry

[2] A tricky question at the heart of the confusion about "second quantization": how come $\hbar$ already appears in Schrödinger's equation if it is a classical wave equation?

[3] A manifold is a topological space that resembles Euclidean space near each point.

of electromagnetism. Symmetries can be continuous (as rotations) or discrete (as space inversion or time reversal). The mathematical tools needed to describe symmetries are groups and their representations.

But what is a symmetry? To answer that question precisely, we first need to define a few important notions.

First, there are transformations, which should be carefully distinguished from symmetries. This is a change in our description of a system. We adopt the passive viewpoint: the system is left unchanged, only the description (a frame, for example) is transformed (the active viewpoint would consist in having a single frame and in applying a transformation to the system). An example is the rotation of a reference frame used to describe a bicycle. A transformation need not be a symmetry. The bicycle is not invariant under arbitrary rotation. Still, it is not forbidden to apply a rotation (transformation) to a bicycle.

Second, there is the notion of invariance under a transformation. Something (an object, the state of a system, etc.) is said to be symmetrical under a transformation (or to admit a transformation as a symmetry) if it is left unchanged by the transformation. For example, a cube is invariant under certain rotations (but not all).

Third, one needs to distinguish the symmetry of an object (or of the state of a system) from the symmetry of a law of physics. In the following, we will be more interested in the symmetry of physical laws than in the symmetry of an object (except in the last chapter on spontaneous symmetry breaking). If a law of physics (or a description of a system) is left unchanged by a transformation, then we say that the law exhibits a symmetry or that the system possesses a symmetry. We will often describe a system by an action S: then, the system exhibits a symmetry if the action is left invariant by a transformation. For example, 3d space is thought to be rotationally invariant. In order to respect that invariance, the fundamental laws of physics (e.g. Newton's second law) have to be written in a covariant manner (i.e. as an equality between objects that transform the same under space symmetries, e.g. vectors $\vec{F} = m\vec{a}$). But this does not preclude the existence of objects (e.g. a cube, a bicycle, etc.) that do not have the full rotational invariance of space. Not all objects are spheres.

Fourth, when a symmetry is present for a physical law, it does not mean that every state of the system will feature that symmetry[4]. Indeed a symmetry can be spontaneously broken: the system may possess the symmetry but this is not necessarily reflected in its state. For example, translational symmetry of space may be apparent in the state of an atomic ensemble (when it is in its gaseous or its liquid phase) or may be spontaneously broken (when it is in its cristalline solid phase). The state of the system may be less symmetric than the laws of physics.

Eventually, a symmetry can also be explicitly broken. For example, full rotational symmetry of space is explicitly broken at the surface of earth by the presence of a gravitation field indicating a preferred direction (vertical direction).

1.4 Natural units, dimensional analysis and orders of magnitude in high-energy physics

We will use natural units such that $\hbar = 1$ and $c = 1$. It is a good exercise to put units back in final expressions. With these units, energy = mass = 1/length = 1/time, which is usually expressed in GeV (1 GeV = 10^9 eV = 1.6×10^{-10} J = 1.8×10^{-27} kg, which is the order of magnitude of the proton or neutron mass). The corresponding length scale is ~ 0.1 fm = 10^{-16} m (femtometer or fermi). The typical size of a nucleus being ~ 1 fm.

When performing dimensional analysis of physical quantities, we will always be interested in knowing their "mass dimension". For example, energy has mass dimension 1, time has mass dimension -1, power has mass dimension 2, action has mass dimension 0, the Lagrangian density has mass dimension $D+1$ (in $D+1$ spacetime), a scalar field has mass dimension $\frac{D-1}{2}$, etc.

At this point it would be good to make connection to the natural playground for relativistic QFT, which is nuclear and subnuclear physics (also called high-energy physics). Unfortunately, we won't have time to

[4]Again, it is important to distinguish the symmetry of a law and that of an object (or of the state of a system).

do that. We therefore only provide what we think is the minimum piece of information. For more details see the introduction chapter in Maggiore [2] or Ryder [3]. For a pleasant evening reading see the popular science account by A. Zee [5].

The elementary particles consist mainly of two categories: matter particles or constituents of matter (fermions) and interaction mediating particles or messengers of interaction (mediating bosons).

The first category splits in leptons (electron, neutrino, etc.: spin 1/2) that do not feel the strong interaction[5] and hadrons (proton, neutron, pion, etc.) that do feel the strong interaction. Hadrons further separate into baryons (proton, neutron, etc.) that are fermions and mesons (pion, etc.) that are bosons[6]. The neutrino is almost massless (less than ~ 1 eV). The electron mass is roughly 0.5 MeV and that of nucleons (proton or neutron) 1 GeV. Entering the world of nuclear physics happens at the mass scale of the pion 0.1 GeV which corresponds to a distance of 1 fm (the typical size of a nucleus).

The four fundamental interactions (electromagnetism, gravity, strong and weak) and their main properties are summarized in Table 1.2. The particles carrying interactions are all bosons. The photon (massless, spin 1, gauge boson) carries the electromagnetic interaction, the intermediate vector bosons W and Z ($80-90$ GeV, spin 1, gauge bosons) carry the weak interaction and the mesons (e.g. the pions, 0.1 GeV) carry the strong interaction (see also a preceding footnote and the caption of Table 1.2). Gravity is supposed to be mediated by a massless spin 2 boson called the graviton, although it has never been observed. Classical gravitational waves have recently been detected by LIGO-Virgo (september 2015). The huge energy scale at which gravity becomes quantum is expected to be the so called Planck mass $\sqrt{\hbar c/G} \sim 10^{19}$ GeV (where G is Newton's constant of gravity) corresponding to a distance of 10^{-20} fm. [7]

The weak interaction is quite peculiar. It is by far the one with the shortest range: a thousand time smaller than the size of a nucleus. Apart from gravity, it is the weakest interaction. Also it violates parity and time-reversal symmetries. And the neutrino only interacts trough the weak interaction (again apart from its small mass, which means a gravitational interaction).

Eventually, there is a third type of particles – on top of fermions and mediating bosons – with a single known member: the Higgs (scalar, spin 0) boson. Its peculiarity comes from its being a scalar (rather than a vector) boson, not mediating an interaction and therefore being closer to being a matter particle. Its mass (125 GeV) is of the same order as that of the intermediate vector bosons (W and Z). It was discovered experimentally in 2012. We will encounter this particle at the end of the course, when discussing spontaneous symmetry breaking. We should also mention that the highest energy currently achieved in particle accelerators (e.g. the large hadron collider LHC at CERN) is of the order of 1 TeV = 1000 GeV (per nucleon) corresponding to a distance of $\sim 10^{-4}$ fm, i.e. 0.01% of the nucleus size. The current status is the following: the standard model seems to explain almost every observed phenomena in high-energy physics (with an extension to include finite neutrinos' masses) and, at the moment, there are no signs of things such as super-symmetry or extra particles.

[5] In addition, the neutrino does not carry an electric charge and is almost massless. Therefore, it essentially only couples to the weak interaction and is very hard to detect.

[6] A potentially confusing fact in the above presentation is that mesons (which are composite bosons) appear both as matter particles (and as such should be fermions) but also as the carriers of the strong interaction at low energy. At a more fundamental level, the correct theory of strong interaction is quantum chromodynamics. The elementary fermions in this theory are called quarks (rather than baryons) and interact via the exchange of massless spin 1 gauge bosons called gluons (rather than mesons). It is only at a phenomenological level that hadrons are formed due to quark and gluon confinement – the strong interaction being so strong at low energy that free quarks or free gluons have never been seen – leaving baryons (bound states of three quarks) as "matter fermions" and mesons (quark anti-quark bound states) as "mediating bosons". At low energy, the strong interaction proceeds via the exchange of mesons (e.g. pions) between baryons (e.g. nucleons) as proposed by Yukawa. Baryons are composite fermions made of three fermions, whereas mesons are composite bosons made of two fermions. This resolve the apparent contradiction of mesons being constituents of matter and bosons at the same time.

[7] Later, we will see that fundamental interactions are described by gauge theories and that carriers of interactions are gauge bosons. Electromagnetism or quantum electrodynamics (QED) will be seen as a $U(1)_Q$ gauge theory (Q is the electric charge). It is an abelian gauge theory. Electroweak interaction or quantum flavor dynamics (QFD) will be seen as an $SU(2)_L \times U(1)_Y$ gauge theory (weak isospin and Y is weak hypercharge). It is a non-abelian or Yang-Mills gauge theory. Upon spontaneous symmetry breaking (Higgs mechanism) $SU(2)_L \times U(1)_Y \to U(1)_Q$. Strong interaction or quantum chromodynamics (QCD) will be seen as a $SU(3)$ gauge theory (the corresponding charge is called color charge). The standard model is built from the gauge group $SU(3) \times SU(2)_L \times U(1)_Y$. And gravitation can also be seen as some kind of gauge theory where spacetime translations are gauged into diffeomorphisms? There is also a covariant derivative, and a Levi-Civita connection, etc.

interaction	realm	strength (at low energy)	range	mass of carriers
em	atoms, chemical binding	weak ($\frac{e^2}{\hbar c} = \frac{1}{137} \sim 10^{-2}$)	long (∞)	0 (photon)
gravity	planets, galaxies, cosmos	weakest (10^{-40} @ 0.1 GeV)	long (∞)	0 (graviton?)
strong	nuclei, nuclear binding	strong (~ 1 @ 0.1 GeV)	short (1 fm)	0.1 GeV (pion)
weak	radioactive β decay, nucleosynthesis	weaker (10^{-7} @ 0.1 GeV)	shortest (10^{-3} fm)	100 GeV (W, Z)

Table 1.2: The four fundamental interactions. Interaction range r and mass of the carrier boson m are related by $r \sim \hbar/mc \sim 1/m$ in natural units (1 fm $\leftrightarrow$ 0.2 GeV). At a more fundamental level, strong interactions are carried by spin 1 massless gauge bosons called gluons. The latter interact very strongly and are confined such that the resulting effective interaction is carried by massive composite particles (called mesons: pions being one example, e.g.) and the effective range of nuclear forces is finite despite the mass of gluons being zero.

1.5 Logic of the course: the menu

Chapter 1: quantum + relativity $\Rightarrow$ QFT as a necessity (also quantum + many bodies $\Rightarrow$ QFT as a convenience)
Chapter 2: space-time symmetries constrain physical theories $\Rightarrow$ relativistic (classical) field theory
Chapter 3-4: scalar field theoryand its canonical quantization $\Rightarrow$ QFT
Chapter 5: symmetries and representation
Chapter 6-7: More general field theories, Dirac Field, Electromagnetism.

Chapter 2

Spacetime and its symmetries

It is expected (and tested to a great accuracy) that the laws which govern our universe are invariant under translations (in space and in time) as well as rotations and what we now call Lorentz boosts. These boosts correspond to a change of coordinates corresponding to two observers moving one from another with a constant velocity. For a long time it was also thought that the laws of physics were invariant under parity (mirror image) and time inversion but experiments showed that the weak interaction actually break these symmetries.

Your first introduction to special relativity probably followed the historical development of the field. Michelson-Morley experiment showed, to the surprise of everyone at that time, that the speed of light was the same in all reference frames, in contradiction with the classical law of compositions of velocities. From this experimental fact, you probably deduced the Lorentz transformation, which relate the coordinates of an event in two reference frames which move at a constant speed one from another. (If one wants to understand something to relativity, we recommend reading Epstein [10]. For a brief recap, we recommend Feynman [11]. And for a more formal presentation with the covariant notation, we suggest reading Boratav and Kerner [12].) Here, we will use a faster track, which makes the geometric structure of spacetime more explicit.

2.1 Spacetime and interval

We consider the 3+1 dimensional Minkowskian space-time. A point in spacetime is called an event. Once we have chosen a reference frame, we can characterize an event by a time coordinate and three space coordinates. [A typical event in Jussieu campus would be: Friday afternoon (1 time coordinate) tower 23 (these are actually 2 space coordinates), 5th floor (third space cordinate)]. We often merge these 4 coordinates in what we call a 4-vector, noted x^μ, with x^0 the time coordinate and x^i with $i = 1, 2, 3$ the three space coordinates. In this course (and in virtually all the literature), greek indices like μ run from 0 to 3 while latin ones (like i) run from 1 to 3.

Now consider two points which are close by and call dx^μ the difference between the coordinates of the two events. The interval between these two events is defined to be

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2). \tag{2.1}$$

Note that ds^2 is not positive in general. If $ds^2 > 0$, the interval is said to be time-like and $ds = \sqrt{ds^2}$ is called the proper time interval. If $ds^2 < 0$, the interval is said to be space-like and $\sqrt{|ds^2|}$ is called the proper distance. If $ds^2 = 0$ the interval is said to be light-like (or null). In our units where the speed of light equals 1, the interval between two events on the line of universe of a ray of light is equal to 0 (hence the name).

The set of events with a vanishing interval with respect to a certain event defines the light-cone of this event. In particular, if we consider two event of the history of a massive particle (whose speed is smaller than 1), the interval is positive. Inside the light cone of an event are other events that can be causally related to it (either before in the "past" or after in the "future"). Outside the light cone are events that may be thought as happening "now" and that are not causally related to the given event.

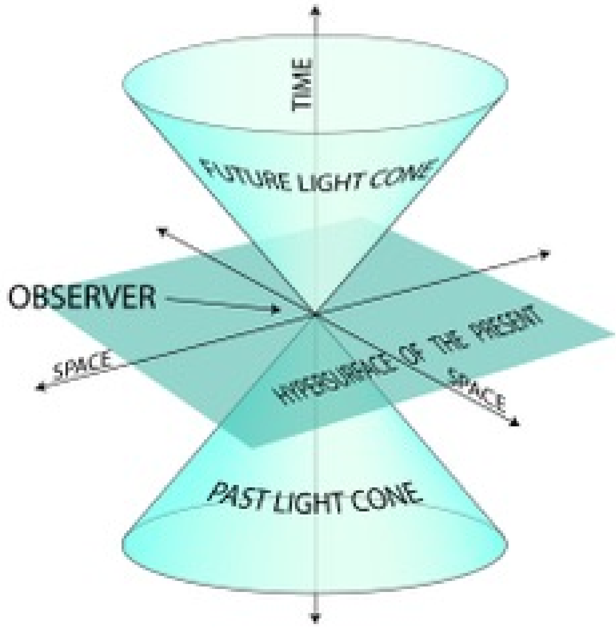

Figure 2.1: Light cone of a given event (or observer). Taken from https://en.wikipedia.org/wiki/Light_cone.

The definition of the interval ds^2 resembles closely the distance in usual euclidean space, except for the minus sign. It is convenient to rewrite it in terms of a metric tensor:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

where the Einstein convention is used (repeated indices are to be summed over) and where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathrm{diag}(1, -1, -1, -1) \tag{2.2}$$

2.2 Symmetries of spacetime

In our fast (but admittedly obscure) way of introducing special relativity, we will postulate that physics is invariant under all changes of coordinates that leave the interval unchanged. All such transformations form the Poincaré group.

Let us characterize the possible changes of coordinates in more detail. The constraint reads: $\eta_{\mu\nu} dx'^\mu dx'^\nu = \eta_{\alpha\beta} dx^\alpha dx^\beta$. Using the fact that $dx'^\mu = (\partial x'^\mu / \partial x^\sigma) dx^\sigma$, we obtain $\eta_{\mu\nu}(\partial x'^\mu/\partial x^\sigma)(\partial x'^\nu/\partial x^\rho) = \eta_{\sigma\rho}$. First, taking the determinant we see that the Jacobian $|\det \frac{\partial x'}{\partial x}| = \pm 1$ so that the matrix $\frac{\partial x'}{\partial x}$ is invertible. Second, taking a further derivative, we have $\partial_\alpha(\eta_{\mu\nu} \partial_\sigma x'^\mu \partial_\rho x'^\nu) = 0$ so that $A_{\alpha\sigma\rho} + A_{\alpha\rho\sigma} = 0$ where we defined $A_{\alpha\sigma\rho} \equiv \eta_{\mu\nu} \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\rho}$. Note that A is symmetric under permutation of the first two indices. Permuting indices we find the three relations: $A_{\alpha\sigma\rho} + A_{\alpha\rho\sigma} = 0$, $A_{\alpha\sigma\rho} + A_{\rho\sigma\alpha} = 0$ and $A_{\rho\sigma\alpha} + A_{\alpha\rho\sigma} = 0$. Now, summing the first two identities and subtracting the last one, we conclude that $A_{\alpha\sigma\rho} = 0$. But since $\frac{\partial x'}{\partial x}$ is invertible, this implies that $\frac{\partial x'^\mu}{\partial x^\sigma \partial x^\alpha} = 0$ so that x' is a linear function of x. Therefore, we introduce the 4×4 matrix $\Lambda^\mu{}_\nu$ and the vector a^μ such that

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,. \tag{2.3}$$

By convention $\Lambda^\mu{}_\nu$ is represented by a matrix for which μ is the row index and ν the column index (note that this convention for Λ differs from the one we have taken for the metric η for which the associated matrix is defined by $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and not by $\eta^\mu{}_\nu$).

The constant shift a^μ corresponds to a spacetime translation, whereas Λ corresponds to a "spacetime rotation" (i.e. either to a space rotation, to a change of inertial frame called a Lorentz boost or simply a boost or to a combination of these). (Λ, a) is an element of the Poincaré group (translations + rotations + boosts), whereas Λ is an element of the Lorentz group (rotations + boosts)[1].

For the moment, we concentrate on the Lorentz group, an element Λ of which acts as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. From the invariance of the interval, we deduce that

$$\eta_{\alpha\beta} = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta, \tag{2.4}$$

Therefore

$$\boxed{\eta = \Lambda^T \eta \Lambda} \tag{2.5}$$

as a matrix identity where η represents the matrix $\eta_{\mu\nu}$, while Λ represents the matrix $\Lambda^\mu{}_\nu$. This relation defines (homogeneous) Lorentz transformations.

Equation (2.5) should be seen as a Minkowskian equivalent of the Euclidian relation $\mathbb{I} = R^T \mathbb{I} R$ showing that rotations are described by orthogonal matrices. Lorentz transformations form a group (show it) called $\mathcal{L}$. This group is actually made of four disconnected pieces due to the presence of discrete transformations. This can be understood by observing that $det\Lambda = \pm 1$ (show this). The Lorentz group is therefore divided in a proper ($det\Lambda = +1$, noted $\mathcal{L}_+$) and an improper ($det\Lambda = -1$, noted $\mathcal{L}_-$) part. Similarly, we can show that $(\Lambda^0{}_0)^2 \geq 1$ so that the Lorentz group is divided in a orthochronous part ($\Lambda^0{}_0 \geq 1$, noted $\mathcal{L}^\uparrow$) and a anti-orthochronous part ($\Lambda^0{}_0 \leq -1$, noted $\mathcal{L}^\uparrow$). You can show that $\mathcal{L}^\uparrow$, $\mathcal{L}_+$ and $\mathcal{L}_+^\uparrow$ are subgroups.

Let us give simple examples of Lorentz transformations to exemplify these discrete operations. Space inversion is $P = \text{diag}(1, -1, -1, -1)$ is improper, orthochronous. Time reversal is $T = \text{diag}(-1, 1, 1, 1)$ is improper anti-ortochronous. Their product is $PT = -\mathbb{I}$ is proper, antiorthochronous. These three discrete transformations P, T and PT are not continuously connected to the identity. P and T change the orientation of spacetime (just like in 3d Euclidian space, space inversion changes the orientation of space). [2]. In the following, we concentrate on $\mathcal{L}_+^\uparrow$, which we will loosely call the Lorentz group.

Other transformations of interest are rotations (which leave the time coordinate unchanged) and Lorentz boosts. Let us give one particular example: you can check that the following choice fulfills the constraint:

$$\begin{cases} \Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\phi \\ \Lambda^2{}_2 = \Lambda^3{}_3 = 1 \\ \Lambda^1{}_0 = \Lambda^0{}_1 = -\sinh\phi \\ \text{all other entries vanish} \end{cases} \tag{2.6}$$

This change of variables corresponds to an observer $\mathcal{O}'$ using coordinates x', which travels at a constant velocity $\tanh\phi$ with respect to the observer $\mathcal{O}$ using coordinates x.

[1] The Poincaré group is also called the inhomogeneous Lorentz group and the Lorentz group is also known as the homogeneous Lorentz group.

[2] The group $\mathcal{L}$ is the semi-direct product of $\mathcal{L}_+^\uparrow andthediscretegroup$ {I,P,T,PT}, $whichisknownastheKleinfour-group$ K$_4$. As an exercise, write the multiplication table of the group K_4 and compare it to that of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{1, -1; \times\}$ or equivalently $\{0, 1; +\}$.

2.2.1 Covariant and contravariant coordinates

We have found the transformation rules of coordinates under the elements of the Lorentz group. We may encounter other sets of 4 quantities which transform as the coordinates. These will be called (by definition) the contravariant components of a 4-vector. Otherwise stated, a 4-vector is by definition an object which transforms as the coordinates, that is:

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \tag{2.7}$$

But beware, we may also encounter object that do not transform as the coordinates!

For instance, assume that A^μ are contravariant coordinates of the 4-vector A and consider the four quantities defined as

$$A_\mu = \eta_{\mu\nu} A^\nu$$

It is a simple exercise to show that these objects do not transform as contravariant coordinates (show it). In fact, we call A_μ the covariant coordinates of the 4-vector A. The notations are such that upper indices indicate contravariant coordinates and lower indices indicate covariant ones. Some tensors have covariant and contravariant indices (such as the Λ matrices).

The metric tensor enables us to transform the contravariant coordinates to covariant ones (in colloquial terms, we say that we lowered the index). The opposite operation (raise the index) is done by using the tensor $\eta^{\mu\nu}$, defined to be the inverse of $\eta_{\mu\nu}$: $\eta_{\mu\nu}\eta^{\nu\rho} = \delta^{\rho}_{\mu}$. This definition ensures that $A^\mu = \eta^{\mu\nu} A_\nu$, as you can readily prove. In actual calculations, it is very easy to transform covariant coordinates to contravariant ones. We just need to change the sign of the space-components of the 4-vectors. For instance, if $x^\mu = \{t, x, y, z\}$, then $x_\mu = \{t, -x, -y, -z\}$.

But how do covariant coordinates transform? A very simple exercise show that $A'_\alpha = \Lambda_\alpha{}^\mu A_\mu$ where we have used our convention for lowering and raising indices (Please pay attention to the horizontal position of the indices!). It is now convenient to rewrite Eq. (2.4) with our convention of raising and lowering indices, as: $\Lambda^\alpha{}_\mu \Lambda_\alpha{}^\nu = \delta^\nu_\mu$. This enables us to rewrite the transformation rule for covariant coordinates as: $\Lambda^\mu{}_\nu A'_\mu = A_\nu$. This closely resembles the transformation rules of contravariant coordinates, see Eq (2.7), except that the Λ matrix changes side in the equation!

Let us consider another set of 4 objects of great importance in what follows, the operators $\frac{\partial}{\partial x^\mu}$. Does it transform as co- or contra-variant components? To answer this, we use the Leibnitz rule of derivatives:

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} \tag{2.8}$$

$$= \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu} \tag{2.9}$$

We recognise the transformation rules of covariant coordinates. For this reason, we note the derivatives with a lower index:

$$\frac{\partial}{\partial x^\mu} = \partial_\mu$$

A word on three words: invariant, covariant, contravariant. Invariant means unaffected by a transformation (e.g. a scalar is invariant under an isometry). Covariant means that something transforms in the same way as something else. For example a covariant equation is such that both sides of the equality are transformed but that the equality remains true after transformation[3]. Covariant and contravariants coordinates are coordinates that transform in a precise way, as described above.

2.3 Scalar, vectors tensors

Suppose that we have two 4-vectors A and B. We can now form new objects out of these. We can look at $A_\mu B^\mu = \eta_{\mu\nu} A^\mu B^\nu$. How does this quantity transform under Lorentz? You can convince yourself that it

[3] For example, an equality between two vectors $\vec{a} = \vec{b}$ is covariant, whereas an equality between a triplet of scalars (a, b, c) and a vector $\vec{d} = (d_x, d_y, d_z)$ is not and is generally wrong, even if it may be true in a particular frame. It is seen to be wrong in a rotation as (a, b, c) is invariant but not $(d_x, d_y, d_z) \to (d'_x, d'_y, d'_z) = R(d_x, d_y, d_z)$, where R is a rotation matrix.

actually does not transform! Such quantities, which take the same value in all reference frames are called scalars. We can also build $C^{\mu\nu} = A^\mu B^\nu$. The transformation rule of this set of 16 numbers transform as: $T'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$. Ths characterizes the transformation of a rank-2 tensor with two covariant indices. More generally, we sometimes need to consider tensors with p contravariant and q covariant indices: $D^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q}$ which transforms as:

$$D'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Lambda^{\mu_1}_{\alpha_1}\cdots\Lambda^{\mu_p}_{\alpha_p}\Lambda_{\nu_1}^{\beta_1}\cdots\Lambda_{\nu_q}^{\beta_q} D^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q}$$

There is a last object to be introduced, for completeness. Indeed, very much as the metric tesor η is the same in all reference frame, there exists a rank-four tensor which is invariant. It is totally antisymmetric rank 4 tensor $\epsilon^{\alpha\beta\gamma\delta}$ such that $\epsilon^{0123} = +1$ (also known as the Levi-Civita symbol). As an exercise, show that it is invariant under Lorentz transformations.[4] There is an important difference between the Levi-Civita symbol with 3 and 4 indices. Indeed, with three indices, $\epsilon^{ijk} = +1$ if and only if (ijk) is an even permutation of (123) (i.e. if it is obtained from an even number of transposition of (123)). This is the same as saying that (ijk) is a circular permutation of (123). With four indices, $\epsilon^{\mu\nu\rho\sigma} = +1$ iff $(\mu\nu\rho\sigma)$ is an even permutation of (0123). But this is not the same as saying that $(\mu\nu\rho\sigma)$ is a circular permutation of (0123). As a counter-example, show that $\epsilon^{3012} = -1$: even if (3012) is a circular permutation of (0123), it is an odd permutation of (0123). Pay also attention to the fact that $\epsilon_{\mu\nu\rho\sigma} \equiv \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}\eta_{\sigma\delta}\epsilon^{\alpha\beta\gamma\delta}$ and the product of four η's gives -1. In other words $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$.

[4] This is actually true for the proper part of the Lorentz group $\mathcal{L}_+^\uparrow$. Improper transformation lead to a change of sign. This means that ϵ is a pseudo-tensor.

Chapter 3

Classical fields, symmetries and conservation laws

A. Zee: "Einstein's legacy: Symmetry dictates design."

The idea now is to use our knowledge of spacetime symmetries to build relativistic classical field theories. We will see that a field theory is conveniently specified by giving its action $S = \int d^4x\mathcal{L}$ in terms of its Lagrangian density $\mathcal{L}$. To require that a field theory has a certain symmetry (for example Poincaré symmetry) amounts to designing an action that is invariant under Poincaré transformations. In others words, the action should be a Poincaré-scalar. The Lagrangian density should therefore be a Lorentz-scalar and should not explictly depend on the spacetime coordinate $\underline{x}$ (in order to have translational invariance). Then the corresponding action (and therefore the field theory) will be invariant under Poincaré transformations, i.e. automatically incorporate the physical facts that space is isotropic, homogeneous, that there is no prefered time origin, that relative motion is undetectable (relativity principle), etc.[1]

3.1 Lagrangian formalism

For this section, it would be good to have analytical mechanics fresh in your mind. The essential forward step here is to go from analytical mechanics of a discrete and finite number of degrees of freedom to that of fields.

3.1.1 Action and Lagrangian

To learn relativistic field theory, we will most often use the simplest example of a real scalar field $\phi(x^\mu)$. Therefore $\phi \in \mathbb{R}$ and $\phi(x) \to \phi'(x') \hat{=} \phi(x)$ under a Lorentz transformation. When we want to be slightly more general, we will take a field with internal indices $\phi_I(x^\mu)$, where $I = 1, ..., N_I$. Here, internal index means that this index does not bear a space-time structure: $\phi'_I(x') \hat{=} \phi_I(x)$ The action

$$S[\phi] = \int dt\ L = \int_R d^4x\ \mathcal{L} \tag{3.1}$$

is a functional of the field ϕ, which is written in terms of the Lagrangian L or of the Lagrangian density $\mathcal{L}$. The integral is over a very large portion R of Minkowski spacetime M (eventually, all of it). A functional is a machine that eats a function (not just a number) and gives a number as an output. It should clearly be distinguished from a function[2].

[1] Well, things are slightly more complex. There are examples where the action is not invariant but the physical laws are... We will talk about that later but, fortunately, We won't encounter such cases in the cases of interest for us.

[2] Indeed $S[\phi]$ depends on the whole function ϕ and not just on the value of the function ϕ at a single given point x.

To exemplify the construction, we will deal explicitly with the simplest field theory, which describes the dynamics of a single field (with no internal index), described by the following lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{3.2}$$

If one insists on staying quadratic (free field) then $V(\phi) \propto \phi^2$: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. At this point m^2 is just the name of a constant that is real but could be positive or negative (later m will be interpreted as a mass). We will motivate this form below (observe for the moment that the Lagrangian density is a scalar under Lorentz transformation and translations).

3.1.2 Equations of motion

The equations of motion of the field are obtained from the principle of least action (Hamilton): the action should be stationary $\delta S = 0$ when we make a small variation of the field $\delta\phi_I(x) \equiv \phi'_I(x) - \phi_I(x)$ [3]. Note the position of the prime and x is a short notation for x^μ. Later we will need to introduce another type of variation $\Delta\phi_I(x) \equiv \phi'_I(x') - \phi_I(x)$ that should not be confused with $\delta\phi_I(x)$. When $\phi_I(x) \to \phi'_I(x)$ the action varies from $S[\phi_I(x)] \to S[\phi'_I(x)] = S[\phi_I(x)] + \delta S$. The principle of least action imposes that $\delta S = 0$ when computed with a field that fulfills the equations of motion.

In many physical applications, the lagrangian density depends only on the field and its first derivatives: $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$, and not of higher derivatives. In this case, $\delta S = \int_R d^4x[\mathcal{L}(\phi'_I, \partial_\mu\phi'_I) - \mathcal{L}(\phi_I, \partial_\mu\phi_I)] = \int_R d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_I}\delta\phi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)}\partial_\mu\delta\phi_I\right] + \mathcal{O}(\delta\phi^2)$. Integration by part[4] allows us to express the integrand in terms of $\delta\phi_I$ only and we obtain $\delta S = \int_R d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_I} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)})\right]\delta\phi_I$. By definition[5], the functional derivative of the action is the prefactor of $\delta\phi_I$ in the integrand, that is $\frac{\delta S}{\delta\phi_I} = \frac{\partial\mathcal{L}}{\partial\phi_I} - \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)})$. When it vanishes, we find the Euler-Lagrange (EL) equations of motion:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi_I} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)}\right) = 0} \tag{3.4}$$

A solution of this equation is called a classical field ϕ_I^{cl}.

As an example, take $\mathcal{L} = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - \frac{m^2}{2}\phi^2 - U(\phi)$, where $U(\phi)$ is a polynomial in ϕ with powers greater than 2: $U(\phi) = a\phi^3 + b\phi^4 + ...$ Show that the EL equation is $(\partial_\mu\partial^\mu + m^2)\phi = -\frac{dU}{d\phi} = -3a\phi^2 + 4b\phi^3 + ...$ A first thing to note is that this equation is linear in ϕ if $U = 0$. The operator $\partial_\mu\partial^\mu$ is often called the d'Alembertian operator and denoted by a square $\Box$ or by ∂^2. If $U = 0$, one obtains the so-called Klein-Gordon (KG) equation $(\Box + m^2)\phi(x^\mu) = 0$. If in addition $m = 0$, then one obtains the d'Alembert wave equation $\Box\phi = 0$ familiar from the study of waves on a string, sound waves and light waves. It describes the propagation of a wave at the velocity of light $c = 1$. A solution of the KG equation as a propagating wave $\phi(x^\mu) = \phi_0 e^{-ik_\mu x^\mu} = \phi_0 e^{-i\omega t}e^{i\vec{k}\cdot\vec{x}}$, where $k^\mu = (k^0, \vec{k}) = (\omega, \vec{k})$, is easily found by Fourier transform and yields the following dispersion relation $\omega^2 = \vec{k}^2 + m^2$ of relativistic flavor.

[3] One should think of x^μ here as an index labelling degrees of freedom. Just like j in q_j labels the number of degrees of freedom in classical mechanics. In other words $\phi(x^\mu)$ is something like $q(j)$. At each x^μ, there is a finite number N_I of degrees of freedom. What we want to vary is ϕ_I not x^μ. Hence $\delta\phi_I(x) = \phi'_I(x) - \phi_I(x)$ and not $\phi'_I(x') - \phi_I(x)$.

[4] We have assumed that boundary terms vanish. Indeed, using Gauss' theorem $\int_R d^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)}\delta\phi_I\right) = \int_{\partial R} d^3S_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_I)}\delta\phi_I = 0$ if $\delta\phi_I = 0$ at the boundary ∂R of R.

[5] The Taylor expansion of a functional reads:

$$S[\phi(x) + \delta\phi(x)] = S[\phi(x)] + \int d^4x \frac{\delta S}{\delta\phi(x)}\delta\phi(x) + \frac{1}{2}\int d^4x \int d^4x' \frac{\delta^2 S}{\delta\phi(x)\delta\phi(x')}\delta\phi(x)\delta\phi(x') + \dots \tag{3.3}$$

This gives a practical definition of the successive derivatives of a functional.

3.1.3 Conjugate field and Hamiltonian

Given a field $\phi_I(x)$ (i.e. the equivalent of a position q in classical mechanics), the canonically conjugate field $\Pi^I(x)$ (i.e. the equivalent of a momentum $p = \partial L/\partial \dot{q}$) is defined by

$$\Pi_{\phi_I}(x) = \Pi^I(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_I(x)} \tag{3.5}$$

where $\dot{\phi}_I \equiv \partial_t \phi_I$. The Hamiltonian density (often simply called Hamiltonian) is

$$\mathcal{H} = \Pi^I(x)\dot{\phi}_I(x) - \mathcal{L} \tag{3.6}$$

and should be expressed in terms of ϕ_I and Π^I rather than ϕ_I and $\dot{\phi}_I$ (this is easy to remember as $H(q,p) = p\dot{q} - L(q,\dot{q})$). The Hamiltonian (not density here) is $H = \int d^3x \mathcal{H}$. Indeed in a Legendre transform $L[\phi_I, \dot{\phi}_I]$ becomes $H[\phi_I, \Pi^I] = \int d^3x \Pi^I \dot{\phi}_I - L[\phi_I, \dot{\phi}_I]$. With our favorite example, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, the conjugate field $\Pi = \dot{\phi}$ and $\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)$. The Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right] \tag{3.7}$$

is the sum of kinetic energy, elastic energy and potential energy, in this order.

3.2 Scalar fields and the Klein-Gordon equation

3.2.1 Real scalar field

One of the simplest Lagrangian for a real scalar field $\phi(x) \in \mathbb{R}$ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - \frac{m^2}{2}\phi^2 \tag{3.8}$$

The corresponding EL equation of motion is

$$(\partial_\mu\partial^\mu + m^2)\phi = (\Box + m^2)\phi = 0 \tag{3.9}$$

which is known as the Klein-Gordon (KG) equation [6]. As we have invariance under space and time translation, we can look for a solution in the form of a plane wave $\phi(x^\mu) = \phi(0)e^{-ik^\mu x_\mu} = \phi(0)e^{i\vec{k}\cdot\vec{x} - i\omega t}$. Injecting this ansatz in the KG equation, we find that

$$\omega^2 = \vec{k}^2 + m^2 \text{ i.e. } \omega = \pm\omega_k \text{ with } \omega_k \equiv \sqrt{\vec{k}^2 + m^2} \tag{3.10}$$

This dispersion relation should be familiar from relativistic mechanics except that it pertains to wave quantities (ω and $\vec{k}$) instead of particle quantities (E and $\vec{p}$). Therefore m is here a gap or a zero momentum frequency $\omega_0 = m$ and not yet a mass. Maybe $\hbar$ is playing a role in this correspondance? The answer will be found in the next chapter.

There is an internal and global symmetry of this Lagrangian. It is given by the discrete transformation $\phi \to -\phi$. This is known as a $\mathbb{Z}_2$ or Ising symmetry. Because this symmetry is discrete and not continuous,

[6] It was actually first discovered in 1926 by Schrödinger as a relativistic version of what is now known as the Schrödinger equation. It was then rediscovered by Klein, Gordon and Fock in 1926. It was first considered as a relativistic and quantum equation describing the motion of a single electron rather than as a field equation. It suffers from two major problems as an equation of quantum mechanics: probability density that is not always positive and negative energy states. Also as an equation describing a single electron it does not include spin and gives energy levels for the hydrogen atom that are not in agreement with experiments. This is why it was discarded by Schrödinger who later realized that the non-relativistic limit was much better behaved.

it can not be made infinitesimal and does not result in a conserved Noether current. In the next section, we will study a complex (rather than real) scalar field that has a more interesting internal symmetry than $\mathbb{Z}_2$.

A general solution of the KG equation can be found as a mode expansion thanks to the Fourier transform: $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\phi(k)$. Injecting in the KG we find that $k^2 = m^2$ and therefore $k_0 = \pm\omega_k$ ($k_0 = \omega$). Therefore a general solution can be written as $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\varphi(k)\delta(k^2-m^2)2\pi e^{-ikx}$. But $\delta(k^2-m^2) = \delta((k_0-\omega_k)(k_0+\omega_k)) = \frac{1}{2\omega_k}[\delta(k_0-\omega_k)+\delta(k_0+\omega_k)]$ using the fact that $\delta(f(x)) = \sum_j \frac{1}{|f'(x_j)|}\delta(x-x_j)$ where x_j are all the roots of f, i.e. $f(x_j) = 0$. Then $\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}[\varphi(\omega_k,\vec{k})e^{-i\omega_k t+i\vec{k}\cdot\vec{x}} + \varphi(-\omega_k,\vec{k})e^{i\omega_k t+i\vec{k}\cdot\vec{x}}] = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\varphi(k)e^{-ik\cdot x}+\varphi(-k)e^{ik\cdot x}\right]_{k_0=\omega_k}$ where in the last expression it is understood that k_0 is not independent of $\vec{k}$ but is actually equal to ω_k. We now use the fact that the field is real $\phi(x)^* = \phi(x)$ which means that $\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\varphi(k)^* e^{ik\cdot x}+\varphi(-k)^* e^{-ik\cdot x}\right] = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\varphi(k)e^{-ik\cdot x}+\varphi(-k)e^{ik\cdot x}\right]$ so that $\varphi(-k) = \varphi(k)^*$. Therefore $\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\varphi(k)e^{-ik\cdot x}+\varphi(k)^* e^{ik\cdot x}\right]$. The usual notation is to call $a(k) \equiv \varphi(k)$ so that the mode expansion reads

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(k)e^{-ik\cdot x}+a(k)^* e^{ik\cdot x}\right] = \int_k [a(k)e^{-ik\cdot x} + \text{c.c.}] \tag{3.11}$$

where we introduced the short hand notation $\int_k \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k}$.

Here are a few remarks on the above expression:

- The $a(k)$'s are just the expansion coefficients of $\phi(x)$ on the plane wave basis.
- The function $a(k)$, despite its name, depends only on $\vec{k}$ and not separatly on k_0 which is fixed to $k_0 = \omega_k$.
- $\frac{d^3k}{(2\pi)^3 2\omega_k}$ is a Lorentz invariant integration measure as it comes from $\int \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)$ which is obviously invariant (show it by computing the Jacobian of the change of variable $k^\mu \to k'^\mu = \Lambda^\mu{}_\nu k^\nu$ where Λ is any Lorentz transform) but d^3k is not.
- Note that we have made no hypothesis on the sign of the energy, although in the end everything only depends on $k_0 = \omega_k \geq 0$. Indeed $e^{-i\omega_k t}$ (positive energies) and $e^{i\omega_k t}$ (negative energies) are both present in (3.11).

3.2.2 Complex scalar field

The field ϕ is now assumed to be complex. The (free) Lagrangian is taken as

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi \tag{3.12}$$

This theory is also equivalent to that of two real fields with a special symmetry relating them. Indeed let $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$, which is just a decomposition of the complex field into its real and imaginary parts (ϕ_1 and $\phi_2 \in \mathbb{R}$). Check that:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{m^2}{2}\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{m^2}{2}\phi_2^2\,. \tag{3.13}$$

The special symmetry is related to having a single coefficient $m^2 = m_1^2 = m_2^2$ instead of two $m_1^2 \neq m_2^2$.

The independent degrees of freedom are ϕ_1 and ϕ_2, but one often does as if ϕ and ϕ^* could be taken as independent. The EL equations of motion are (check it in two different ways):

$$(\Box + m^2)\phi = 0 \text{ and } (\Box + m^2)\phi^* = 0\,. \tag{3.14}$$

We now study the above-mentioned special symmetry. It is a transformation $\phi(x) \to \phi'(x) = e^{-i\chi}\phi(x)$ that leaves the action (actually the Lagrangian density) invariant. Here χ is a phase, which is independent of the spacetime point x. It is therefore a global (not local) symmetry. In addition it mixes the two internal components of the complex field. It is therefore an internal symmetry. As $e^{-i\chi} \in U(1)$, it is called a global internal $U(1)$ symmetry. $U(1)$ is an abelian Lie group (it has a single generator) and $U(1) \approx SO(2)$. The transformation can be thought of as either a global phase change for the complex scalar field ϕ (notation $U(1)$) or as a global rotation in the (ϕ_1, ϕ_2) plane around a perpendicular axis (notation $SO(2)$). Show that the Lagrangian is invariant under the above $U(1)$ transformation.

3.3 Symmetries and conservation laws

What are symmetries? Let us take the example of invariance under translations. We can state that translations are symmetries if performing an experiment in different positions leads to the same results. In particular, assume that $\vec{r}(t)$ represents the trajectory of an object in free fall in the gravitational field of earth, then invariance under translation states that $\vec{r}(t) + \vec{a}$ should also be an acceptable trajectory. In this sense, a symmetry enables to transform a solution of the equations of motions into another solution.

The lagrangian approach is particularly well suited to encode symmetries. As we shall see in a moment, imposing a symmetry is easily done by choosing an action which is invariant under the symmetry transformation considered. Indeed, suppose that the action is invariant under some transformation, which means it takes the same value after transforming the trajectory: $S[\vec{r}] = S[\vec{r}']$. Then, if $\vec{r}_c$ is an extremum of S, so is $\vec{r}_c'$![7]

This construction holds true for trajectories of particles as for fields. In what follows, we will build actions which are invariant under Poncaré group. To do so, we should write lagrangian densities with no explicit dependence in x^μ in order to ensure invariance under translation. Moreover, we impose Lorentz invariance, at the level of a scalar field, by contracting co- and contra-variant indices as described in the previous section.

We will see that <u>continuous and global symmetries of the action imply conservation laws</u>. This is a fundamental result that was obtained by Emmy Noether around 1918.

3.3.1 Symmetries of the action

We start again from an action functional $S[\phi_I(x^\mu)]$ where $I = 1, ..., N_I$ labels the components of the field whatever their origin

We consider a transformation that acts as

$$\phi_I(x) \quad \rightarrow \quad \phi_I'(x) = \phi_I(x) + \Delta\phi_I(x) \tag{3.15}$$

We now make the transformation continuous and infinitesimal. We assume that it depends on N_a independent real, infinitesimal, parameters ϵ^a (labelled by $a = 1, ..., N_a$, where N_a is the number of generators of the group of transformations):

$$\phi_I(x) \quad \rightarrow \quad \phi_I'(x) = \phi_I(x) + \epsilon^a F_{I,a} \tag{3.16}$$

If ϵ^a does not depend on x^μ, the symmetry is said to be <u>global</u>. If $\epsilon^a(x) \neq$ const. does depend on the spacetime event, the symmetry is said to be <u>local</u>. An <u>internal</u> symmetry is such that F only depends on the field, not of its derivatives.

3.3.2 Noether's theorem

We first present an elegant but somewhat tricky proof (apparently due to Steven Weinberg). It proceeds in five steps:

Step 1: Suppose the action is invariant under a global ($\epsilon^a =$ const.) transformation such as (3.16) (i.e. continuous and infinitesimal).

[7] Note that if the action is symmetric, the proof ensures that the transformed trajectory is also an extremum. It may happen that the action is modified by a transformation, even though the transformation relates two acceptable trajectories. This is in particular the case if the transformation of the action is proportional to the equations of motion.

Step 2: Now, consider the same transformation (3.16) but with $\epsilon^a(x) \neq$ cst. The transformation is no longer a symmetry and the action is therefore modified by this transformation:

$$\Delta S = \int d^4x \ \epsilon^a(k) K_a \neq 0 \tag{3.17}$$

However, if we choose a constant ϵ^a, we really work with a symmetry an the variation of the action vanishes. This means that Eq. (3.17) has a very particular form: It is a functional of ϵ which must vanish for constant ϵ. The only way to fulfill this constraint is that K is a divergence. Therefore, Eq. (3.17) rewrites:

$$\Delta S = -\int d^4x \ \epsilon^a(k) \partial_\mu J^\mu_a \tag{3.18}$$

(the sign is purely conventional.)

Step 3: we are now evaluating the previous equation in a field configuration which fulfills the equations of motion. Recall that the variational principle tells us that, in a solution of the field equation, any variation of the field leads to no variation of the action (at leading order in $\delta\phi$). But the transformation $\phi_I \to \phi_I + \epsilon_a(x) F_{I,a}$ is itself a variation of the field (It is not the most general field transformation, of course). This implies that, when the equations of motions are fulfilled, Eq. (3.18) vanishes for any $\epsilon_a(x)$. Otherwise stated,

$$\partial_\mu J^\mu_a = 0 \tag{3.19}$$

To summarize, we have found that for each generator (labelled by a) of a global and continuous symmetry group, there is a divergenceless current $J^\mu_a(x)$. Equation (3.19) should be familiar from the continuity equation, e.g. local charge conservation in electromagnetism $\partial_\mu J^\mu = \partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$ where $J^\mu = (\rho, \vec{J})$, ρ is the electric charge density and $\vec{J}$ the electric current density.

The quantity

$$\boxed{Q_a \equiv \int d^3x J^0_a(t, \vec{x})} \tag{3.20}$$

is called a (Noether) charge. It is also said to be conserved (meaning time-independent here) because when integrated over most of space R_s

$$\frac{dQ_a}{dt} = \int_{R_s} d^3x \partial_0 J^0_a(x) = -\int_{R_s} d^3x \partial_i J^i_a(x) = -\int_{\partial R_s} d^2S_i J^i_a(x) = 0, \tag{3.21}$$

where we used Gauss' theorem and the fact that the field (and therefore the current) vanishes sufficiently fast at the boundary ∂R_s. This is now a global statement. For each generator of the continuous and global symmetry group, there is a quantity that when computed for the whole space (universe?) is a constant in time. Remember electric charge conservation: $Q = \int d^3x\rho$ is supposed to be a constant in the entire universe.

3.3.3 Some generalization

This section may be skipped in a first lecture. We discuss here the point raised in Footnote 1. We recall that a symmetry in classical mechanics corresponds to a transformation of coordinates which maps a solution of the equations of motion to another solution. A simple way of generating a symmetric theory is to use an action which is invariant under the transformation. But could we have a system which is symmetric without being described by a symmetric action? THe answer is yes. Consider a particle whose action is given by $S = \int dt(\dot{q} - \dot{q}\log\dot{q})$. The minimization of the action imposes the equation of motion $\ddot{q}\log\dot{q} = 0$ whose most general solution is $\ddot{q} = 0$. The action describes a particle at a constant speed, whose solutions are clearly invariant under the transformations $q(t) \to q'(t) = q(t) + a + vt$. a corresponds to a translation and v to a change of reference frame. Let us make an infinitesimal transformation with $a = 0$ and v infinitesimal.

The variation of the action is $\delta S = v \int dt \log \dot{q}$ which is clearly nonzero. We therefore have an example of a system whose trajectories have a symmetry but whose action is not.

To clarify this apparent paradox, let us upgrade the infinitesimal parameter to a function: $v \to v(t)$. The variation of the action writes $\delta S = -\int dt\, v(t) t \frac{\ddot{q}}{\dot{q}}$. This is clearly nonzero in general. We observe however that it does vanish when we impose the equation of motion $\ddot{q} = 0$. We can then follow the proof of Noether's theorem and conclude that the conserved charge associated with the change of reference frame is indeed conserved.

In general, if the variation of the action under a change of coordinates is proportional to the equations of motion, this is sufficient to ensure that the system is invariant under this transformation and that a conserved charge can be built.

Another (more interesting) example is the one of a particle moving in the (xy) plane, in the presence of a constant magnetic field pointing in the z direction. The action is given by $S = \int dt[\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \frac{\omega}{2}(\dot{x}y - \dot{y}x)]$. The lagrangian explicitly depends on the position and therefore the invariance under translation is not obvious. You can check for yourself that, upgrading the infinitesimal translation to a function, $a^i \to a^i(x)$, the variation of the action vanishes when we impose the equation of motion. You can therefore deduce that there is a conserved Noether charge, that you can compute explicitly.

3.3.4 Examples

To get more familiar with Noether currents, we consider a few concrete examples.

Internal global symmetry

Consider a field theory with two real fields ϕ_1 and ϕ_1, with a Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - V(\vec{\phi}^2) \tag{3.22}$$

which is invariant under the transformation:

$$\phi_1(x) \to \phi_1(x) + \epsilon \phi_2(x) \tag{3.23}$$
$$\phi_2(x) \to \phi_2(x) - \epsilon \phi_1(x) \tag{3.24}$$
$$\tag{3.25}$$

which we can also write as:

$$\phi_I(x) \to \phi_I(x) + \epsilon \epsilon_{IJ} \phi_J(x) \tag{3.26}$$

(sorry for the notations... ϵ_{IJ} is the 2-dimensional Levi-Civita tensor and ϵ is the small parameter of the transformation...) There is one conserved current, which reads:

$$J^\mu = \phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1 \tag{3.27}$$

Check that this is indeed divergenceless when we impose the equations of motion.

Spacetime translations

How should we modify a field under a translation? We concentrate on a single field. the translated field ϕ' at point x^μ is equal to the original field at point $(x+a)^\mu$

$$\phi'(x) = \phi(x+a) \tag{3.28}$$

Taking infinitesimal translation (here a^μ plays the rôle of ϵ_a in our general discussion), then

$$\phi'(x) = \phi(x) + a^\mu \partial_\mu \phi(x). \tag{3.29}$$

(check that this is indeed a symmetry of the Klein-Gordon action (3.1)). You can work out the Noether curent for yourself. Note that there are 4 Noether currents because we can translate in any spacetime direction. The current is therefore a rank-2 tensor, which reads:

$$\theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} \tag{3.30}$$

The conserved charges are

$$P^\nu = \int d^3x\theta^{0\nu} \tag{3.31}$$

correspond to energy and momentum of the field. The energy part is particularly interesting to write:

$$P^0 = \int d^3x\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{3.32}$$

This corresponds exactly to the hamiltonian of the Klein-Gordon field.

Lorentz transformations

Here, for simplicity, we restrict to a scalar field. A Lorentz transformation acts as $x'^\mu = \Lambda^\mu{}_\nu x^\nu \approx x^\mu + \omega^\mu{}_\nu x^\nu$ and $\phi'(x') = \phi(x)$ (scalar field). Therefore, here $F_a = 0$ and we have a little bit of work to identify A^μ_a, where a labels the generators of the Lorentz group. For an infinitesimal transformation, we have shown that $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$. Therefore $\delta x^\mu = x_\sigma\omega^{\mu\sigma} = \delta^\mu_\rho x_\sigma\omega^{\rho\sigma}$ from which we identify ϵ^a as $\omega^{\rho\sigma}$ and A^μ_a as $\delta^\mu_\rho x_\sigma$. Remember that $\omega^{\rho\sigma} = -\omega^{\sigma\rho}$ is antisymmetric and therefore $a = (\rho\sigma)$ only takes 6 different values (01,02,03,12,13,23). Because of antisymmetry, we can also write $\delta x^\mu = \frac{1}{2}(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho)\omega^{\rho\sigma}$ and $A^\mu{}_{(\rho\sigma)} = \frac{1}{2}(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho)$ (the notation meaning that this tensor is antisymmetric with respect to its last two indices $A^\mu{}_{\rho\sigma} = -A^\mu{}_{\sigma\rho}$). The Noether current is then $J^\mu{}_{(\rho\sigma)} = \theta^\mu{}_\nu A^\nu{}_{(\rho\sigma)} = -\frac{1}{2}(\theta^\mu{}_\nu x_\rho - \theta^\mu{}_\rho x_\sigma)$. The divergenceless current can therefore be taken as

$$\mathcal{M}^{\mu(\rho\sigma)} = x^\rho\theta^{\mu\sigma} - x^\sigma\theta^{\mu\rho} \tag{3.33}$$

with $\partial_\mu\mathcal{M}^{\mu(\rho\sigma)} = 0$. As $\mathcal{M}^{\mu\rho\sigma} = -\mathcal{M}^{\mu\sigma\rho}$, there are 6 independent divergenceless currents and hence 6 conserved charges

$$M^{\rho\sigma} = \int d^3x\mathcal{M}^{0\rho\sigma} = \int d^3x(x^\rho\theta^{0\sigma} - x^\sigma\theta^{0\rho}) \tag{3.34}$$

such that $dM^{\rho\sigma}/dt = 0$. These charges are the angular momentum

$$M^{ij} = -i\int d^3x\Pi[x^i(-i\partial_j) - x^j(-i\partial_i)]\phi, \tag{3.35}$$

where $L^{ij} = x^i(-i\partial_j) - x^j(-i\partial_i)$ is the angular momentum operator (generator of rotations)[8], and a vectorial quantity $\vec{G}$ with no definite name and related to boosts generators (how?):

$$G^i = M^{0i} = \int d^3x(x^i\theta^{00} - x^0\theta^{0i}) = \int d^3x[\mathcal{H}x^i - t\Pi\partial^i\phi]\,. \tag{3.36}$$

In other words $\vec{G} = -t\vec{P} + \int d^3x\vec{x}\mathcal{H}$. The conservation of this last quantity is perhaps less familiar then that of angular momentum or energy or linear momentum. Let us give an example in the simplest possible case of one-dimensional relativistic mechanics of a massive particle: the energy is conserved and $H = E = \sqrt{p^2 + m^2}$ (as $c = 1$). The Hamilton equations of motion are $\dot{p} = -\partial_x H = 0$ and $\dot{x} = \partial_p H = p/E = v$ is a constant. Then $G = Ex - pt$ is indeed a conserved quantity as $\dot{G} = \frac{d}{dt}(Ex - tp) = E\dot{x} - p = E\frac{p}{E} - p = 0$. This conserved quantity is related to the uniform motion of the center of mass. Indeed, if at $t = 0$ the particle is in $x = x_0$

[8] M^{ij} is an anti-symmetric tensor and therefore contains 3 independent entries, which is equivalent to an axial vector usually called the angular momentum $\vec{M}$.

then the conservation law is $G = Ex - pt = Ex_0 = \text{const.}$ which means that $x - \frac{p}{E}t = x - vt = x_0 = \text{const.}$ The conserved quantity is unusual as it explicitly depends on time (it is also unusual in having no clear name...). In the non-relativistic limit $v = p/E \ll 1$, a Lorentz boost becomes a Galilean boosts, the energy $E \approx m$ and the conserved quantity is $G = mx - pt$. The conservation law is usually written as $x - vt = \text{const.}$ See the discussion on the conservation law of the center of mass in §8 of Ref. [14].

Let us now show that invariance under the Poincaré group implies that the energy-momentum tensor has to be a symmetric tensor. Invariance under Lorentz transformations implies that $\partial_\mu \mathcal{M}^{\mu\nu\rho} = 0 = \partial_\mu(\theta^{\mu\sigma}x^\rho - \theta^{\mu\rho}x^\sigma) = \theta^{\mu\sigma}\partial_\mu x^\rho - \theta^{\mu\rho}\partial_\mu x^\sigma$, where we used that $\partial_\mu\theta^{\mu\nu} = 0$ as a consequence of invariance under spacetime translations. Given that $\partial_\mu x^\nu = \delta^\nu_\mu$, we obtain that $0 = \theta^{\rho\sigma} - \theta^{\sigma\rho}$, qed. See the symmetrization procedure from $\theta^{\mu\nu}$ to $T^{\mu\nu}$ discussed above.

Remark (to be skipped in a first reading): another reason for wanting an energy-momentum tensor that is symmetric $T^{\mu\nu} = T^{\nu\mu}$ comes from general relativity. General relativity is basically the double statement that (1) energy-momentum curves space-time (Einstein field equations) and (2) that in the absence of forces (gravity being considered not as a force here), particles follow geodesics in this curved space-time (equation of geodesics). The first equation reads:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -\frac{8\pi G}{c^4}T^{\mu\nu} \tag{3.37}$$

where $g^{\mu\nu}$ is the metric for curved space-time (it replaces $\eta^{\mu\nu}$ defined for flat space-time) and the quantities $R^{\mu\nu}$ and R are curvatures of space-time defined from the metric tensor[9]. The above equation states that space-time curvature (left hand side) is proportional to the energy-momentum tensor (right hand side). The proportionality constant depends on Newton's gravitation constant G and on the velocity of light c. It turns out that the left hand side is (by construction) symmetric with respect to its two indices μ, ν. Therefore the energy-momentum tensor also has to be symmetric. Actually, when studying field theory in curved spacetime, the definition of the energy-momentum tensor becomes that it equals the variation of the action[10] with respect to the metric:

$$T^{\mu\nu} = -\frac{2}{\sqrt{-\det g_{\mu\nu}}}\frac{\delta S}{\delta g_{\mu\nu}} \propto \frac{\delta S}{\delta g_{\mu\nu}} \tag{3.38}$$

See for example page 78 in Zee [4]. Again, this last definition makes it obvious that the energy-momentum tensor has to be symmetric in $\mu\nu$.

As a summary of this section: the Poincaré group has 10 generators, which implies 10 conserved charges and the corresponding divergenceless currents. Invariance under time translation gives the conservation of energy H, invariance under space translations gives the conservation of momentum $\vec{P}$. Overall $\partial_\mu T^{\mu\nu} = 0$. Invariance under Lorentz transformations give 6 other conserved charges: space rotations implies the conservation of angular momentum $\vec{M}$ (related to the rotation generator $\vec{L}$); Lorentz boosts implies the conservation of $\vec{G}$ (related to boosts generators $\vec{K}$), known as the conservation of the center of mass. Overall $\partial_\mu \mathcal{M}^{\mu\rho\sigma} = 0$.

As a word of caution, we note that the conserved quantities are strongly related to the corresponding generators (and are often called by the same name!) but should still be carefully distinguished from them. For example, the energy of the field $H = \int d^3x\mathcal{H} = -i\int d^3x\Pi(i\partial_t)\phi - L$ is different from the time translation operator $i\partial_t$. Or the momentum of the field $\vec{P}$ is different from the momentum operator (generator of space translations) $-i\vec{\nabla}$. This difference is similar to the one that exists between the total momentum of a gas

[9] More precisely $R^\lambda_{\mu\nu\kappa} = \partial_\nu\Gamma^\lambda_{\mu\kappa} - \partial_\kappa\Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\kappa}\Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\kappa\sigma}$ is the Riemann curvature tensor defined in terms of Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})$, $R_{\mu\nu} \equiv R^\lambda_{\mu\lambda\nu}$ is the Ricci curvature tensor and $R \equiv R_{\mu\nu}g^{\mu\nu}$ is the scalar curvature. With hindsight and after studying the electromagnetic field, you will recognize that the Christoffel symbol is essentially a connection just as the 4-vector gauge potential A^μ and that the Riemann curvature tensor is essentially a curvature just as the electromagnetic field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. For more details, see A. Zee [4] chapter VIII.1, page 419.

[10] This action S being that for a field in a curved spacetime excluding the gravitational field itself that has its own action S_g. In plain words the total action should be $S_{\text{tot}} = S + S_g$ and here we only vary S with respect to the metric $g_{\mu\nu}$ and not S_{tot}.

of particles [a.k.a. the many-body momentum and equivalent to $\vec{P} = -i\int d^3x\Pi(-i\vec{\nabla})\phi$] and the individual momenta [a.k.a. the single-particle momenta and equivalent to eigenvalues of $-i\vec{\nabla}$].

The next sections are about building field theories that respect the spacetime symmetries described above. The strategy is to write actions that are Lorentz invariants so that the EL equations of motion obtained by minimizing the action will automatically be covariant. We will study first the scalar (Klein-Gordon) fields.

Chapter 4

Canonical quantization of scalar fields

In this chapter, we will turn the classical field theories discussed previously into quantum field theories. For that, we will use the machinery of "canonical quantization" (and not the alternative machinery of "path integral quantization", for example). The idea is to identify pairs of canonically conjugate variables (such as q and p) in the classical field theory and then to impose that their (equal-time) commutator is non-vanishing and proportional to $i\hbar$ (i.e. $[\hat{q},\hat{p}] = i\hbar$). It is the usual idea of replacing a Poisson bracket $\{q,p\}_{PB} = 1$ with a commutator $(i\hbar)^{-1}[\hat{q},\hat{p}] = 1$, which turns c-numbers (i.e. commuting numbers q, p) into q-numbers (i.e. operators $\hat{q}$, $\hat{p}$)[1]. Canonical quantization relies on the Hamiltonian formulation of classical field theory, whereas path integral quantization relies on the Lagrangian formulation. Canonical quantization is not explicitly covariant as time plays a special role in the Hamilton formalism.

We deal exclusively with the Klein-Gordon field for the time being...

4.1 Real scalar fields

[see Ryder [3], pages 129-139 and Maggiore [2], pages 83-88]

4.1.1 Quantum field

We consider a real scalar field $\phi(x)$ with Lagrangian $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2$. The variable $\phi(x)$ is associated to its conjugate $\Pi(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi(x))} = \partial_0\phi(x)$. The Hamiltonian is $H = \int d^3x \frac{1}{2}\left(\Pi(x)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$. For each point $\vec{x}$ of space, we have a generalized coordinate $\phi(x) = \phi(t,\vec{x})$ (you should think of it as $q_{\vec{x}}(t)$ with $\vec{x}$ labeling the degrees of freedom) and a generalized conjugate momentum $\Pi(x) = \Pi(t,\vec{x})$ (think of it as $p_{\vec{x}}(t)$). We now impose the following equal-time commutation relations (ETCR):

$$[\phi(t,\vec{x}),\Pi(t,\vec{x}')] = i\hbar\delta(\vec{x}-\vec{x}')$$

and

$$[\phi(t,\vec{x}),\phi(t,\vec{x}')] = [\Pi(t,\vec{x}),\Pi(t,\vec{x}')] = 0 \tag{4.1}$$

Note that the above ETCR are not manifestly Lorentz covariant as t plays a special role. Note also that $\hbar \to 0$ just affects the first commutator. Once every commutator vanishes, we are back to a classical field theory. From now on, we will use units such that $\hbar = 1$ on top of $c = 1$.

The consequence of imposing ETCR is that the classical field $\phi(x)$ has been promoted to a quantum field, i.e. to an operator (and called a quantum field operator). We don't write hats on ϕ (for simplicity), but we

[1] We are used to distinguish commuting numbers (ordinary numbers, c-numbers, classical numbers) from operators (q-numbers, quantum numbers), which usually do not commute. A third even stranger category is anti-commuting numbers (called Grassmann variables): these are not operators but numbers, but they anticommute rather than commute as ordinary numbers do. They are useful in the path integral quantization of fermionic fields.

mean it! The quantum field $\phi(x)$ is a hermitian operator $\phi^\dagger(x) = \phi(x)$ because the classical field was real. It is an operator in the Heisenberg representation $\phi(t, \vec{x})$ with an explicit time-dependence. It is therefore very different from a wavefunction or a quantum state (a ket)[2]. The EL equation of motion $(\Box + m^2)\phi(x) = 0$ is now re-interpreted as a Heisenberg equation of motion[3] for the time-dependent operator $\phi(t, \vec{x})$. In the Heisenberg picture $\phi(t, \vec{x}) = e^{iHt}\phi(0, \vec{x})e^{-iHt}$ and therefore $\partial_t\phi(x) = -i[\phi(x), H]$.
Exercise: From $\dot{\phi} = -i[\phi, H]$, $\ddot{\phi} = -i[\dot{\phi}, H]$ and $[\phi(t, \vec{x}), \dot{\phi}(t, \vec{x}')] = i\delta(\vec{x} - \vec{x}')$, show that $(\Box + m^2)\phi(x) = 0$.

We recall the mode expansion of a classical free and real scalar field

$$\phi(x) = \int_k \left(a(k)e^{-ik\cdot x} + a^*(k)e^{ik\cdot x}\right) \tag{4.2}$$

where it is understood that in the above expression $k^\mu = (k^0, \vec{k})$ with $k^0 = \omega_k \equiv \sqrt{\vec{k}^2 + m^2}$ and the shorthand notation $\int_k \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k}$. At this point $a(k)$ is just the name of the coefficient of the Fourier expansion of a field $\phi(x)$ satisfying the massive KG equation $(\Box + m^2)\phi = 0$. Because of quantization, the mode expansion of the quantum field now reads

$$\phi(x) = \int_k \left(a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right) \tag{4.3}$$

and $a(k)$ and $a^\dagger(k)$ become operators as well (but non-hermitian as the classical $a(k)$ is a complex and not a real number). For the conjugate field operator

$$\Pi(x) = \partial_0\phi(x) = \int_k (-i\omega_k)\left(a(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right) \tag{4.4}$$

Check that the ETCR of ϕ and Π imply that

$$[a(k), a^\dagger(k')] = 2\omega_k(2\pi)^3\delta(\vec{k} - \vec{k}') \tag{4.5}$$

and

$$[a(k), a(k')] = 0 = [a^\dagger(k), a^\dagger(k')] \tag{4.6}$$

This starts to smell like the algebra of annihilation and creation operators (i.e. $[a, a^\dagger] = 1$, $[a, a] = 0 = [a^\dagger, a^\dagger]$) familiar from the quantum mechanical harmonic oscillator. Hence the name $a(k)$ for the coefficients in the mode expansion of the field $\phi(x)$.

4.1.2 Particle interpretation and Fock space

In order to construct the Fock space and to strengthen the analogy with the harmonic oscillator, it is easier to work with a finite spatial volume $V = L^3$ and periodic boundary conditions. Therefore

$$\int \frac{d^3k}{(2\pi)^3} \to \frac{1}{V}\sum_{\vec{k}} \text{ and } (2\pi)^3\delta(\vec{k} - \vec{k}') \to V\delta_{\vec{k},\vec{k}'} \tag{4.7}$$

with $\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$ where $n_j \in \mathbb{Z}$. Let $a_{\vec{k}} \equiv \frac{1}{\sqrt{2\omega_k V}}a(k)$ so that $\phi(x) = \sum_{\vec{k}} \frac{1}{\sqrt{V 2\omega_k}}\left(a_{\vec{k}}e^{-ik\cdot x} + h.c.\right)$ instead of (4.3). Then the commutation relations for the $a_{\vec{k}}$ operators read

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \text{ and } [a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] \tag{4.8}$$

Now, we really have a clear analogy with annihilation and creation operators of the harmonic oscillator, except that we have one such harmonic oscillator (one such mode) for each $\vec{k}$.

[2] In particular, it is not a wavefunction that has been quantized a second time. It is not a quantum wavefunction with a hat on top. Hence, the dislike with the terminology of "second quantization". Much better would be "occupation number representation" or "annihilation/creation formalism".

[3] Remember that, in quantum mechanics, an operator A in the usual (Schrödinger) picture – for simplicity, we assume that it does not explicitly depend on time – becomes $A(t) \equiv e^{iHt}Ae^{-iHt}$ in the Heisenberg picture and satisfies the Heisenberg equation of motion $\dot{A}(t) = -i[A(t), H]$.

Detour by the 1D quantum harmonic oscillator

The 1D quantum mechanical harmonic oscillator has a Hamiltonian $H = \frac{p^2}{2m} + \frac{m\omega_0^2 q^2}{2}$ with $[q,p] = i\hbar$ where q is the position and p the canonically conjugated momentum. One usually defines $a \equiv \frac{q+ip}{\sqrt{2}}$ and $a^\dagger = \frac{q-ip}{\sqrt{2}}$ (in units such that the characteristic length $\sqrt{\frac{\hbar}{m\omega_0}} = 1$ and $\hbar = 1$), which satisfy the following algebra $[a,a^\dagger] = 1$ and $[a,a] = 0 = [a^\dagger,a^\dagger]$ as a consequence of $[q,p] = i$ and $[q,q] = 0 = [p,p]$. The Hamiltonian is then rewritten as $H = \frac{\omega_0}{2}(a^\dagger a + aa^\dagger) = \omega_0(a^\dagger a + \frac{1}{2})$. We know that $[a,a^\dagger] = 1$ implies that $n = a^\dagger a$ is the number operator. Indeed, call $|n=0\rangle = |vac\rangle$ the vacuum state (i.e. the groundstate of the harmonic oscillator, $H|0\rangle = \frac{\omega_0}{2}|0\rangle$), which is defined by $a|0\rangle = 0$. Then $[n,a] = -a$ and $[n,a^\dagger] = a^\dagger$ such that $[n,a^\dagger]|0\rangle = a^\dagger|0\rangle$, which means that $n(a^\dagger|0\rangle) = 1(a^\dagger|0\rangle)$. In other words $a^\dagger|0\rangle \propto |n=1\rangle$. We can choose $a^\dagger|0\rangle = |n=1\rangle$, which shows that $a^\dagger$ is a creation operator for an excitation quantum of the harmonic oscillator. Continuing this construction, we arrive at

$$a|n\rangle = \sqrt{n}|n-1\rangle \ , \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \text{ and } a^\dagger a|n\rangle = n|n\rangle \tag{4.9}$$

which confirms that $n = a^\dagger a$ is the number operator (its eigenvalues are in $\mathbb{N}$), $a^\dagger$ is a creation operator (it creates a single excitation of the harmonic oscillator) and a is an annihilation operator (it destroys a single excitation of the harmonic oscillator). The Hilbert space of states for the harmonic oscillator[4] can be thought of as a Fock space[5] for excitation quanta (i.e. particles that you may call phonons): the number of excitations quanta is not fixed and an orthonormal basis is $\{|n\rangle, n \in \mathbb{N}\}$. Because the number of excitation quanta that can be accommodated in this single mode is not bounded, we also know that we are dealing with bosons. In the case of a single harmonic oscillator, the particles (i.e. the excitation quanta) have no dispersion relation (here, the phonons are located at one point in space and can not move) and are characterized by a single frequency ω_0.

Here, we would like to make a connection with quantum statistics as seen in lectures on statistical mechanics. The statistics for such phonons is the Bose-Einstein statistics at zero chemical potential (also known as Planck's distribution). In the groundstate, there are no phonons and when the temperature increases, the number of phonons increases. The number of phonons is not conserved. We are obviously not describing matter particles. One should have in mind the two equivalent descriptions: either a single massive particle in a harmonic oscillator (Hilbert space of a single 1D particle) or an ideal gas of non-conserved bosons which are the excitation quanta of the harmonic oscillator (Fock space for 0D excitation quanta). In the first picture, we are doing quantum mechanics of a single particle in 1D space. In the second picture, we are doing QFT of a 0D field (an infinite number of degrees of freedom but all at the same spatial position).

Remark: In non-relativistic quantum mechanics of identical particles (many-body physics), the idea of working with number representation (i.e. with occupation numbers of different modes) is known as second quantization formalism. It is an alternative to first quantization formalism in which particles carry labels (i.e. particles are numbered), are assigned to specific single-particle states and the many-body wavefunction is then symmetrized (for bosons) or antisymmetrized (for fermions). Here we have a very simple example, with a unique 1D massive particle in a harmonic oscillator, which can be either described by a wavefunction such as $\langle q|1:\chi_n\rangle$ (1: meaning the particle number 1) or alternatively described by the occupation number $n_{\omega_0} = a^\dagger a$ of its single mode of frequency ω_0 given a wavefunction such as $\langle q|n_{\omega_0} = n\rangle$. The two kets are indeed equal $|1:\chi_n\rangle = |n_{\omega_0} = n\rangle$ but the first refers to a single massive particle in 1D and the second to a 0D ensemble of n excitation quanta (phonons).

[4]This is actually not $\mathcal{H} = \{|q\rangle, q \in \mathbb{R}\}$ as localized states far from the minimum $q = 0$ of the harmonic potential are not allowed. It is a smaller Hilbert space $\mathcal{H} = \{|\chi_n\rangle, n \in \mathbb{N}\}$ if we call $\chi_n(q) = \langle q|n\rangle \propto H_n(q)e^{-q^2/2}$ the eigen-wavefunctions, where H_n are the Hermite polynomials and $H|\chi_n\rangle = (n+1/2)\omega_0|\chi_n\rangle$.

[5]The Fock space is $\mathcal{F} = \oplus_{n=1}^{\infty}\mathcal{E}_n$ where $\mathcal{E}_n$ is a one-dimensional Hilbert space generated by $|n\rangle$. One has that $\mathcal{H} = \mathcal{F}$.

Back to quantum field theory

Going back to the quantized scalar field, show that the Hamiltonian can be rewritten as

$$H = \int d^3x \frac{1}{2}\left((\partial_0\phi)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right) = \sum_{\vec{k}} \frac{\omega_k}{2}\left(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}}\right) = \sum_{\vec{k}} \omega_k \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\right) \tag{4.10}$$

This looks like a sum of independent harmonic oscillators (one for each $\vec{k}$) and each with its own frequency ω_k. We also see that, despite the fact that the massive KG equations gives a dispersion relation $\omega = \pm\omega_k$ with positive and negative branches, the energy of the field is always positive[6]. Indeed, the expectation value of the Hamiltonian operator in any state is

$$\langle H \rangle = \sum_{\vec{k}} \omega_k \left(\langle a^\dagger_{\vec{k}} a_{\vec{k}}\rangle + \frac{1}{2}\right) \geq \sum_{\vec{k}} \frac{\omega_k}{2} > 0 \tag{4.11}$$

as $\langle a^\dagger_{\vec{k}} a_{\vec{k}}\rangle \geq 0$.

The Fock space is constructed by analogy with the harmonic oscillator. For each mode $\vec{k}$ of frequency ω_k, there is a pair of creation/annihilation operators $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ such that $n_{\vec{k}} = a^\dagger_{\vec{k}} a_{\vec{k}}$ is the occupation number operator in that mode. The operator $N = \sum_{\vec{k}} n_{\vec{k}}$ gives the total number of excitation quanta (i.e. the total number of particles) contained in the field. The vacuum state is defined as being annihilated by all annihilation operators $a_{\vec{k}}|vac\rangle = 0$, $\forall\vec{k}$: in other words, the vacuum contains no particles (no excitation quanta). A single particle state is $|\vec{k}\rangle = |N_{\vec{k}} = 1\rangle = a^\dagger_{\vec{k}}|vac\rangle$. Show that it is an energy eigenstate with energy ω_k above that of the vacuum.

4.1.3 Vacuum energy and normal ordering

The vacuum energy is obtained from $H|vac\rangle = (\sum_{\vec{k}} \frac{\omega_k}{2})|vac\rangle$. The energy $\langle vac|H|vac\rangle = \sum_{\vec{k}} \frac{\omega_k}{2}$ is infinite[7]. It comes from the zero-point motion of each mode. This absolute energy is unobservable (but energy differences are: see the exercise sheet on the Casimir effect). It is usually subtracted by redefining the zero of energy as being the energy of the vacuum state[8]. Then

$$H \equiv H - \langle vac|H|vac\rangle = \sum_{\vec{k}} \omega_k a^\dagger_{\vec{k}} a_{\vec{k}} \tag{4.12}$$

Another way to look at the same issue is to realize that when going from the classical to the quantum theory, there is an ambiguity. Indeed, take a complex field ϕ: when in the classical theory one has $\phi^*\phi = \phi\phi^* = |\phi|^2$ in the Hamiltonian density, should it be quantized as $\phi^\dagger\phi$ or $\phi\phi^\dagger$ or $\frac{1}{2}(\phi^\dagger\phi + \phi\phi^\dagger)$ or $f\phi^\dagger\phi + (1-f)\phi\phi^\dagger$ with f an arbitrary number between 0 and 1? We therefore fix the following prescription known as normal ordering (and symbolized by a pair of colons $A \rightarrow : A :$): upon quantization, the field operators have to be normal ordered, i.e. creation operators are to be moved to the left of annihilation operators

[6] Discuss this point in more details with the remark of J.Y. Ollitrault. There is a choice behind the way the mode expansion is interpreted which leads to positive energy. Note also that, contrary to the Dirac interpretation of the Dirac equation, we did not need to invoke the Pauli principle to fill the negative energy states and have a lower bound for energy. This is good news because Dirac's argument would not work for bosons. Here the formalism of quantum field theory allows one to obtain a total energy that is positive despite the fact that it is built out of modes that have positive and negative energy branches.

[7] Even the energy density is infinite as $\frac{1}{V}\sum_{\vec{k}} \frac{\omega_k}{2} = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2} \sim \int_0^\Lambda dk k^2 \omega_k \sim \int^\Lambda dk k^3 \sim \Lambda^4$ diverges when the UV cutoff $\Lambda \to \infty$. The UV cutoff $\Lambda \sim 1/a$ is related to the short distance structure of space (where a is an artificial lattice spacing) and was here introduced in order to show the nature of the divergence. Don't mix it with the large distance L box that controls the IR behavior, i.e. the large distance behavior.

[8] Is this without physical consequence? Even in the presence of gravitation? What about the gravitational constant and the expansion of the universe?

(while preserving the order among creation operators and separately also among annihilation operators). The quantum Hamiltonian is then

$$H = \int d^3x \frac{1}{2} : \Big((\partial_0 \phi)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \Big) := \sum_{\vec{k}} \frac{\omega_k}{2} : \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}} \Big) := \sum_{\vec{k}} \omega_k a^\dagger_{\vec{k}} a_{\vec{k}} \tag{4.13}$$

which is equivalent to removing the vacuum energy. Now $H|vac\rangle = 0$. A precise definition of normal ordering is that

$$: a(k_1) a^\dagger(k_2) := a^\dagger(k_2) a(k_1) \text{ and } : a(k_1) a(k_2) a(k_3)^\dagger a(k_4) := a(k_3)^\dagger a(k_1) a(k_2) a(k_4) \tag{4.14}$$

The 3-momentum is classically $\vec{P} = -i \int d^3x \Pi(-i\vec{\nabla})\phi$ (remember that this is just the Noether charge associated to the space translation symmetry). After quantization, it becomes

$$\vec{P} = -i \int d^3x : \Pi(-i\vec{\nabla})\phi := \sum_{\vec{k}} \frac{\vec{k}}{2} : \Big(a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}} \Big) := \sum_{\vec{k}} \vec{k} a^\dagger_{\vec{k}} a_{\vec{k}} \tag{4.15}$$

so that the vacuum has zero momentum $\vec{P}|vac\rangle = 0$.

Note that there is the 3-momentum operator $-i\vec{\nabla}$, which is the space translation generator. And there is the 3-momentum of the whole field $\vec{P}$, which upon canonical quantization also becomes an operator. The first is an operator in the sense that it acts on the field seen as a function (it is a gradient). Whereas the second is a many-body operator (it concerns the whole gas of particles): it acts in Fock space. In the language of second quantization, the first would be called a single-particle operator (acting in the Hilbert space for a single-particle) and the second a many-body operator (acting in Fock space).

4.1.4 Fock space and Bose-Einstein statistics

A single particle state is $|\vec{k}\rangle = |n_{\vec{k}} = 1\rangle = a^\dagger_{\vec{k}}|vac\rangle$. It is indeed an eigenvector of the number operator N with eigenvalue 1. It is also an eigenvector of H with eigenvalue ω_k (after normal ordering) and an eigenvector of $\vec{P}$ with eigenvalue $\vec{k}$ (after normal ordering). Notice that particles (i.e. field excitation quanta) only have positive energy despite the fact that the dispersion relation has positive and negative energy branches.

A multi-particle state is $|\vec{k}_1, ..., \vec{k}_n\rangle = a^\dagger_{\vec{k}_1} ....a^\dagger_{\vec{k}_n}|vac\rangle$. It is an eigenvector of N with eigenvalue n, an energy eigenvector with eigenvalue $\omega_{k_1} + ... + \omega_{k_n}$ and a 3-momentum eigenvector with eigenvalue $\vec{k}_1 + ... + \vec{k}_n$.

Because of $[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0$, the multiparticle state $|\vec{k}_1, ..., \vec{k}_n\rangle = a^\dagger_{\vec{k}_1} ....a^\dagger_{\vec{k}_n}|vac\rangle$ is symmetric under any exchange of particle, which means that the particles (excitation quanta of the scalar field) are bosons. There is actually a general connection between the fact that an integer spin field (here scalar field means spin 0) is quantized with commutators and the fact that the excitation quanta obey Bose-Einstein statistics.

Making a connection to statistical mechanics, the average occupation of a mode $\vec{k}$ in an equilibrium state at temperature T (often simply called a thermal state) is given by

$$n_{\vec{k}} \equiv \langle a^\dagger_{\vec{k}} a_{\vec{k}} \rangle_T = \frac{1}{e^{\omega_k/T} - 1} \tag{4.16}$$

which is the Planck occupation factor (i.e. the Bose-Einstein distribution at zero chemical potential)[9]. The number of particles is not conserved: it changes with temperature. Indeed, the total number of particles is $\langle N \rangle_T = \sum_{\vec{k}} \frac{1}{e^{\omega_k/T}-1}$, which goes as $T^3 \to \infty$ when $T \gg m$ and as $m^3 e^{-m/T} \to 0$ when $T \ll m$.

[9] We are taking units such that Boltzmann's constant $k_B = 1$ in addition to $\hbar = 1$ and $c = 1$. All these fundamental constants are merely conversion factors that can safely be taken to be equal to 1 (k_B from energy to temperature, c from length to time, $\hbar$ from energy to frequency or from momentum to wavevector) unlike coupling strengths (as the electric charge unit e or the gravitation constant G), which are also fundamental constants but of a different nature. They are actually not constants (cf. renormalization group and the flow of "coupling constants") and their value has a meaning (the fine structure constant, which is essentially e^2 as $\alpha = \frac{e^2}{\hbar c} = e^2 \approx \frac{1}{137} \ll 1$, can obviously not be taken to equal one).

4.1.5 Summary on canonical quantization

Canonical quantization is the usual way of going from classical to quantum mechanics by turning Poisson brackets into commutators, here extended from mechanics to fields. One has to identify a field and its conjugate field and then to impose equal-time commutation relations. The field becomes a quantum operator. Other operators defined from the field (often bilinears in the fields such as the energy, the momentum, the angular momentum, etc.) have to be normal ordered in order to be unambiguously defined upon quantization. This procedure of normal ordering also removes unobservable infinites (such as the vacuum energy).

4.2 Complex scalar field and anti-particles

It is worth considering the complex scalar field $\phi(x)$ as it brings an essential novelty compared to the real field. The novelty is related to its $U(1)$ internal symmetry. The Lagrangian is $\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2\phi^*\phi$, with $\phi^*(x) \neq \phi(x)$. The conjugate field is now $\Pi = \partial_0\phi^*$. We impose ETCR, which have the same expression as before (exercise by writing all of them). Upon quantization the conjugate field $\Pi = \partial_0\phi^\dagger$.

The mode expansion is slightly different from the case of the real scalar field as one does not impose $\phi^\dagger(x) = \phi(x)$. Therefore

$$\phi(x) = \int_k \left(a(k)e^{-ik\cdot x} + b^\dagger(k)e^{ik\cdot x}\right) \tag{4.17}$$

where as usual $k^\mu = (\omega_k, \vec{k})$ here. The coefficient $a(k)$ and $b^\dagger(k)$ are both related to the field $\phi(x)$. Actually, they are related to its Fourier transform $\varphi(k)$ by $a(k) \equiv \varphi(k)$ and $b^\dagger(k) \equiv \varphi(-k)$. We introduced the notation b because here $\varphi^\dagger(-k)$ (i.e. $b(k)$) is different from $\varphi(k)$ (i.e. $a(k)$). In other words, a corresponds to plane waves at positive energy and b to plane waves at negative energy. And in the case of a complex field, the two are not identical, meaning that $b^\dagger(k) = a(-k) \neq a^\dagger(k)$.

On the $a(k)$ and $b(k)$ operators, the ETCR become

$$[a(k), a^\dagger(k')] = 2\omega_k(2\pi)^2\delta(\vec{k} - \vec{k}') = [b(k), b^\dagger(k')] \tag{4.18}$$

and all other commutators vanish. This last point is important, it concerns commutators involving a and a, those involving $a^\dagger$ and $a^\dagger$ but also those involving a and b. Don't consider these vanishing commutators as a triviality, otherwise you will face big difficulties when discussing fermions later. Remember also that in these commutators k_0 is always equal to ω_k.

Upon normal ordering, the Hamiltonian becomes

$$H = \int d^3x : \left(\partial_0\phi^*\partial_0\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + m^2\phi^*\phi\right) := \sum_{\vec{k}} \omega_k(a^\dagger_{\vec{k}}a_{\vec{k}} + b^\dagger_{\vec{k}}b_{\vec{k}}) = \int_k \omega_k(a(k)^\dagger a(k) + b(k)^\dagger b(k)) \tag{4.19}$$

(without normal ordering, the vacuum energy would be $\sum_{\vec{k}} \omega_k$) and the 3-momentum becomes

$$\vec{P} = \sum_{\vec{k}} \vec{k}(a^\dagger_{\vec{k}}a_{\vec{k}} + b^\dagger_{\vec{k}}b_{\vec{k}}) = \int_k \vec{k}(a^\dagger(k)a(k) + b^\dagger(k)b(k)) \tag{4.20}$$

The total number operator is

$$N = \sum_{\vec{k}}(a^\dagger_{\vec{k}}a_{\vec{k}} + b^\dagger_{\vec{k}}b_{\vec{k}}) = \int_k (a^\dagger(k)a(k) + b^\dagger(k)b(k)) = N_a + N_b \tag{4.21}$$

With these operators, we can again construct a Fock space. The novelty is that we have two types of particles, because for each $\vec{k}$ we have two modes with the same frequency ω_k (i.e. with the same mass m). We have two pairs of creation/annihilation operators i.e. $a^\dagger/a$ and $b^\dagger/b$. The vacuum is defined as being annihilated by all $a_{\vec{k}}$ and all $b_{\vec{k}}$. It has $H|vac\rangle = 0$, $\vec{P}|vac\rangle = 0$ and $N|vac\rangle = 0$.

Because the complex field has twice as many degrees of freedom as the real scalar field, it was expected that there would be twice as many modes. The two species of particles that appear are both quanta of the same field ϕ and have the same mass as a consequence of the $U(1)$ symmetry. What distinguishes them? Remember that associated to the $U(1)$ global internal symmetry, there is a classically conserved current $J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$. The corresponding classical charge is $Q = \int d^3x (i\phi^* \partial_0 \phi + c.c.)$. Upon quantization, it becomes

$$\begin{aligned} Q &= \int d^3x : (i\phi^\dagger \partial_0 \phi + h.c.) := \sum_{\vec{k}} : \left(a^\dagger_{\vec{k}} a_{\vec{k}} - b_{\vec{k}} b^\dagger_{\vec{k}} \right) := \sum_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}} \right) \\ &= \int_k (a^\dagger(k) a(k) - b^\dagger(k) b(k)) = N_a - N_b \end{aligned} \tag{4.22}$$

At this moment charge simply means difference in number of a-type and b-type particles, which is an integer. The vacuum is uncharged $Q|vac\rangle = 0$ thanks to normal ordering. A single a-type particle has a charge $Q|n_{a,\vec{k}} = 1\rangle = Q a^\dagger_{\vec{k}} |vac\rangle = +1 |n_{a,\vec{k}} = 1\rangle$, whereas a single b-type particle state has a charge $Q|n_{b,\vec{k}} = 1\rangle = Q b^\dagger_{\vec{k}} |vac\rangle = -1 |n_{b,\vec{k}} = 1\rangle$. Therefore the two types of particles are distinguished by their charge being $+1$ (a-type) or -1 (b-type). Type a is actually called a particle and type b is called an antiparticle. They are distinguished by their charge. And the total charge counts the total number of $+$ charge minus the total number of $-$ charges.

4.3 Microcausality

So far, we have concentrated on equal time commutation relation. We now want to consider the commutator of operators at different times. Let us first consider the commutator $[\phi(x), \phi^\dagger(y)]$ where $x - y$ is spacelike. By using the commutation relations of the annihilation and creation operators, we fnd:

$$[\phi(x), \phi^\dagger(y)] = D(x-y) - D^\star(x-y)$$

with

$$D(x) = \int_k e^{-ikx}$$

We recall that in these equations, the temporal part of the 4-vector k is $k^0 = \omega_k$.

To go further, we should make a change of variables in the previous equation. More explicitly, we will consider the new variable of integration $k'_x = \cosh\eta k_x + \sinh\eta \omega_k$, k_y and k_y being unchanged. A simple algebra shows that $\omega_{k'} = \cosh\eta \omega_k + \sinh\eta k_x$. All in all, $(\omega_k, \vec{k})$ transforms as a 4-vector under our change of variables. Moreover, $d^3k/\omega_k = d^3k'/\omega_{k'}$. This means that the measure is invariant under Lorentz transformation. This should not come as a surprise because the measure was introduced in a covariant way, as $d^4k\delta(k^2 - m^2)$...

If we perform this change of variables in the integral defining D above, we realize that it boils down to performing a Lorentz transformations of its coordinates. The function does not depend on coordinates but only on x^2! as a consequence, the function for spacelike interval $x - y$ boils down to the case $x^0 = y^0$. But the equal time commutator vanishes. We conclude that $[\phi(x), \phi^\dagger(y)] = 0$. Deriving the previous equation with respect to the time components x^0 and/or y^0, we also find that $[\phi(x), \Pi(y)] = [\Pi(x), \Pi(y)] = 0$, etc.

This property has important physical consequences! Recall that operators which commute have a special status in quantum mechanics. As you well know, if A and B do not commute, then a measure of A may influence the result of the measure of B. The fact that fields at events which are spacelike commute means that a measure of the field at one event cannot modify its measure at the other event. This generalizes to all local observables which are built from the fields and its derivatives. We conclude that a measure performed in a region of specetime cannot influence another measure performed out of the light cone of the first experiment. This is how causality enters into the game in quantum field theory.

4.4 Correlations in the vacuum

We conclude this chapter by some more physical observations in our theory. In quantum mechanics, as you well know, we can compute averages of some operators in a given state. We concentrate here on the vacuum properties and first look at

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = D(x-y) \tag{4.23}$$

where we have used the fact that the operators a and b annihilate the vacuum. We can interpret this equation by saying that the field $\phi^\dagger(y)$ creates a particle of type a, which is annihilated by $\phi(x)$. If the interval between the two events is spacelike, we can make a Lorentz transformation such that $x^0 = y^0$. For distant points, we find that the correlations decrease exponentially:

$$D(x) \sim e^{-mx} \qquad \text{if } x \text{ is spacelike and large.}$$

Reciprocally, if the interval is timelike, we perform a Lorentz transformation such that $\vec{x} = \vec{y}$. For distant times, we the correlations oscillates very fast:

$$D(x) \sim e^{imt} \qquad \text{if } x \text{ is timelike and large.}$$

Reciprocally, the correlation

$$\langle 0|\phi^\dagger(y)\phi(x)|0\rangle = D^\star(x-y) \tag{4.24}$$

can be interpreted as follows: a b excitation is created by the field $\phi(x)$, which is annihilated by $\phi^\dagger(y)$. The fact that the commutator $[\phi(x), \phi^\dagger(y)]$ vanishes results from a cancellation of contributions from the a and b particles. For this to happen, it is important that the a and b particles have the same mass. A model where the masses of a and b are different would not be causal. This implies that, in order to have causality in the theory, we need to have particles and associated antiparticles at the same time. The case of the real Klein-Gordon field is too simple in this sense because the particles have no charge and are therefore their own antiparticles!

Chapter 5

Space-time symmetries and geometrical objects

A. Zee: "The issue of symmetry is whether different observers perceive the same structure of physical reality."

The aim of the present chapter is to study symmetries of 3+1 dimensional space-time using group theory and representations. This will allow us to identify natural objects, that have well defined transformation properties in space-time (just as scalars, vectors, etc. are natural objects of 3d space). There is little physics in this chapter (except for the structure of spacetime) and the focus is mainly on group theory as a way to recognize well behaved geometrical objects. We start with a familiar example – that of the rotations of 3d space – in order to recall important concepts of group theory as applied to the study of symmetries (such as Lie group, Lie algebra, linear representations, etc).

5.1 Rotation group

5.1.1 $O(3)$ and $SO(3)$ groups

This is intended to be a warm up section with the aim of reviewing basic notions of groups and representations. We consider the 3d Euclidian space. The group of isometries (i.e. transformations preserving distances $dl^2 = dx^2 + dy^2 + dz^2$) of this space is the $O(3)$ group (here we do not consider translations), which is also that of orthogonal real 3×3 matrices. Let us see that. Let $\vec{r'} = R\vec{r}$ such that $\vec{r'} \cdot \vec{r'} = \vec{r} \cdot \vec{r}$ defines a rotation R for a position vector $\vec{r}$. The matrix R is 3×3 and encodes a linear transformation. For simplicity, we will also write it as $r' = Rr$. Then $r'^T r' = (Rr)^T Rr = r^T r$ so that $r^T R^T Rr = r^T r$ for all vector r. Therefore $R^T R = 1$, which shows that R is an orthogonal matrix. In addition by its definition as a rotation matrix $\vec{r'} = R\vec{r}$ it is obvious that it is a real matrix. Therefore $R \in O(3)$. Let us see that $O(3)$ has a group structure[1]. (i) If $R \in O(3)$ and $R' \in O(3)$, $(RR')^T(RR') = R'^T R^T RR' = R'^T R' = 1$ so that $RR' \in O(3)$: closure. (ii) If $R \in O(3)$ then $R^T \in O(3)$ and $R^T R = RR^T = 1$ so that $R^{-1} = R^T$: the inverse of R belongs to $O(3)$. (iii) The unit 3×3 matrix $\mathbb{I}$ is a neutral element as $\mathbb{I}R = R\mathbb{I} = R$ (for simplicity we will often write 1 instead of $\mathbb{I}$). (iv) $R(R'R") = (RR')R"$ (associativity). Therefore $O(3)$ is a group. It is

[1] In a pragmatic view, a group is a set G of elements g equiped with a composition law, the properties of which are specified by a multiplication table. For example the group $\mathbb{Z}_2 = \{1, -1; \times\}$ contains two elements $(1, -1)$ and has a composition law denoted by $\times$ that obeys the following multiplication table $\begin{array}{c|cc} \times & 1 & -1 \\ \hline 1 & 1 & -1 \\ -1 & -1 & 1 \end{array}$. Alternatively, the same group can also be written $\{0, 1; +\}$ with a multiplication table that reads $\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$. Its elements are the integers modulo 2 justifying its name $\mathbb{Z}_2$.

non-abelian (or non-commutative) as in general $RR' \neq R'R$, which is a well known property of matrices. As $\det(R^T R) = 1 = (\det R)^2$ and $\det R$ is real so that $\det R = \pm 1$. $SO(3)$ (special orthogonal group) is a subgroup made of $\det R = 1$ matrices (show it). The subset of $O(3)$ made of $\det R = -1$ matrices is not a subgroup (why?). $SO(3)$ is the group of proper rotations. It contains the identity $\mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ but not the space inversion[2] $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. $R \in SO(3)$ a priori depends on $3^2 = 9$ real elements but $R^T R = 1$ means that $R_{ki} R_{kj} = \delta_{ij}$ (summation over repeated indices implied), which gives 6 independent conditions (because the two equations $R_{ki} R_{kj} = 0$ when $i \neq j$ and $R_{kj} R_{ki} = 0$ are the same) so that in the end there are only 3 independent (real) parameters [3]. The group is therefore continuous and the three parameters can be taken as 3 angles: 2 angles to specify a direction in 3D space (an axis) and a last angle to specify the amount of rotation around that axis. When the elements of the group depend in a continuous and differentiable way on a set of real parameters, the group is called a Lie group. The parameter space of the $SO(3)$ Lie group is 3-dimensional and made of 3 angles.

Any element of $O(3)$ can be obtained as the product of an element of $SO(3)$ with either the identity $\mathbb{I}$ or the space inversion P [4]. $SO(3)$ is the part of $O(3)$ that is continuously connected to the identity. In the following we concentrate on $SO(3)$.

Generators of rotations and Lie algebra

To be concrete, we now build the rotation matrix $R_z(\psi)$ for a rotation of angle ψ around the z axis in the passive viewpoint (there is a single vector and two different frames). We consider a fixed vector $\vec{V}$ and describe it in two orthonormal frames $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ and $\{\vec{e'}_x, \vec{e'}_y, \vec{e'}_z\}$. In the first frame $\vec{V} = V_i \vec{e}_i$ and in the second $\vec{V} = V'_i \vec{e'}_i$ (summation over repeated indices). The second frame is rotated with respect to the first one around the z axis. As $\vec{e'}_x = \cos\psi \vec{e}_x + \sin\psi \vec{e}_y$, $\vec{e'}_y = \cos\psi \vec{e}_y - \sin\psi \vec{e}_x$ and $\vec{e'}_z = \vec{e}_z$, we obtain that the coordinates of the vector $\vec{V}$ transform as

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = R_z(\psi) \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \tag{5.1}$$

It is important to realize that basis vectors do not transform as vectors. In the present passive viewpoint, vectors do not transform at all $\vec{V'} = \vec{V}$, unlike their coordinates and basis vectors, which both transform in the same way. Indeed

$$\begin{pmatrix} \vec{e'}_x \\ \vec{e'}_y \\ \vec{e'}_z \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} \tag{5.2}$$

[2] Space inversion and parity are not exactly the same thing. Space inversion or point reflection means that $\vec{r} = (x, y, z) \to -\vec{r} = (-x, -y, -z)$. Parity means that the direction of space is inverted, which can be realized by reverting either one – as in $(x, y, z) \to (-x, y, z)$ – or three – as in $(x, y, z) \to (-x, -y, -z)$ – components of a vector. Therefore parity needs to be clearly defined. In 3 space dimensions, one usually takes parity to be the same as space inversion $\vec{r} = (x, y, z) \to -\vec{r} = (-x, -y, -z)$. But the general definition of parity is that of the change of sign of a single component. Physically, it corresponds to a mirror reflection. For example, in 2 space dimensions, one should carefully distinguish space inversion – $(x, y) \to (-x, -y)$ – from parity – either $(x, y) \to (-x, y)$ or $(x, y) \to (x, -y)$. Indeed, in 2 space dimensions, space inversion is equivalent to a rotation by π around an axis perpendicular to the xy plane and does not revert the direction of space. Whereas parity does revert the direction of space. Parity should have $\det P = -1$.

[3] The fact that the determinant be $+1$ rather than -1 allows one to choose a sign but does not lead to an extra independent linear equation that would reduce the number of independent parameters.

[4] The zeroth homotopy group $\Pi_0(V)$ of a manifold V is the set of connected components. We have $\Pi_0(O(3)) = \mathbb{Z}_2$ and $\Pi_0(SO(3)) = 0$.

In an infinitesimal rotation $\psi \to 0$, $R_z(\psi) \approx \begin{pmatrix} 1 & \psi & 0 \\ -\psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 + i\psi J_z$ with $J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -i\frac{d}{d\psi}R_z(\psi)|_{\psi\to 0}$, which is called the generator of rotations around the z axis. For a finite rotation $R_z(\psi) = \lim_{N\to\infty}\left[R_z(\psi/N)\right]^N = \lim_{N\to\infty}(1 + iJ_z\psi/N)^N = e^{i\psi J_z}$.

As an exercise, show that for a rotation of angle ψ around x, one obtains $R_x(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix} = e^{i\psi J_x}$ with the generator $J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$. And for a rotation around y, $R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix} = e^{i\psi J_y}$ with the generator $J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$.

In the end, there are as many generator (J_x, J_y, J_z) as independent continuous parameters of the Lie group. The generators are hermitian matrices $J_i^\dagger = J_i$ with $i = 1, 2, 3$ for x, y, z. You can check that, $(J_i)_{jk} = -i\epsilon_{ijk}$ where i indicates the generator (either $J_1 = J_x$ or $J_2 = J_y$ or $J_3 = J_z$), j is the row index and k the column index of the matrix. And ϵ_{ijk} is the fully antisymmetric tensor such that $\epsilon_{123} = +1$ (as an exercise show that only 6 out of its 27 elements are non-zero and try to represent this tensor as a "3D matrix"). The generators do not commute but satisfy the following commutation relation $[J_x, J_y] = iJ_z$ or more generally

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{5.3}$$

where ϵ_{ijk}.

This structure enables us to build what is called a Lie algebra. We consider the set of matrices obtained by multiplying the commutators by some real number: $\theta_i J_i$. This set is a vector space (summing to elements, or multiplying an element by a real number, we remain in this set). We have one more operation at our disposal: taking two elements m_1 and m_2 in the vector space, we can obtain a third by applying the commutator. This last property upgrades our vector space to a Lie algebra. In order to distinguish the Lie group $SO(3)$ from its algebra, the latter is usually written $so(3)$.

A general proper rotation[5] can be written as

$$R_{\vec{n}}(\psi) = R(\vec{\psi}) = e^{i\psi\vec{n}\cdot\vec{J}}, \tag{5.5}$$

where the unit vector $\vec{n}$ – with coordinates $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$ – defines the rotation axis and ψ the rotation angle (sometimes with the notation $\vec{\psi} = \psi\vec{n}$). Note that this is not the same as $e^{i\psi n_x J_x}e^{i\psi n_y J_y}e^{i\psi n_z J_z}$ (to convince yourself, think of the Baker-Campbell-Haussdorff formula for the product of exponentials of two operators that do not commute: $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])+\cdots} \neq e^{X+Y}$).

Parameter space and the topology of $SO(3)$

Because $R_{\vec{n}}(\psi + \pi) = R_{-\vec{n}}(\pi - \psi)$ (show it), the rotation angle is such that $0 \le \psi \le \pi$ and not $0 \le \psi \le 2\pi$. The three real parameters needed to specify a proper rotation are therefore (ψ, θ, ϕ) to be taken in the parameter space $[0, \pi] \times [0, \pi] \times [0, 2\pi]$. A Lie group can also be seen as a differentiable manifold, that can

[5] A general formula due to B. Olinde Rodrigues is (here we are momentarily using the active viewpoint)

$$\vec{x}' = R_{\vec{n}}(\psi)\vec{x} = \cos\psi\,\vec{x} + (1-\cos\psi)(\vec{x}\cdot\vec{n})\vec{n} + \sin\psi(\vec{n}\times\vec{x}) \tag{5.4}$$

where $\vec{x} = \vec{x}_\parallel + \vec{x}_\perp = (\vec{x}\cdot\vec{n})\vec{n} + [\vec{x} - (\vec{x}\cdot\vec{n})\vec{n}]$ and $\vec{x}' = \vec{x}_\parallel + \cos\psi\,\vec{x}_\perp + \sin\psi\,(\vec{n}\times\vec{x}_\perp)$. Note that $\vec{n} = \vec{x}_\parallel/x_\parallel$, $\vec{x}_\perp/x_\perp$ and $\vec{n}\times\vec{x}_\perp/x_\perp$ form a direct orthonormal basis. The Rodrigues formula clearly shows that the rotation matrix only depends on $\cos\psi$ and $\sin\psi$ (and not on $\cos\frac{\psi}{2}$ and $\sin\frac{\psi}{2}$ for example: this will play an important role later). In other words, ψ only matters modulo 2π. As an exercise, write explicitly the rotation matrix for a general rotation parametrized by (ψ, θ, ϕ).

be characterized topologically. The parameters of the Lie group live in a parameter space which is roughly a ball (i.e. a filled sphere S^2 in $\mathbb{R}^3$) of radius π when ψ is interpreted as a radial coordinate and (θ, ϕ) as spherical coordinates. It can also be seen as a kind of S^3 sphere in $\mathbb{R}^4$, but see below. The Lie group $SO(3)$ is said to be compact as $[0, \pi] \times [0, \pi] \times [0, 2\pi]$ is compact (as it is a closed and bounded interval of $\mathbb{R}^3$). It is connected as the parameter space is made of a single piece. Let's take a closer look at what happens on the surface of this ball of radius π. As $R_{\vec{n}}(\pi) = R_{-\vec{n}}(\pi)$ (corresponding to $\psi = 0$ in the above equation $R_{\vec{n}}(\psi + \pi) = R_{-\vec{n}}(\pi - \psi)$) for any unit vector $\vec{n}$, antipodal points on the surface of the ball have to be identified. Then this parameter space is not simply connected. Actually, it is doubly connected: there are two homotopy classes (two classes of closed paths that can not be continuously deformed into one another). One class uses the identification of opposite points on the surface of the ball (paths in this class can not be smoothly contracted to a null path), the other does not (paths in this second class can be smoothly deformed to the null path, i.e. the identity). The fundamental group of $SO(3)$ is therefore $\mathbb{Z}_2$, which is usually written as $\Pi_1(SO(3)) = \mathbb{Z}_2$. $SO(3)$ as a topological manifold is similar to the projective space RP^3, which is the 3-sphere S^3 with antipodal points identified. This is usually written as $SO(3) \approx RP^3 \approx S^3/\mathbb{Z}_2$.

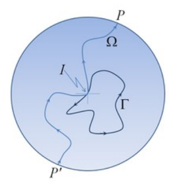

Figure 5.1: This picture illustrates the topology of the $SO(3)$ manifold. It is roughly a ball (a filled sphere S^2) of radius π, but antipodal points on the surface of the sphere (such as P and P') have to be identified. There are therefore two homotopy classes (i.e. the fundamental homotopy group $\Pi_1(SO(3)) = \mathbb{Z}_2$). Image taken from http://physics.stackexchange.com/questions/76096/lie-groups-and-group-extensions.

For more information on homotopy groups and basic notions of topology, you can consult Ryder [3] (section on "Topology of the vacuum: the Bohm-Aharonov effect") or Altland and Simons [6] (section 9.2 on homotopy groups in chapter 9 "Topology") or J. Sethna [7] (section "IV. Classify the topological defects") or Nakahara [8]. We will essentially need the homotopy groups of the sphere [9]. You should understand statements such as $\Pi_1(S^1) = \mathbb{Z}$ (winding number), $\Pi_1(S^2) = 0$, $\Pi_2(S^2) = \mathbb{Z}$ (wrapping number), etc.

5.1.2 Representations of $SO(3)$

First, we review a few results from representation theory. A group can be thought of as an abstract object[6] $(G, \cdot)$ made of a set G of elements g and a composition law (symbolized by the dot $\cdot$) with closure, inverse, neutral element and associativity. Now, we would like to know how a group of transformations acts on physical quantities. We therefore introduce linear[7] representations of this group. Let's take a physical object with n components (for example the electric field with 3 components). A representation built in this representation space (or base space) is such that for each $g \in G$ there is an $n \times n$ matrix $T(g)$ acting on physical quantities such that $T(g_1 g_2) = T(g_1)T(g_2)$. In other words:

$$T : G \quad \rightarrow \quad V \tag{5.6}$$

$$g \quad \rightarrow \quad T(g) \tag{5.7}$$

where V is a linear (vector) space called the representation space and $T(g)$ is a matrix (if V is of finite dimension) or a linear operator (if V is infinite dimensional). A representation such that $T(g_1 g_2) = T(g_1)T(g_2)$ is

[6] Here, by calling the group an abstract object, we want to emphasize that having constructed $SO(3)$ from 3×3 matrices, we should now forget this construction and think of this group as an abstract group. Its elements are no longer matrices but abstract elements that obey a certain "multiplication table".

[7] Why do we restrict ourselves to linear representations? See [1] for some ideas about this question.

called a faithful representation, whereas one such that $T(g_1g_2) = e^{i\phi(g_1,g_2)}T(g_1)T(g_2)$ is called a projective representation (or a representation up to a phase).

An irreducible representation is such that it contains no non-trivial subspace of V which is left invariant by all $T(g)$. Otherwise, a representation is said to be reducible. For example, for $SO(3)$ a quadruplet (a, V_x, V_y, V_z) made of a scalar a and a vector $\vec{V}$ defines a reducible representation. For a compact group such as $SO(3)$, we can restrict ourselves to irreducible representations which are unitary and of finite dimension (see [1] encadrés I and III). Then $T(g)$ are unitary matrices (i.e. $T(g)^{-1} = T(g)^\dagger$) and the corresponding generators are hermitian. For a non-compact group, there are no unitary representations of finite dimension (apart from the trivial one).

A representation of the Lie group can be obtained from a representation of its Lie algebra. As in the case of the SO(3) group, it is convenient to refer to the Lie algebra associated with a Lie group G by the lower case symbol g. The strategy is as follows. suppose we find 3 matrices j_i which fulfill the same commutation relations as the J_i: $[j_i, j_j] = i\epsilon_{ijk}j_k$. Note that these matrices need not be 3×3. We then build the function

$$T_{\hat{n}}(\theta) = e^{i\theta j_i \hat{n}_i}$$

by analogy with the exponential formula used to represent the rotation matrix associated with a ψ angle around the direction $\hat{n}$. This gives us a representation of the rotation group. This can be shown by using the Baker-Campbell-Hausdorff formula, that we quote again

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+\cdots}$$

where the dots involve more and more commutators. Let us apply this formula to the rotation matrices

$$\begin{aligned}
R_{\hat{n}}(\theta).R_{\hat{m}}(\psi) &= e^{i\theta n_i J_i} e^{i\psi m_i J_i} && (5.8)\\
&= e^{iJ_i(\theta n_i + \psi m_i + \frac{1}{2}\theta\psi\epsilon ijk n_i m_j + \cdots)} && (5.9)\\
&= e^{iJ_i \tau l_i} && (5.10)\\
&= R_{\hat{l}}(\tau) && (5.11)
\end{aligned}$$

This formula formally gives us the multiplication table of two rotation matrices and although it is difficult to determine explicitly the angle τ and direction $\hat{l}$ resulting from the product of two rotations, we know at least how to define these quantities.

Let us repeat the discussion with the T's. Actually, since the commutation relations for the j and the J are the same, we can readily write that:

$$T_{\hat{n}}(\theta).T_{\hat{m}}(\psi) = T_{\hat{l}}(\tau)$$

where τ and $\hat{l}$ are exactly those already computed! This concludes our proof that indeed T is a representation of the rotation group.

The conclusion is that finding a representation of the group boils down to finding a set of 3 matrices with the right commutation relations, which is much more simple![8].

There is however a piece of information that may be lost in this procedure. On the one hand, the Lie group $SO(3)$ made of elements $R_{\vec{n}}(\psi) = e^{i\psi\vec{n}\cdot\vec{J}}$ has a global structure (for example, we also discussed it as a topological manifold being doubly connected). On the other hand, the Lie algebra $so(3)$ defined by $[J_i, J_j] = i\epsilon_{ijk}J_k$ only reflects the local structure of the group $SO(3)$ close to the identity $\mathbb{I}$ but not its global structure[9].

We now concentrate on irreducible representations of the Lie algebra $so(3)$. We already said that these representations are unitary and of finite dimension.

[8]Modulo the fact that two groups (e.g. $SO(3)$ and $SU(2)$) can have the same algebra but different faithful representations. In addition, this statement turns out to be true only for compact groups. See below.

[9]We will soon see that $SU(2)$ is a different group from $SO(3)$ – the former is simply connected $\Pi_1(SU(2)) = 0$, while the latter is doubly connected $\Pi_1(SO(3)) = \mathbb{Z}_2$ – but it has the same Lie algebra $su(2) = so(3)$.

Dimension 1

Let us start by trying to construct a representation of dimension 1. $[J_i, J_j] = i\epsilon_{ijk}J_k$ can be satisfied by 1×1 matrices $J_i = 0$ so that $R(\vec{\psi}) = e^{i\vec{\psi}\cdot\vec{J}} = 1$. This is the scalar (or trivial) representation. A scalar quantity s transforms indeed as $s' = 1s = s$.

Dimension 3

We already found a representation of dimension 3 when first constructing the rotation group. The generators are 3×3 matrices such that $(J_i)_{jk} = -i\epsilon_{ijk}$. This is the vector (or fundamental or defining) representation of $SO(3)$. Hence, for us a vector is now a triplet of numbers (V_x, V_y, V_z) that transforms under a rotation according to $\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = e^{i\vec{\psi}\cdot\vec{J}} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ where $e^{i\vec{\psi}\cdot\vec{J}}$ is a 3×3 matrix here. Remember that a triplet of numbers that has no specified transformation property under spatial rotations is just a triplet of numbers, not a vector.

Dimension 2

M. Atiyah: "No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors."

Let us now try to find a two dimensional representation of the Lie algebra. Pauli matrices are 2×2 hermitian matrices that do satisfy the $so(3)$ algebra (up to a a factor 2, see below). We recall that $\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the three Pauli matrices. Check that indeed $[\sigma_i/2, \sigma_j/2] = i\epsilon_{ijk}\sigma_k/2$. Another property of Pauli matrices is that they anticommute. Show that $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ (this last property defines a Clifford algebra, which we will encounter again when discussing the Dirac equation). The corresponding rotation matrix is

$$U(\vec{\psi}) = e^{i\vec{\psi}\cdot\vec{\sigma}/2} = \cos\frac{\psi}{2}\sigma_0 + i\sin\frac{\psi}{2}\vec{n}\cdot\vec{\sigma} \tag{5.12}$$

where $\vec{\psi} = \psi\vec{n}$ and $\sigma_0 = \mathbb{I}$ is here the 2×2 unit matrix and $\sigma_i/2$ are the generators for the representation of dimension 2. Note that, unlike $R(\vec{\psi})$ in the representation of dimension 3, $U(\vec{\psi})$ depends on $\cos\frac{\psi}{2}$ and $\sin\frac{\psi}{2}$, rather than on $\cos\psi$ and $\sin\psi$. The matrix $U(\vec{\psi})$ is a 2×2 complex matrix that depends on three real parameters $(\psi, \vec{n}(\theta,\phi))$. In addition $U(\vec{\psi})^\dagger U(\vec{\psi}) = 1$ and $\det U(\vec{\psi}) = +1$, which means that $U(\vec{\psi}) \in SU(2)$ the group of special (det$= +1$) unitary matrices.

The $SU(2)$ Lie group is different from $SO(3)$. It has the same local structure (i.e. the same Lie algebra) as $SO(3)$ but a different global structure. Indeed, for each element of $SO(3)$, there are actually two elements of $SU(2)$. Let's see that. Consider $\vec{\psi} = \psi\vec{n}$ and $\underline{\vec{\psi}} = \underline{\psi}\vec{n} = (\psi + 2\pi)\vec{n}$. Then $U(\underline{\vec{\psi}}) = e^{i\underline{\vec{\psi}}\cdot\vec{\sigma}/2} = \cos\frac{\underline{\psi}}{2} + i\sin\frac{\underline{\psi}}{2}\vec{n}\cdot\vec{\sigma} = -e^{i\vec{\psi}\cdot\vec{\sigma}/2} = -U(\vec{\psi})$, whereas $R(\underline{\vec{\psi}}) = e^{i\underline{\vec{\psi}}\cdot\vec{J}} = e^{i\vec{\psi}\cdot\vec{J}} = R(\vec{\psi})$. This is a consequence of $e^{i\underline{\psi}} = e^{i\psi}$ and $e^{i\underline{\psi}/2} = -e^{i\psi/2}$. The parameter space of $SU(2)$ is $(\psi,\theta,\phi) \in [0,2\pi]\times[0,\pi]\times[0,2\pi]$ as $U_{\vec{n}}(\psi+2\pi) = -U_{\vec{n}}(\psi) = U_{-\vec{n}}(2\pi - \psi)$ (compare with the discussion of $R_{\vec{n}}(\psi)$). This parameter space is twice as large as that of $SO(3)$: for each element $R_{\vec{n}}(\psi)$ of $SO(3)$ defined by the triplet $(\psi,\theta,\phi) \in [0,\pi]\times[0,\pi]\times[0,2\pi]$, there are two distinct elements $U_{\vec{n}}(\psi)$ and $U_{\vec{n}}(\psi+2\pi) = -U_{\vec{n}}(\psi)$ of $SU(2)$ corresponding to the triplets of parameters (ψ,θ,ϕ) and $(\psi+2\pi,\theta,\phi) \equiv (2\pi-\psi, \pi-\theta, \phi+\pi)$. In summary $\pm U \in SU(2) \leftrightarrow R \in SO(3)$.

As a manifold $SU(2) \approx S^3$ which is compact and simply connected (i.e. the fundamental homotopy group $\Pi_1(S^3) = 0$)[10]. $SU(2)$ is known as the universal cover of $SO(3)$ which is written as $\overline{SO(3)} = SU(2)$.

[10] To see that directly, note that a generic $SU(2)$ matrix can be written $U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ where a and b are complex

The representation we have just found is a faithful representation of $SU(2)$ but not of $SO(3)$. Let's see that it is actually a projective representation of $SO(3)$. Take the following two elements of $G = SO(3)$: $g_1 = R_z(\pi)$ and $g_2 = R_z(\pi)$, then $g_1 \cdot g_2 = R_z(2\pi) = \mathbb{I} = e$ but $U(g_1)U(g_2) = e^{i\pi\sigma_z/2}e^{i\pi\sigma_z/2} = e^{i\pi\sigma_z} = -\mathbb{I} = -U(e)$ where e is the neutral element of the group. Here the phase in $T(g_1)T(g_2) = e^{i\phi(g_1,g_2)}T(g_1g_2)$ is π.

This representation is called the spinor (or fundamental or defining) representation of $SU(2)$. Note how by allowing projective representations, we have moved from the study of the group $SO(3)$ to that of another group, $SU(2)$. Objects that transform according to this representation are called spinors. They are doublets of complex numbers (z_1, z_2) that transform according to

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = e^{i\vec{\psi}\cdot\vec{\sigma}/2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = U(\vec{\psi}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{5.13}$$

Just like any triplet of numbers is not a vector, any doublet is not a spinor. A characteristic feature of a spinor is that it gets a minus sign in a rotation of 2π and only comes back to itself after a 4π rotation. Classical objects that show such a behavior can be constructed with a belt or scissors or a glass of water (show some of the tricks in the classroom: plate, Dirac's string or belt, Balinese cup, etc.). Note how in the present discussion of representations of the rotation group, spinors appear as geometrical objects that *a priori* have nothing to do with quantum physics. Spinors were actually discovered by the mathematician Élie Cartan in 1913 when studying linear representations of groups, before they appeared in quantum mechanics with Wolfgang Pauli in 1927. The connection between spinors and quantum mechanics comes from allowing for the presence of representations *up to a phase*.

Tensor product

Tensor products allow one to construct representations of higher dimension from representations of smaller dimension. Let's study a concrete example: that of the tensor product of two vectors (i.e. two representations of spin 1). Let $\vec{V}$ be a vector, i.e. its coordinates transform as $V_i' = R_{ij}V_j$ with $R = e^{i\vec{\psi}\cdot\vec{J}}$. Let T_{ij} be an array of 9 numbers. If it transforms as $T_{ij}' = R_{ik}R_{jl}T_{jl}$, then it is said to be a rank 2 tensor. For example, $T_{ij} = V_iV_j$. This defines a reducible representation of $SO(3)$ of dimension 9. From your knowledge of the composition of angular momentum in quantum mechanics[11], you know that composing two spin 1 (i.e. vectors) gives rise to a spin 0 (i.e. scalar), a spin 1 and a spin 2. In terms of dimensions of the representations, this is usually written as $3\otimes 3 = 1\oplus 3\oplus 5$, which means that the tensor product of two vector representations is reducible and decomposes into the direct sum of 3 irreducibles representations (irreps): one of dimension 1 (spin 0, scalar, trace of the tensor), one of dimension 3 (spin 1, vector, antisymmetric part of the tensor) and one of dimension 5 (spin 2, traceless symmetric part of the tensor).

Another example is the composition of two spin 1/2 representations: $2 \otimes 2 = 1 \oplus 3$. It is a reducible 4-dimensional representation that splits into a one-dimensional (spin 0) irrep and a three-dimensional (spin 1) irrep. We can continue this game and compose a spin 1/2 with a spin 1: $2 \otimes 3 = 2 \oplus 4$. It is a reducible six-dimensional representation that splits into a two-dimensional irrep (spin 1/2) and a four-dimensional irrep (spin 3/2). Repeating this procedure, we can obtain all the irreps of $SU(2)$.

Casimir operator

A Casimir is an operator that commutes with all generators of the Lie group. In each irrep, it is proportional to the identity (Schur's lemma). Its eigenvalues are used to label the irrep. For the $so(3) = su(2)$ algebra, $\vec{J}^2 = J_iJ_i$ is the only Casimir operator. Check that $[\vec{J}^2, J_i] = 0$ and that $\vec{J}^2 = j(j+1)\mathbb{I}$ in the j^{th} irrep of $SU(2)$. An irrep of $SU(2)$ (or of $SO(3)$) is therefore labeled by its spin j.

numbers such that $|a|^2 + |b|^2 = 1$ (indeed $UU^\dagger = 1$ and $\det U = +1$). Writing $a = x + iy$ and $b = z + it$, we see that $x^2 + y^2 + z^2 + t^2 = 1$ which is the definition of a sphere S^3 of radius 1 in $\mathbb{R}^4$. This shows that $SU(2) \approx S^3$. As a side remark, note that a general $U(2)$ matrix can be written $U = e^{i\chi}e^{i\psi\vec{n}\cdot\vec{\sigma}/2} = e^{i\chi}[\cos\frac{\psi}{2} + i\sin\frac{\psi}{2}\vec{n}\cdot\vec{\sigma}] = e^{i\chi}[\gamma_0 + i\vec{\gamma}\cdot\vec{\sigma}]$ where γ_0 and $(\gamma_x, \gamma_y, \gamma_z)$ all $\in \mathbb{R}$ such that $\gamma_0^2 + \vec{\gamma}^2 = 1$. First, one recognizes the above four real numbers such that the sum of their square equals one. Second, one sees that $U(2) \approx U(1) \times SU(2)$ where $e^{i\chi}$ is the $U(1)$ phase.

[11] When composing a spin j_1 with a spin j_2, one obtains all the spins in between $|j_1 - j_2|$ and $j_1 + j_2$.

Summary

All irreducible representations (faithful or projective) of $SO(3)$ can be found as faithful irreps of $SU(2)$. Irreps of $SU(2)$ are labelled by a half integer $j = 0, 1/2, 1, 3/2, ...$, which is known as the spin of the representation. The dimension of the representation is $2j + 1$.
$j = 0$: spin 0, scalar, dimension = 1
$j = 1/2$: spin 1/2, spinor, dimension = 2
$j = 1$: spin 1, vector, dimension = 3
$j = 3/2$: spin 3/2, dimension = 4
$j = 2$: spin 2, traceless symmetric rank 2 tensor, dimension = 5
....
Faithful representations of $SO(3)$ are that of integer spin $j \in \mathbb{N}$. Projective representations of $SO(3)$ are that of $\frac{1}{2}+$ integer spin $j = 1/2, 3/2, ...$

Remark: from the above construction, keep in mind the difference between a rank 2 tensor and a spin 2. The former corresponds to 9 numbers and induces a reducible representation that splits into the direct sum of three irreps: its trace is a scalar (spin 0), its antisymmetric part is a vector (spin 1) and its traceless symmetric part is a spin 2. $T = \frac{\mathrm{tr}T}{3}\mathbb{I} + A + S$ where $A_{ij} = (T_{ij} - T_{ji})/2$ and $S_{ij} = (T_{ij} + T_{ji})/2 - \frac{\mathrm{tr}T}{3}$.

Space inversion (parity)

Space inversion P does not belong to $SO(3)$ but to $O(3)$. It further distinguishes between a true and a pseudo-scalar or between a true (or polar) and a pseudo- (or axial) vector. Let $\vec{V}$ be a true vector. Under P it transforms as $V_i \to -V_i$. With two vectors $\vec{V}$ and $\vec{W}$, we can make the scalar product $\vec{V}\cdot\vec{W}$ which transforms as $V_iW_i \to (-V_i)(-W_i) = V_iW_i$, which defines a true scalar. We can also consider the cross product $\vec{V} \times \vec{W}$ which transforms as $\epsilon_{ijk}V_jW_k \to \epsilon_{ijk}(-V_j)(-W_k) = \epsilon_{ijk}V_jW_k$, which is therefore not a true vector but a pseudo-vector. The mixed product $(\vec{V} \times \vec{W}) \cdot \vec{U}$ of three vectors transforms as $\epsilon_{ijk}V_jW_kU_i \to -\epsilon_{ijk}V_jW_kU_i$, which is a pseudo-scalar.

If one is interested in going further in this direction of refining the classification of “vectors”, I suggest reading J. Hlinka, Phys. Rev. Lett. **113**, 165502 (2014) (see arXiv:1312.0548), who considers the direct product of $SO(3)$ and $\{\mathbb{I}, P, T, PT\}$ (where P is the space inversion and T the time reversal operator) to build 8 types of directional quantities (polar vector, axial vector, nematic director, etc.) and 4 scalar quantities (time-even scalar, time-even pseudo-scalar, time-odd scalar, time-odd pseudo-scalar).

Question: studying the projective representations of $O(3)$ (i.e. $SO(3)$ and parity P), is it possible to discover the existence of two types of spinors (left-chiral spinors and right-chiral spinors)? These two types of spinors are discussed in the next section and appear as projective irreps of $\mathcal{L}_+^\uparrow$.

5.1.3 Lorentz algebra

The Lorentz group is a Lie group. It is somewhat similar to $SO(4)$ – the group of rotations in a 4-dimensional Euclidian space. The latter has 6 independent rotations (in the planes x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3 and x_2x_3) and therefore 6 generators. Another way to see it is to realize that a 4×4 real matrix has 16 real parameters but that the defining condition $\Lambda^T\eta\Lambda = \eta$ gives 10 independent constraints leaving 6 independent parameters. An important difference with $SO(4)$, that will reveal crucial when contrsucting representations, is that the Lorentz group, which also has 6 generators, is non-compact.

Lorentz boost

To be more concrete, we now construct specific Lorentz transformations in order to obtain the generators and their algebra. Among the 6 independent transformations, there are 3 space rotations in the planes xy, xz and yz and 3 boosts (changes of inertial frame) in the planes tx, ty and tz. As the rotations are similar to the ones studied in $SO(3)$, we focus on a boost in the tx plane. This is a transformation $\underline{x}' = \Lambda\underline{x}$ such that $y' = y$, $z' = z$ and only t and x mixes. In addition, we know that Λ is linear so that t' and x' are linear combinations of t and x. We also know that the transformation preserves intervals (isometry) so that $t'^2 - x'^2 = t^2 - x^2$. In addition $\det \Lambda = +1$ (proper) and $\Lambda^0{}_0 \geq 1$ (orthochronous)[12]. One finds that $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ reads

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{5.14}$$

which is parametrized by a single quantity called the rapidity ϕ. An important point is that $\phi \in \mathbb{R}$ is unbounded and not closed (unlike rotations which are parametrized by an angle $\psi \in [0, \pi]$). The rapidity ϕ is not an angle. Therefore the Lorentz group is non-compact. Physically, this transformation describes a boost along the x direction with a velocity $v = \tanh\phi$. We characterize it by $\vec{\phi} = \phi \vec{e}_x$. We have $x' = \cosh\phi x - \sinh\phi t$ so that the origin of the primed frame at $x' = 0$ moves such that $x = \tanh\phi \times t$ showing that the velocity is indeed $\tanh\phi$ [13]. To make the connection with standard notations of special relativity, note that $\cosh\phi = \gamma = 1/\sqrt{1-v^2}$ [14] and $\sinh\phi = \gamma v = v/\sqrt{1-v^2}$. In the non-relativistic limit $v/c = \tanh\phi \approx \phi \ll 1$, we find that $\begin{pmatrix} ct' \\ x' \end{pmatrix} \approx \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$ i.e. $t' \approx t - vx/c^2 \approx t$ and $x' \approx x - vt$ as expected for a Galilean boost (time is absolute $t' = t$ and velocities simply add $x'/t = x/t - v$).

Generators

We can now obtain the boost generator K_x from

$$K_x \equiv -i \frac{d\Lambda}{d\phi}|_{\phi=0} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.15}$$

which is anti-hermitian $K_x^\dagger = -K_x$ [15]. Similarly, one can work out the boosts in the y and z directions and obtain the corresponding generators:

$$K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{5.16}$$

The three rotation generators are already known from our study of $SO(3)$ (excluding the first row and column of the following 4×4 matrices, you should recognize the 3×3 submatrices $(J_i)_{jk} = -i\epsilon_{ijk}$):

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \; J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \text{ and } J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.17}$$

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[12] An alternative way is so search for a matrix $\Lambda^\mu{}_\nu = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ such that $\Lambda^T \eta \Lambda = \eta$.

[13] If one takes $\vec{v} = v\vec{e}_x$ as the parameters defining the boost, then $v \in]-1, 1[$ which is bounded but not closed and hence not compact. Physically, it means that it is not possible to go in an inertial frame moving at the velocity of light $|v| = 1$ compared to another inertial frame.

[14] The parameter $\gamma = 1/\sqrt{1-v^2}$ is the usual factor describing time dilatation – a moving clock runs slower – and length contraction in the direction of motion.

[15] In the literature, you will also find definitions such that the generator is hermitian, having absorbed an i factor.

One has $K_i^\dagger = -K_i$ and $J_i^\dagger = J_i$. The parameter space of the Lorentz group is made of 3 rapidities and 3 angles[16] such that $(\phi_x, \phi_y, \phi_z, \theta_x, \theta_y, \theta_z) \in \mathbb{R}^3 \times [0,\pi]^3$. The parameter space is non-compact (it is unbounded and not closed).

Lorentz algebra

The Lie algebra $\mathfrak{L}$ of the Lorentz group $\mathcal{L}$ can be obtained from computing the commutators of the generators. Check that $[J_x, J_y] = iJ_z$, $[J_x, K_y] = iK_z$ and $[K_x, K_y] = -iJ_z$. The Lorentz algebra is therefore[17]

$$\begin{aligned} \left[J^i, J^j\right] &= i\epsilon^{ijk} J^k & (5.18)\\ \left[J^i, K^j\right] &= i\epsilon^{ijk} K^k & (5.19)\\ \left[K^i, K^j\right] &= -i\epsilon^{ijk} J^k & (5.20) \end{aligned}$$

The first equation means that rotation generators form a closed sub-algebra (which we recognize as $so(3)$). The second line means that the triplet (K_x, K_y, K_z) transforms as a vector (here meaning a 3-vector, a vector under space rotations). Therefore the first equation also means that the triplet (J_x, J_y, J_z) itself transforms as a vector. Hence the notations $\vec{K}$ and $\vec{J}$. The third equation is more subtle: it means that boosts do not form a closed sub-algebra. The fact that the commutator of two boosts is related to a rotation in the third space direction is at the origin of the Thomas precession effect. This is a relativistic effect.

An infinitesimal Lorentz transformation parametrized by $(\vec{\phi} \to 0, \vec{\theta} \to 0)$ is therefore $\Lambda = 1 + i\vec{\phi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{J}$. Using the usual trick $\lim_{N\to\infty}(1 + x/N)^N = e^N$, we find for a finite transformation that[18]

$$\Lambda = \exp\left(i\vec{\phi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{J}\right) . \tag{5.21}$$

Covariant notation

It may be disturbing that the equation $\Lambda = 1 + i\vec{\phi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{J}$ is not written itself in a covariant fashion (indeed $\vec{\phi}\cdot\vec{K}$ is the scalar product in 3d Euclidian space, therefore $\vec{\phi}\cdot\vec{K}$ is a 3-scalar but not a 4-scalar). For an infinitesimal Lorentz transformation (i.e. close to the identity), we can write $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu + \omega^\mu{}_\nu$. Comparing $\omega^\mu{}_\nu$ with $\Lambda - 1 = i\vec{\phi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{J}$, we find that $\omega^\mu{}_\nu = \begin{pmatrix} 0 & -\phi^1 & -\phi^2 & -\phi^3 \\ -\phi^1 & 0 & \theta^3 & -\theta^2 \\ -\phi^2 & -\theta^3 & 0 & \theta^1 \\ -\phi^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}$ so that $\omega_{\mu\nu} \equiv \eta_{\mu\sigma}\omega^\sigma{}_\nu = \begin{pmatrix} 0 & -\phi^1 & -\phi^2 & -\phi^3 \\ \phi^1 & 0 & -\theta^3 & \theta^2 \\ \phi^2 & \theta^3 & 0 & -\theta^1 \\ \phi^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix} = -\omega_{\nu\mu}$. This antisymmetric rank 2 tensor contains the 6 parameters specifying a generic Lorentz transformation. We define $J^{\sigma\rho}$ such that $\Lambda = 1 - i\omega_{\sigma\rho}J^{\sigma\rho}/2$ when $\omega_{\sigma\rho} \to 0$. It can be chosen to be antisymmetric as well because any symmetric part would vanish in the contraction with the antisymmetric tensor $\omega_{\sigma\rho}$. This antisymmetric tensor $J^{\sigma\rho}$ contains the 6 generators. By equating the two different expressions for Λ, we find that $-i\omega_{\sigma\rho}(J^{\sigma\rho})^\mu{}_\nu/2 = \omega^\mu{}_\nu$ where $\sigma\rho$ labels the

[16] From now on, we call θ the angle giving the magnitude of rotation. It used to be called ψ.

[17] With $J^1 = J_x$, $J^2 = J_y$, etc. Note that when using latin indices $i, j, ... = 1, 2, 3$, it means that we restrict to space rather than spacetime. In 3d space the metric is Euclidian. There is therefore, in that case, no reason in making a difference between upstairs and downstairs indices. When writing $\epsilon^{ijk}J^k$, the summation over repeated indices is implied.

[18] Actually, this is not entirely true in the case of a non-compact group. The issue is that exponentiating the generators, one does not recover all of the elements of the group. All elements might be recovered as products of such exponentials (?). Matthieu Tissier has a nice counter-example involving the non compact Lie group $SL(2,\mathbb{C})$, which is the group of 2×2 complex matrices with det $= +1$. The matrix $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ belongs to $SL(2,\mathbb{C})$ but can not be written as an exponential. However it can be written as the product of two matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ that each can be written as an exponential $e^{i\pi\sigma_z}e^{i(\sigma_y + i\sigma_x)/2}$. Check that by using the Baker-Campbell-Hausdorff formula.

generators and μ is the row index and ν the column index in the matrix representation. In other words, each $J^{\sigma\rho}$ with fixed $\sigma\rho$ is itself a 4×4 matrix. One can show that $(J^{\sigma\rho})^{\mu}{}_{\nu} = i(\eta^{\sigma\mu}\delta^{\rho}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}_{\nu})$, which is the $3+1$ version of $(J_i)_{jk} = -i\epsilon_{ijk}$. Let's have a closer look at the 6 generators hidden in $J^{\sigma\rho}$. Among the 16 entries of $J^{\sigma\rho}$ only 6 are independent because of antisymmetry. Check that $K^i = J^{0i}$ and $J^i = \epsilon^{ijk}J_{jk}/2$, i.e. $J^{0i} = K^i$ and $J_{ij} = \epsilon_{ijk}J^k$. Therefore

$$J^{\sigma\rho} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \tag{5.22}$$

In covariant notation, the Lorentz algebra reads:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(-\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho}\right) \tag{5.23}$$

and an element of the Lorentz group is $\Lambda = e^{-i\frac{\omega_{\mu\nu}}{2}J^{\mu\nu}}$ (see however, the preceding footnote).

5.1.4 Representations of the Lorentz group

Following a strategy similar to that used for the rotation group, we construct representations of the Lie algebra to obtain that of the group. We first rewrite the Lorentz algebra by defining $\vec{N}_{\pm} \equiv (\vec{J} \pm i\vec{K})/2$ such that $\vec{N}_{\pm}^{\dagger} = \vec{N}_{\pm}$. Now, the algebra reads

$$\left[N^i_+, N^j_+\right] = i\epsilon^{ijk}N^k_+ \tag{5.24}$$

$$\left[N^i_-, N^j_-\right] = i\epsilon^{ijk}N^k_- \tag{5.25}$$

$$\left[N^i_+, N^j_-\right] = 0 \tag{5.26}$$

which means that it splits in two decoupled $su(2)$ algebras, i.e. $\mathfrak{L} = su(2) \oplus su(2)$. Therefore irreps of $\mathfrak{L}$ can be obtained as irreps of $su(2) \oplus su(2)$. The latter are labelled by (j_+, j_-) with $j_\pm = 0, 1/2, 1, ...$ and are of dimension $(2j_+ + 1)(2j_- + 1)$. $\vec{N}^2_{\pm}$ are the two Casimir operators: they are proportional to the identity in each irrep. In (j_+, j_-), $\vec{N}^2_+ = j_+(j_+ + 1)\mathbb{I}$ and $\vec{N}^2_- = j_-(j_- + 1)\mathbb{I}$. From their definition, we also know that $\vec{J} = \vec{N}_+ + \vec{N}_-$ and $\vec{K} = -i(\vec{N}_+ - \vec{N}_-)$. The rotation generator $\vec{J}$ is therefore the sum of two angular momenta $\vec{N}_+$ and $\vec{N}_-$. From the familiar composition law of angular momentum applied to $\vec{J} = \vec{N}_+ + \vec{N}_-$, we know that the spin j corresponding to $\vec{J}^2$ goes in unit steps from $|j_+ - j_-|$ to $|j_+ + j_-|$.

In the general case, the generators $N_{\pm}i$ act on an object with <u>two</u> indices, say ψ_{ab} where the first index is summed with the indices of N_+ and the second with the indices of N_-. As a consequence, for the representation (j_+, j_-), the indices a and b run from 1 to $(2j_+ + 1)$ and from 1 to $(2j_- + 1)$ respectively. Therefore, the J and K are in general objects with 4 indices! If $j_+ = 0$ or $j- = 0$, the situation simplifies because one of the indices actually takes just one value and we are endowed not to write it.

Representation $(j_+, j_-) = (0, 0)$

This is the <u>4-scalar</u> representation. It has dimension 1, $\vec{J} = 0$, $\vec{K} = 0$. It is also a 3-scalar under rotations as $j = 0$.

Representation $(1/2, 0)$

This representation has dimension 2 and behaves as a spinor under rotations as $j = 1/2$. Here $\vec{N}_+ = \vec{\sigma}/2$ and $\vec{N}_- = 0$ so that $\vec{J} = \vec{\sigma}/2$ and $\vec{K} = -i\vec{\sigma}/2$. Therefore $\Lambda_L = e^{i\vec{\phi}\cdot\vec{K} + i\vec{\theta}\cdot\vec{J}} = e^{\vec{\sigma}/2\cdot(i\vec{\theta} + \vec{\phi})}$. Doublets of complex numbers ψ_L that transform in such a representation are called <u>left-handed Weyl spinors</u>. The transformation law is $\psi'_L = \Lambda_L\psi_L$.

Representation $(0, 1/2)$

This representation has dimension 2 and also behaves as a spinor under rotations as $j = 1/2$. However it is inequivalent to the previous representation as we will see. Here $\vec{N}_+ = 0$ and $\vec{N}_- = \vec{\sigma}/2$ so that $\vec{J} = \vec{\sigma}/2$ and $\vec{K} = i\vec{\sigma}/2$. Therefore $\Lambda_R = e^{i\vec{\phi}\cdot\vec{K}+i\vec{\theta}\cdot\vec{J}} = e^{\vec{\sigma}/2\cdot(i\vec{\theta}-\vec{\phi})}$. Doublets of complex numbers ψ_R that transform in such a representation are called right-handed Weyl spinors (in the old literature, left and right spinors are also called dotted and undotted spinors). The transformation law is $\psi'_R = \Lambda_R\psi_R$.

Left and right spinors both behave as spinors ($j = 1/2$) under rotations but behave differently under boosts. The two representations are inequivalent because one can not find a 2×2 matrix S (a similarity matrix) such that $\Lambda_R = S\Lambda_L S^{-1}$ (this being the definition of equivalent representations).

Actually $\sigma_2\Lambda_L^*\sigma_2 = \Lambda_R$. The proof goes as follows. First $\sigma_2(e^{\vec{\sigma}/2\cdot(i\vec{\theta}+\vec{\phi})})^*\sigma_2 = \sigma_2 e^{\vec{\sigma}^*\cdot(-i\vec{\theta}+\vec{\phi})/2}\sigma_2$. Then one uses the general matrix formula

$$Be^AB^{-1} = B\sum_{n=0}^{\infty}\frac{A^n}{n!}B^{-1} = 1 + BAB^{-1} + \frac{BA(B^{-1}B)AB^{-1}}{2!} + ... = e^{BAB^{-1}} \tag{5.27}$$

to show that $\sigma_2 e^{\vec{\sigma}^*\cdot(-i\vec{\theta}+\vec{\phi})/2}\sigma_2 = e^{\sigma_2\vec{\sigma}^*\sigma_2\cdot(-i\vec{\theta}+\vec{\phi})/2}$. Finally, using $\sigma_2\sigma_{1,3}^*\sigma_2 = \sigma_2\sigma_{1,3}\sigma_2 = -\sigma_{1,3}$ and $\sigma_2\sigma_2^*\sigma_2 = \sigma_2(-\sigma_2)\sigma_2 = -\sigma_2$, we see that $\sigma_2\vec{\sigma}^*\sigma_2 = -\vec{\sigma}$, which completes the proof as $\Lambda_R = e^{\vec{\sigma}/2\cdot(i\vec{\theta}-\vec{\phi})}$. In other words $\sigma_2 K$ where K is the operation of complex conjugation (it takes the complex conjugate of everything to its right) would be a candidate for S except that $\sigma_2 K$ is an anti-unitary operator and the latter can not be represented by a matrix[19].

Also Λ_L and Λ_R do not belong to $\mathcal{L}_+^\uparrow$: they are complex 2×2 matrices with det $= +1$, but they are not unitary. Indeed $\Lambda_L^\dagger\Lambda_L = e^{\vec{\sigma}\cdot\vec{\phi}} \neq 1$. This non-unitarity can be traced back to the fact that $\mathcal{L}_+^\uparrow$ is non-compact, i.e. to the existence of boosts which are described by anti-hermitian generators so that $(e^{i\vec{\phi}\cdot\vec{K}})^\dagger = e^{i\vec{\phi}\cdot\vec{K}} \neq e^{-i\vec{\phi}\cdot\vec{K}}$. Rather, Λ_L and Λ_R belong to the group $SL(2,\mathbb{C})$ (the special linear group of complex matrices of size 2×2). The latter is the covering group of $\mathcal{L}_+^\uparrow$ – noted as $SL(2,\mathbb{C}) = \overline{\mathcal{L}_+^\uparrow}$ – and the Weyl spinors realize projective (and not faithful) representations of $\mathcal{L}_+^\uparrow$ (similarly to the relation between $SO(3)$ and $SU(2)$: $SU(2) = \overline{SO(3)}$).

Representation $(1/2, 0) \oplus (0, 1/2)$

What is the effect of space inversion P on the Weyl spinors? $\vec{J} \to \vec{J}$ is a pseudo-vector, whereas $\vec{K} \to -\vec{K}$ is a true vector. This can be checked from $PJ^iP = J^i$ and $PK^iP = -K^i$ [20]. Therefore $\vec{N}_\pm \to \vec{N}_\mp$. This shows that the irreps $(1/2, 0)$ and $(0, 1/2)$ are exchanged under parity so that a left spinor becomes a right spinor and vice-versa. Let us therefore glue a left and a right Weyl spinor into a quadruplet $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$ known as a bispinor or Dirac spinor (here written in the so-called chiral basis). It has four complex components. It realizes a 4-dimensional reducible (and projective) representation of $SO^+(3,1)$ – that's the meaning of $(1/2, 0) \oplus (0, 1/2)$ – and an irreducible (and projective) representation when parity is added. A Dirac spinor transforms as $\begin{pmatrix}\psi'_L\\ \psi'_R\end{pmatrix} = \begin{pmatrix}\Lambda_L & 0\\ 0 & \Lambda_R\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$, which we write $\psi' = S(\Lambda)\psi$, under a

[19] Indeed K is not a linear operator and can therefore not be represented by a matrix. It is actually an anti-linear operator. The most famous example of such an operator is the quantum mechanical time reversal operator T which is anti-unitary (as shown by Wigner in 1932). It acts (in Hilbert space) on a superposition of quantum states as $T(a|\psi\rangle + b|\chi\rangle) = a^*|\psi\rangle + b^*|\chi\rangle$ and satisfies $T^\dagger T = TT^\dagger = 1$ and $T^2 = 1$ or -1. It is usually written as the product of a unitary (and linear) operator U and the complex conjugation operator K: $T = UK$. For a brief introduction to the time-reversal operator see pages 99-100 in [4].

[20] Is it clear in the present context why the generators should transform as $J^i \to PJ^iP^\dagger$ under parity? This is the transformation law for a matrix $A \to A' = SAS^{-1}$ and here the role of S is played by $P = P^\dagger = P^{-1}$. Intuitively, the generator $\vec{J}$ is an angular momentum and we know that the orbital angular momentum $\vec{L} = \vec{x}\times\vec{P}$ (see the section on field representations) is a pseudo-vector because position $\vec{x}$ and momentum $\vec{P}$ are true vectors. Also $\vec{K} = t\vec{P} - E\vec{x}$ as we will see in the section on field representations: therefore $\vec{K} \to -\vec{K}$ under parity, as a true vector.

Lorentz transformation. And as $\begin{pmatrix} \psi'_L \\ \psi'_R \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, which we write $\psi' = \gamma^0\psi$, under a parity transformation. We introduce the notation $\overline{\psi} \equiv \psi^\dagger\gamma^0$ called the conjugate of a Dirac spinor. The reason for this is that $\overline{\psi}\psi$ is a Lorentz scalar (prove it), whereas $\psi^\dagger\psi$ is not. This peculiarity can again be traced back to having $\eta_{\mu\nu}$ as a metric instead of $\delta_{\mu\nu}$.

Representation $(1/2, 1/2)$

This representation has dimension 4. It is the defining or fundamental or 4-vector representation of the Lorentz group. It is actually the representation we have obtained in constructing the Lorentz group, when working with 4×4 matrices (section called Lorentz algebra). The contravariant coordinates A^μ of a 4-vector $\underline{A}$ transforms as $A'^\mu = \Lambda^\mu{}_\nu A^\nu$. It can be shown that a 4-vector under the Lorentz group splits into a scalar and a 3-vector under the rotation group (for details, see Maggiore [2] page 28).

It is important to realize that although a Dirac spinor is a quadruplet (it has 4 components), it is not a Lorentz 4-vector. Indeed the transformation law $\psi' = S(\Lambda)\psi$ is different from $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

Tensor representations

The 4-vector representation is the fundamental representation of the Lorentz group (but not of its covering group $SL(2,\mathbb{C})$). From it, we can play the game of obtaining larger representations by using tensor products. Let's do a useful case: the tensor product of two 4-vector representations. A rank 2 tensor $T^{\mu\nu} = A^\mu B^\nu$ transforms as $T'^{\mu\nu} = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma T^{\rho\sigma}$ (by definition) under a Lorentz transformation. This realizes a 16-dimensional representation. However, it is reducible. Indeed if $T^{\rho\sigma}$ is antisymmetric then $T'^{\mu\nu}$ as well. Similarly if $T^{\rho\sigma}$ is symmetric then $T'^{\mu\nu}$ is also symmetric. Therefore the 16-dimensional representation splits into a 6-dimensional antisymmetric representation $A^{\mu\nu} = (T^{\mu\nu} - T^{\nu\mu})/2$ and a 10-dimensional symmetric representation $S^{\mu\nu} = (T^{\mu\nu} + T^{\nu\mu})/2$. But the trace of the tensor $\mathrm{tr}T = \eta_{\mu\nu}T^{\mu\nu} = \eta_{\mu\nu}S^{\mu\nu}$ is a scalar: indeed $\mathrm{tr}T = \eta_{\mu\nu}S^{\mu\nu} \to \eta_{\mu\nu}S'^{\mu\nu} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma S^{\rho\sigma} = \eta_{\rho\sigma}S^{\rho\sigma}$ is Lorentz-invariant. Therfore the 10-dimensional representation is also reducible into a 1-dimensional (scalar $\mathrm{tr}T = \eta_{\mu\nu}S^{\mu\nu}$) and a 9-dimensional (traceless symmetric rank 2 tensor $S^{\mu\nu} - \eta^{\mu\nu}\mathrm{tr}T/4$) representations. This can be summarized by saying that $4\otimes4 = 1\oplus6\oplus9$, i.e. the rank 2 tensor representation decomposes into 3 irreducible representations: a scalar (the trace), an antisymmetric part and a traceless symmetric part. For more details, see Maggiore [2] page 20.

A particular symmetric rank 2 tensor is the metric tensor $\eta_{\mu\nu}$. It has a specific property: under a Lorentz transformation $\eta_{\mu\nu} \to \eta'_{\mu\nu} = \Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\eta_{\rho\sigma} = \eta_{\mu\nu}$ by virtue of equation (2.4), i.e. $\eta_{\rho\sigma} = \eta_{\alpha\beta}\Lambda^\alpha{}_\rho\Lambda^\beta{}_\sigma$, which defines Lorentz transformations, and the fact that $\Lambda^\mu{}_\rho\Lambda_\nu{}^\rho = \delta^\mu_\nu$. The metric tensor has the same expression in every frame: it is called an invariant tensor.

5.2 Poincaré group and field representations

After having studied the homogeneous Lorentz group consisting of rotations and boosts, we want to study the inhomogeneous Lorentz group also known as the Poincaré group. In addition to rotations and boosts, this group contains translations in spacetime. There are 4 extra generators corresponding to the three translations in space and to time translation. Therefore the Poincaré group has 10 generators.

An element of the group is noted $g = (\Lambda, \underline{a})$ such that $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$. Show that the composition law is $(\tilde{\Lambda}, \underline{\tilde{a}})(\Lambda, \underline{a}) = (\tilde{\Lambda}\Lambda, \tilde{\Lambda}\underline{a} + \underline{\tilde{a}})$. Show that Lorentz transformations $(\Lambda, \underline{0})$ form a subgroup, i.e. $\mathcal{L}$ and that spacetime translations $(\mathbb{I}, \underline{a})$ constitute a subgroup usually called $\mathbb{R}^{1,3}$. The Poincaré group is then seen as the semi-direct product of these two subgroups: $SO^+(3,1) \times \mathbb{R}^{1,3}$.

5.2.1 Spacetime translations and field representations

The aim is now to find the 10 generators of the Poincaré group. As we already know the 6 generators of the Lorentz group, we will concentrate on the 4 generators of the spacetime translation group. The parameter space for the latter is $\mathbb{R}^4$, which is not compact. It turns out that a representation of the spacetime translations can therefore only be found using fields (i.e. functions of spacetime) rather than multiplets with a discrete and finite number of components[21]. In other words, spacetime translations will be represented by operators (such as gradients) and not by matrices. Consider the translation $(\mathbb{I}, \underline{a})$ acting as $x'^\mu = x^\mu + a^\mu$ on coordinates. What is its action on a spacetime function $\phi(x)$ [22]? It is simply $\phi'(x') = \phi(x)$. Indeed, in the passive viewpoint, the point of spacetime at which the field ϕ is evaluated is unchanged only its coordinates are changed from x to x' by the transformation. In other words, every field behaves as a "scalar" under a spacetime translation. In an infinitesimal translation, $\phi(x^\mu) = \phi'(x'^\mu) = \phi'(x^\mu + a^\mu) \approx \phi'(x^\mu) + a^\nu \partial_\nu \phi'(x^\mu) \approx \phi'(x^\mu) + a^\nu \partial_\nu \phi(x^\mu)$ when $a^\mu \to 0$. Therefore $\phi'(x) \approx \phi(x) - a^\nu \partial_\nu \phi(x) = [1 - a^\nu \partial_\nu]\phi(x)$, from which the generators are obtained as usual from $-i \frac{d[1-a^\nu \partial_\nu]}{da^\nu}|_{a^\nu=0} = i\partial_\nu$. The generators of translations are usually called

$$P_\nu \equiv i\partial_\nu \tag{5.28}$$

and there are indeed four of them. A finite translation is represented by $\phi'(x) = e^{ia^\nu P_\nu}\phi(x)$. Because $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, the algebra of translation generators is trivial $[P_\mu, P_\nu] = 0$ and the translation group is abelian[23]. Later, we will recognize $P^0 = i\partial_t$ as the Hamiltonian operator, $\vec{P} = -i\vec{\nabla}$ as the momentum operator and therefore $P^\mu = (P^0, \vec{P})$ as the 4-momentum operator.

An important point to understand is that in a field representation we are comparing $\phi'(x)$ with $\phi(x)$ rather than with $\phi'(x')$. To make the discussion more general, consider a field of a more complex nature having internal degrees of freedom: $\phi^I(x)$ with $I = 1, ..., N_I$ labeling internal degrees of freedom. For example, the field could be a 4-vector (then $N_I = 4$) or a Weyl spinor (then $N_I = 2$) or a Dirac spinor (then $N_I = 4$) etc. If we were comparing $\phi^I(x)$ with $\phi'^I(x')$ we would be studying a fixed point (or event) of spacetime (in the passive viewpoint, the event is fixed but its coordinates are changing from x^μ to x'^μ) and how the N_I degrees of freedom ϕ^I are mixed by the transformation. In the case of a field which is itself a scalar ($N_I = 1$), the field is invariant, there is a single degree of freedom and therefore $\phi'(x') = \phi(x)$. In a more general case of a field with $N_I > 1$, the internal components are mixed in a transformation $\phi'^I(x') = [S(g)]^I{}_J \phi^J(x)$ where g is an element of the Poincaré group and $S(g)$ is a $N_I \times N_I$ matrix. Doing this, there would be no interest of using fields $\phi^I(x)$ rather than similar objects ϕ^I without the dependence on spacetime coordinates. Therefore, what we are now doing is to compare $\phi'^I(x)$ with $\phi^I(x)$ i.e. comparing the transformed field at another point of spacetime (having the same coordinates in two different frames) with the original field at the original point of spacetime. Because we are going from one event to another, there is now an infinite number of degrees of freedom involved (even if $N_I = 1$). And hence, the field representation is of infinite dimension. See the corresponding discussion on page 30 of Maggiore [2].

5.2.2 Lorentz transformation of a scalar field

Here we want to study how a Lorentz transformation $(\Lambda, \underline{0})$ acts on fields. Let's first consider a scalar field (i.e. a field that is itself a scalar rather than a vector or a spinor). One has $\phi'(x') = \phi(x)$, as before and by definition of a scalar field, except that now $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ instead of $x'^\mu = x^\mu + a^\mu$. In an infinitesimal transformation $x'^\mu \approx x^\mu + \omega^\mu{}_\nu x^\nu$ (when discussing the Lorentz group, we saw that $\Lambda^\mu{}_\nu \approx \delta^\mu_\nu + \omega^\mu{}_\nu$), therefore $\phi'(x'^\mu) \approx \phi'(x^\mu) + \omega^\rho{}_\nu x^\nu \partial_\rho \phi'(x^\mu) \approx \phi'(x^\mu) + \omega^\rho{}_\nu x^\nu \partial_\rho \phi(x^\mu)$. Using $\phi'(x') = \phi(x)$, we obtain $\phi'(x) \approx \phi(x) - \omega^\rho{}_\nu x^\nu \partial_\rho \phi(x) = [1 - \omega^{\rho\nu} x_\nu \partial_\rho]\phi(x)$. As $\omega^{\rho\nu} = -\omega^{\nu\rho}$ is antisymmetric, we can keep only the antisymmetric part of $x_\nu \partial_\rho$. Therefore $\phi'(x) \approx [1 - \omega^{\rho\nu}(x_\nu \partial_\rho - x_\rho \partial_\nu)/2]\phi(x)$. This is similar to $x'^\mu \approx x^\mu - \frac{i}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu x^\nu$, and we can read the generators from the previous expression

$$L_{\nu\rho} \equiv x_\nu i\partial_\rho - x_\rho i\partial_\nu = x_\nu P_\rho - x_\rho P_\nu \,. \tag{5.29}$$

Later, we will identify L^{ij} as the (extrinsic or orbital) angular momentum operator. For a finite transformation, we have $\phi'(x) = \exp\left(-i\frac{\omega_{\mu\nu}}{2}L^{\mu\nu}\right)\phi(x)$. From its definition, $L^{\mu\nu}$ is antisymmetric $L^{\mu\nu} = -L^{\nu\mu}$ and there-

[21] Under translations, objects such as scalars, 4-vectors, spinors, etc. are all left unchanged. That's the reason why we have to search for different type of objects to build representations of the spacetime translation group.

[22] For shortness, we write $\phi(x)$ instead of $\phi(t, x, y, z)$ or $\phi(x^\mu)$.

[23] All irreducible representations of an abelian group are one-dimensional.

fore contains only 6 independent entries corresponding to the 6 generators (3 boosts: $K^i = L^{0i} = x^0 P^i - x^i P^0$ i.e. $\vec{K} = t\vec{P} - H\vec{x}$ and 3 rotations: $L^i = \epsilon^{ijk} L_{jk}/2$ i.e. $\vec{L} = \vec{x} \times \vec{P}$) of the Lorentz group.

5.2.3 Poincaré algebra

We are now in position to obtain the complete Poincaré algebra. We already know the spacetime translation algebra

$$[P^\mu, P^\nu] = 0 \tag{5.30}$$

and also the Lorentz algebra (see the section on the Lorentz group)

$$[L^{\mu\nu}, L^{\rho\sigma}] = i\left(-\eta^{\mu\rho} L^{\nu\sigma} - \eta^{\nu\sigma} L^{\mu\rho} + \eta^{\nu\rho} L^{\mu\sigma} + \eta^{\mu\sigma} L^{\nu\rho}\right) \tag{5.31}$$

What remains to be computed is the commutator of P^μ and $L^{\rho\sigma}$: $[P^\mu, L^{\rho\sigma}]\phi(x) = i\partial^\mu(x^\rho i\partial^\sigma - x^\sigma i\partial^\rho)\phi(x) - (x^\rho i\partial^\sigma - x^\sigma i\partial^\rho) i\partial^\mu \phi(x) = i(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho)\phi(x)$. Therefore

$$[P^\mu, L^{\rho\sigma}] = i(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho) \tag{5.32}$$

which tells us that P^μ behaves as a 4-vector under Lorentz transformations. These three equations constitute the Poincaré algebra written in covariant notation. It can also be more explicitly written as

$$\left[L^i, L^j\right] = i\epsilon^{ijk} L^k, \;\; \left[L^i, K^j\right] = i\epsilon^{ijk} K^k, \;\; \left[L^i, P^j\right] = i\epsilon^{ijk} P^k \tag{5.33}$$

$$\left[K^i, K^j\right] = -i\epsilon^{ijk} L^k, \;\; \left[P^i, P^j\right] = 0, \;\; \left[K^i, P^j\right] = iP^0 \delta^{ij} \tag{5.34}$$

$$\left[L^i, P^0\right] = 0, \;\; \left[P^i, P^0\right] = 0, \;\; \left[P^0, P^0\right] = 0, \;\; \left[K^i, P^0\right] = iP^i \tag{5.35}$$

with $L^i = \epsilon^{ijk} L_{jk}/2$ and $K^i = L^{0i}$.

5.2.4 Lorentz transformation of a 4-vector field

Let us consider a more involved situation with a field with internal indices $\phi^I(x)$, where $I = 1, ..., N_I$. Under a Lorentz transformation $g = (\Lambda, \underline{0})$, both x^μ and ϕ^I are transformed: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ and $\phi'^I(x') = [S(g)]^I{}_J \phi^J(x)$ where $S(g)$ is an $N_I \times N_I$ matrix (for example if ϕ^I is a left Weyl spinor, $N_I = 2$ and $S(g) = \Lambda_L$; if ϕ^I is a scalar, $N_I = 1$ and $S(g) = 1$). The question we ask is: how is $\phi'^I(x)$ related to $\phi^I(x)$?

To be more concrete, we consider the example of a 4-vector field $\phi^\mu(x)$ (here the internal index I is called μ and $N_I = 4$). In an infinitesimal transformation, $\phi'^\mu(x') = \Lambda^\mu{}_\nu \phi^\nu(x) \approx \phi^\mu(x) + \omega^\mu{}_\nu \phi^\nu(x)$ (i.e. $[S(g)]^\mu{}_\nu = \Lambda^\mu{}_\nu$ here) where $x'^\mu \approx x^\mu + \omega^\mu{}_\nu x^\nu$. Here $\Lambda^\mu{}_\nu = [\exp(-i\omega_{\rho\sigma} S^{\rho\sigma}/2)]^\mu{}_\nu \approx \delta^\mu_\nu + \omega^\mu{}_\nu$ where $[S^{\rho\sigma}]^\mu{}_\nu = i(\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu)$ (note that $S^{\rho\sigma}$ was previously called $J^{\rho\sigma}$). Doing a Taylor expansion, we find $\phi'^\rho(x^\mu + \omega^\mu{}_\nu x^\nu) \approx \phi'^\rho(x^\mu) + \omega^\mu{}_\nu x^\nu \partial_\mu \phi^\rho(x^\mu)$. But $\phi'^\rho(x'^\mu) \approx \phi^\rho(x^\mu) + \omega^\rho{}_\nu \phi^\nu(x^\mu)$ so that $\phi'^\rho(x^\mu) \approx \phi^\rho(x^\mu) + \omega^\rho{}_\nu \phi^\nu(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi^\rho(x^\mu) = [\mathbb{I} - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu} + L^{\mu\nu})]^\rho{}_\sigma \phi^\sigma(x^\mu)$. By definition we now call $J^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$ the total generator of Lorentz transformations. It has two parts: $S^{\mu\nu}$ generates the transformation on the nature of the field (it is the intrinsic generator of Lorentz transformations) whereas $L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ generates the transformation on the field as being a function of spacetime (it is the extrinsic generator of Lorentz transformations). This should be familiar from the decomposition $\vec{J} = \vec{S} + \vec{L}$ of the total angular momentum operator into its intrinsic (spin) and extrinsic (orbital) parts in non-relativistic quantum mechanics.

5.2.5 Summary

In the case of an arbitrary field $\phi^I(x)$, at the same point in spacetime, we have

$$\phi'^I(x') = [S(\Lambda)]^I{}_J \phi^J(x) \text{ where } S(\Lambda) = e^{-i\frac{\omega_{\mu\nu}}{2} S^{\mu\nu}} \tag{5.36}$$

with $S^{\mu\nu}$ depending on the representation (0 for a scalar field, $i(\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu)$ for a 4-vector field, $\sigma^{\mu\nu}/2 = [\gamma^\mu, \gamma^\nu]/4$ for a Dirac spinor field, $S(\Lambda) = \Lambda_L$ for a left Weyl spinor field, etc). And at the same coordinates (but at different points in spacetime), we have

$$\phi'^I(x) = [e^{-i\frac{\omega_{\mu\nu}}{2} J^{\mu\nu}}]^I{}_J \phi^J(x) \text{ where } J^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu} \tag{5.37}$$

where $L^{\mu\nu}$ is always given by $x^\mu P^\nu - x^\nu P^\mu$ with $P^\mu = i\partial^\mu$.

5.2.6 Lorentz transformation of a spinor field

Exercise: write the two equations (5.36) and (5.37) in the case of a left Weyl spinor and also in the case of a Dirac spinor in order to identify the corresponding $S^{\mu\nu}$. You should find that, for the left spinors

$$S^{0i} = -i\frac{\sigma^i}{2} \tag{5.38}$$

$$S^{ij} = \epsilon^{ijk}\frac{\sigma^k}{2} \tag{5.39}$$

Similar relations hold for the right Weils spinor except that the sign of S^{0i} is changed.

5.2.7 Representations of the Poincaré group on single particle states

A subject that would be worth studying here, especially in preparation of the appearance of particles as excitation quanta of the fields, – but which we omit because we feel it does not belong to a classical (i.e. non-quantum) discussion of the Poincaré group – is the representation of the Poincaré group on single particle states in Hilbert space (see e.g. Maggiore [2] pages 36-40 or Weinberg [13], pages 62-74). This is a famous work of Wigner (1939). He showed that irreps are classified by the mass and the spin of particles. More precisely, for a massive particle, irreps are classified by spin $j = 0, 1/2, 1, 3/2, ...$ and have a dimension $2j+1$. And for a massless particle, irreps are classified by the helicity $h = \pm 1/2, \pm 1, ...$ and have dimension 1. In case parity is conserved, one may build two-dimensional representations by grouping $h = +1$ and $h = -1$ for the photon for example.

5.3 Conclusion

Scalars, vectors, spinors, tensors, etc. are simply natural objects (i.e. irreducible representations of the symmetry group of space-time) that emerge from the geometry of space-time. They have well-defined transformation properties under the symmetries of space-time. Almost no physics at this point. At least no dynamics, no particles, etc.

Chapter 6

Dirac field

6.1 Massless spinor fields and the Weyl equations

see Maggiore [2] pages 54-56

In this section, we build classical theories for spinor fields using symmetries. We will therefore be quite far from the historical appearance of the Dirac equation (in the context of single particle relativistic quantum mechanics) or the Weyl equation. We have in mind the description of spin 1/2 particles such as the neutrino or the electron (even if for the moment we describe fields not yet particles, which will only emerge upon quantization).

6.1.1 Left Weyl field, helicity and the Weyl equation

Consider a single left Weyl field $\psi_L(x)$. Let $\bar\sigma^\mu \equiv (\mathbb{I}, -\sigma^i) = (\mathbb{I}, -\vec\sigma)$ in any frame be a quadruplet of 2×2 matrices (and $\sigma^\mu \equiv (\mathbb{I}, \sigma^i)$) built from the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and the 2×2 identity. Then $\psi_L^\dagger \bar\sigma^\mu \psi_L$ is a 4-vector. Check it (i.e. show that, under a Lorentz transformation Λ, it becomes $\psi_L^\dagger \Lambda_L^\dagger \bar\sigma^\mu \Lambda_L \psi_L = \Lambda^\mu{}_\nu \psi_L^\dagger \bar\sigma^\nu \psi_L$). Is $\psi_L^\dagger \sigma^\mu \psi_L$ also a 4-vector? From this knowledge, we can build a Lorentz invariant kinetic (and elastic) term that is first order in derivative:

$$\mathcal{L}_L = i\psi_L^\dagger \bar\sigma^\mu \partial_\mu \psi_L \tag{6.1}$$

and such that the corresponding action is also real (check it). It is not possible to write a mass term just for a left Weyl field (for example, $\psi_L^\dagger \psi_L$ is not Lorentz invariant – check it).

The EL equations give $0 = \partial_\mu(i\psi_L^\dagger \bar\sigma^\mu)$ and upon taking the hermitian conjugate and using $(\bar\sigma^\mu)^\dagger = \bar\sigma^\mu$, we obtain

$$\bar\sigma^\mu \partial_\mu \psi_L = 0 \tag{6.2}$$

which is known as the Weyl equation (1929). The latter can also be written as $i\partial_t \psi_L = -\vec\sigma \cdot (-i\vec\nabla)\psi_L$, which may remind you of the low-energy description of graphene (albeit in 3 space dimension rather than 2), or that of a 3D Weyl semimetal. We can "take the square" of the Weyl equation to obtain

$$\Box \psi_L = 0\,. \tag{6.3}$$

This follows from $\sigma^\nu \partial_\nu \bar\sigma^\mu \partial_\mu = (\partial_0 + \sigma^j \partial_j)(\partial_0 - \sigma^i \partial_i) = \partial_0^2 - \sigma^j \sigma^i \partial_j \partial_i = \partial_0^2 - \partial_j \partial_j = \partial^\mu \partial_\mu = \Box$ as $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$. The left Weyl spinor therefore satisfies the d'Alembert (or massless KG) equation. Actually, the spinor has two complex components. Each component of the doublet satisfies the massless KG equation (6.3). But in addition the doublet has to satisfy the Weyl equation (6.2). We will see that the latter is a projection equation that gets rid of one of the two (complex) degrees of freedom.

Take a plane wave solution $\psi_L(x) = u_L e^{-ik\cdot x}$ where $u_L(x) = u_L$ is a constant spinor. If we inject it in (6.3), we find that $k_0^2 = \vec k^2$ i.e. the dispersion relation $\omega = \pm|\vec k|$ (identical to that of light). If we inject in (6.2), we find that $\vec{k}\cdot\vec{\sigma}u_L = -\omega u_L$. Using the two equations and choosing[1] the positive branch of the dispersion relation $\omega = |\vec{k}|$, we find that $\hat{k}\cdot\vec{\sigma}u_L = -u_L$ (where $\hat{k} \equiv \vec{k}/|\vec{k}|$). But the spin operator $\vec{S} = \vec{\sigma}/2$ for a spin 1/2 and therefore

$$\hat{k}\cdot\vec{S}u_L = -\frac{1}{2}u_L\,. \tag{6.4}$$

This means that the left Weyl field has an helicity of $-1/2$. The helicity is the projection of the spin (or intrinsic angular momentum) on the direction of motion. This last equation should be thought of as an equation that projects out half of the apparent degrees of freedom. A Weyl field seems to have two (complex) degrees of freedom. But, because it is an helicity eigenstate, it only has a single (complex) degree of freedom. To make this point clear, imagine the motion is along the z direction, so that $\vec{k} = |\vec{k}|\vec{e}_z$. Then $\sigma_z u_L = -u_L$ meaning that u_L is the spinor $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ having a single non-zero component.

Another Lagrangian – which one obtains by integration by part in the action, which is more symmetrical, gives the same equation of motion and is therefore equivalent – is:

$$\mathcal{L}'_L = \frac{i}{2}\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - \frac{i}{2}(\partial_\mu\psi_L^\dagger)\bar{\sigma}^\mu\psi_L = i\psi_L^\dagger\bar{\sigma}^\mu\overleftrightarrow{\partial_\mu}\psi_L \tag{6.5}$$

where we introduced the following notation $A\overleftrightarrow{\partial_\mu}B \equiv \frac{1}{2}A(\partial_\mu B) - \frac{1}{2}(\partial_\mu A)B$ or $\overleftrightarrow{\partial_\mu} \equiv \frac{1}{2}\overrightarrow{\partial_\mu} - \frac{1}{2}\overleftarrow{\partial_\mu}$.

As a conclusion to this part, it is essential to realize that the Weyl equation is more than just the dispersion relation $\omega = \pm|\vec{k}|$ (which is essentially what the KG equation for a massless field is). It is an equation which is first order in gradient and which enforces a relation between the components of the spinor doublet. Physically it projects out unwanted (or extra or redundant) degrees of freedom.

6.1.2 Right Weyl field

A similar construction can be made for a right Weyl spinor field $\psi_R(x)$. It gives a Lagrangian

$$\mathcal{L}_R = i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R \tag{6.6}$$

(or equivalently $\mathcal{L}'_R = i\psi_R^\dagger\sigma^\mu\overleftrightarrow{\partial_\mu}\psi_R$), a Weyl equation

$$\sigma^\mu\partial_\mu\psi_R = 0 \tag{6.7}$$

and, as a consequence, a massless KG equation $\Box\psi_R = 0$. When one injects a plane wave solution, one obtains the dispersion relation $\omega = \pm|\vec{k}|$ and the helicity equation $\hat{k}\cdot\vec{S}u_R = +\frac{1}{2}u_R$.

Remark: In Nature, the neutrino (if massless) could be described by a left Weyl spinor field (it has an helicity of $-1/2$). While the anti-neutrino (if it exists) would correspond to a right Weyl spinor field (of helicity $+1/2$). It is now known that the neutrino has a finite tiny mass (see the phenomenon of neutrino oscillations). However it is unclear whether it is identical or different from its antiparticle, i.e. whether it is a Majorana particle or not.

6.2 Spinor field and the Dirac equation

see Maggiore [2] pages 56-65

Remember that ψ_L and ψ_R are exchanged under parity. Therefore, if we insist on having a parity-invariant action, we need to take both a left $\psi_L(x)$ and a right $\psi_R(x)$ Weyl spinor fields and glue them into a bispinor (i.e. a quadruplet of complex fields) to make a Dirac field $\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix}$. This is the representation $(1/2,0)\oplus(0,1/2)$.

[1]The important point is not the choice of the positive branch. The argument goes as well with the negative branch. The point is that there is no freedom in the projection of the spin once the orbital motion is given. Hence, we do not have two complex degrees of freedom but a single complex one. The spin projection is locked to the momentum. This is the notion of helicity eigenstate.

6.2.1 Parity and the Dirac Lagrangian

The game is now to build a Lagrangian, which is both Lorentz and parity invariant. It will now be possible to construct a simple mass term. Indeed, under a Lorentz transformation Λ, $\psi_L \to \Lambda_L \psi_L$ and $\psi_R \to \Lambda_R \psi_R$ with $\Lambda_L = e^{(i\vec{\theta}+\vec{\phi})\cdot\vec{\sigma}/2}$ and $\Lambda_R = e^{(i\vec{\theta}-\vec{\phi})\cdot\vec{\sigma}/2}$. The bilinears $\psi_L^\dagger \psi_L$ and $\psi_R^\dagger \psi_R$ are not invariant but $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$ are (check it).

Under parity $x^\mu = (t, \vec{x}) \to x'^\mu = (t, -\vec{x})$ and $\psi_{L/R}(x) \to \psi'_{L/R}(x') = \psi_{R/L}(x')$. So that $\psi_L^\dagger \psi_R \to \psi_R^\dagger \psi_L$ and $\psi_R^\dagger \psi_L \to \psi_L^\dagger \psi_R$. Therefore $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ is parity-invariant: it is a true scalar (whereas $\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L$ is a pseudo-scalar).

Therefore, a free, Lorentz and parity invariant Lagrangian is

$$\mathcal{L}_D = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)\,, \tag{6.8}$$

which is known as the Dirac Lagrangian. It is obviously Lorentz invariant. Let's check its behavior under parity: $\partial_i \to -\partial_i$ so that $\bar{\sigma}^\mu \partial_\mu \to \sigma^\mu \partial_\mu$ and $\psi_L \leftrightarrow \psi_R$ so that indeed $\mathcal{L}_D \to \mathcal{L}_D$.

6.2.2 The Dirac equation and γ matrices

Dirac to Feynman: "I have an equation. Do you have one too?"

The EL equation $\frac{\partial \mathcal{L}_D}{\partial \psi_L^\dagger} = \partial_\mu \frac{\mathcal{L}_D}{\partial(\partial_\mu \psi_L^\dagger)}$ gives $-m\psi_R = -i\bar{\sigma}^\nu \partial_\mu \psi_L$. Similarly $\frac{\partial \mathcal{L}_D}{\partial \psi_R^\dagger} = \partial_\mu \frac{\mathcal{L}_D}{\partial(\partial_\mu \psi_R^\dagger)}$ gives $-m\psi_L = -i(\partial_\mu \sigma^\mu \psi_R)$. The Dirac equation is therefore

$$\begin{aligned} i\bar{\sigma}^\mu \partial_\mu \psi_L &= m\psi_R \\ i\sigma^\mu \partial_\mu \psi_R &= m\psi_L\,, \end{aligned} \tag{6.9}$$

which shows that the left and right spinors inside the Dirac bispinor are coupled.

Here it is also possible to take the square of this first order equation by applying $i\sigma^\mu \partial_\mu$ to the left of $i\bar{\sigma}^\nu \partial_\nu \psi_L = m\psi_R$ to find $(i\sigma^\mu \partial_\mu)(i\bar{\sigma}^\nu \partial_\nu \psi_L) = -\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu \psi_L = -\frac{1}{2}(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu)\partial_\mu \partial_\nu \psi_L$. Now, we note that $\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu}\mathbb{I}$ (pay attention to the fact that $\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu$ is not the anticommutator $\{\sigma^\mu, \bar{\sigma}^\nu\} = \sigma^\mu \bar{\sigma}^\nu + \bar{\sigma}^\nu \sigma^\mu$). Therefore $-\partial_\mu \partial^\mu \psi_L = (i\sigma^\mu \partial_\mu)(m\psi_R) = mi\sigma^\mu \partial_\mu \psi_R = m^2 \psi_L$ (where, in the last step, we used the second equation (6.9)). We eventually arrive at $(\Box + m^2)\psi_L = 0$, which is the massive KG equation for ψ_L. We could as well obtain $(\Box + m^2)\psi_R = 0$ so that in the end, each of the four componenents of the Dirac spinor ψ satisfies the massive KG equation:

$$(\Box + m^2)\psi = 0\,. \tag{6.10}$$

We can rewrite the above results using the Dirac spinor notation $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, which is here written in the chiral representation (meaning that its first two components behave as a left Weyl spinor and its last two as a right Weyl spinor). But we could have chosen to define e.g. $\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_L + \psi_R \\ \psi_L - \psi_R \end{pmatrix}$ (see below for another representation). In the chiral representation, the Dirac equation reads:

$$\begin{pmatrix} 0 & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} i\partial_\mu \psi = m\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m\psi \tag{6.11}$$

We define the γ matrices in the chiral representation by $\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ so that the Dirac equation becomes

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0 \text{ or } (i\not{\partial} - m)\psi = 0 \text{ or } (\not{p} - m)\psi = 0} \tag{6.12}$$

with the Feynman slash notation $\not{r} \equiv \gamma^\mu r_\mu$ and $p_\mu = i\partial_\mu$ is here the 4-momentum operator.

Remarks:
- the $\bar{\sigma}^\mu$ and σ^μ are quadruplets of 2×2 (Pauli) matrices but they do not transform as 4-vectors (despite their notation). They are invariant under a Lorentz transformation.
- Similarly, the γ^μ are a quadruplet of 4×4 matrices (known as gamma matrices or Dirac matrices) but they do not transform as a 4-vector. They are unchanged in a Lorentz transformation. Here, we have given their expression in the so-called chiral basis.

The γ matrices satisfy the following algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}} \tag{6.13}$$

known as the Clifford algebra. It implies that $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$. This algebra will actually be taken as the definition of γ matrices. Applying $i\gamma^\nu\partial_\nu$ to the left of $i\gamma^\mu\partial_\mu\psi = m\psi$ and using the Clifford algebra, we find that $(\Box + m^2)\psi = 0$ as expected. Each of the 4 components of the Dirac bispinor satisfies the same KG equation.

The Dirac Lagrangian can be rewritten

$$\boxed{\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi} \text{ or } \mathcal{L}'_D = \bar{\psi}(i\overleftrightarrow{\partial\!\!\!/} - m)\psi \tag{6.14}$$

with $\bar{\psi} \equiv \psi^\dagger\gamma^0$ the conjugate Dirac spinor (which we already defined; remember that its main property is that $\bar{\psi}\psi$ is a Lorentz scalar whereas $\psi^\dagger\psi$ is not).

The spin operator is $\frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ so that the helicity operator is $\frac{1}{2}\vec{\Sigma} \cdot \frac{\vec{p}}{|\vec{p}|}$ where $\vec{p} = -i\vec{\nabla}$ is here the 3-momentum operator.

6.2.3 Chirality operator

In addition to γ^μ with $\mu = 0, 1, 2, 3$, we also define $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5$. In the chiral basis, it is $\gamma_5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}$, which we recognize as the chirality operator, i.e. the operator which distinguishes left from right Weyl spinors. A Dirac bispinor purely made of a left (resp. right) Weyl spinor is an eigenvector of γ^5 with eigenvalue -1 (resp. $+1$). Its status as a γ matrix comes from the fact that it shares a few properties with the other 4 gamma matrices: it anticommutes with the other γ matrices $\{\gamma^5, \gamma^\mu\} = 0$ (check it) and it squares to one $(\gamma^5)^2 = \mathbb{I}$ (check it). The reason for avoiding γ^4 as a name is because it already exists in some conventions in which $\mu = 1, 2, 3, 4 = x, y, z, t$ instead of $\mu = 0, 1, 2, 3 = t, x, y, z$. The chirality projectors are $\frac{1\pm\gamma_5}{2}$. Indeed they are projectors $\left(\frac{1\pm\gamma_5}{2}\right)^2 = \frac{1\pm\gamma_5}{2}$ and they project onto chirality eigenstates $\frac{1-\gamma_5}{2}\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ (left) and $\frac{1+\gamma_5}{2}\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$ (right) because $\gamma^5\begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = -\begin{pmatrix} \psi_L \\ 0 \end{pmatrix}$ and $\gamma^5\begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = +\begin{pmatrix} 0 \\ \psi_R \end{pmatrix}$. Note that parity $\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ in the chiral basis such that $\gamma^0\begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_L \end{pmatrix}$ and $\gamma^0\begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ 0 \end{pmatrix}$. Parity exchanges left and right chirality eigenstates.

Chirality refers to left and right spinors, which are spinors distinguished by their behavior under a Lorentz boost but not under a rotation. In addition, a left spinor becomes a right spinor under parity transformation and vice-versa. The word "chiral" actually means an object that is not identical – even after a rotation – to its mirror image (like a right hand and a left hand). The parity transformation is essentially a mirror reflection.

6.2.4 The Clifford algebra and different representations of the γ matrices

The γ matrices are defined by the Clifford algebra and therefore exist in several representations. In this course, we have first presented them in the chiral basis. Let $U \in U(4)$ be a unitary transformation such that

$\psi' = U\psi$ and $\gamma'^{\mu} = U\gamma^{\mu}U^{-1} = U\gamma^{\mu}U^{\dagger}$. Note that U is independent of the spacetime point whereas $\psi(x)$ is a field even if we write ψ for conciseness. Let's check that this is a do-nothing transformation (similar to a canonical transformation in classical mechanics: we just change our way of representing a given problem). From $(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$ and $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$, show that $(i\gamma'^{\mu}\partial_{\mu} - m)\psi' = 0$ and $\{\gamma'^{\mu}, \gamma'^{\nu}\} = 2\eta^{\mu\nu}$. The different representations of the γ matrices correspond to different representations of the Clifford algebra.

Two important representations are the chiral (or Weyl) one, which we already discussed, and the standard (or Dirac) representation, that we discuss below[2]. The chiral representation has a diagonal chirality operator γ^5 and an off-diagonal parity operator γ^0. It is convenient to study massless or almost massless fields (such as the neutrino field) and also for the transformation properties of ψ under a Lorentz transformation. The γ matrices are

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \left[\text{i.e. } \gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}\right] \text{ and } \gamma^5 = \begin{pmatrix} -\mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} \tag{6.15}$$

and the Dirac spinor transforms as follows

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{6.16}$$

under a Lorentz transformation Λ, where $\Lambda_{L/R} = e^{\vec{\sigma}/2\cdot(i\vec{\theta}\pm\vec{\phi})}$.

The standard (or parity) representation has a diagonal parity operator γ^0 and an off-diagonal chirality operator γ^5. This representation is obtained from the chiral one through the unitary transformation $U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I} & \mathbb{I} \\ -\mathbb{I} & \mathbb{I} \end{pmatrix}$ so that

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \text{ and } \gamma^5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{6.17}$$

and

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_R + \psi_L \\ \psi_R - \psi_L \end{pmatrix} \tag{6.18}$$

This representation of γ matrices is convenient to discuss the non-relativistic limit of a massive field (such as the electronic field). The two spinors ϕ and χ are then known as the large and small components of the Dirac bispinor.

We summarize some general properties of the γ matrices:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}, (\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0, \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5, \{\gamma^5, \gamma^{\mu}\} = 0, (\gamma^5)^{\dagger} = \gamma^5 \text{ and } (\gamma^5)^2 = 1 \tag{6.19}$$

Note the similarity between γ^5 and γ^0: they both square to 1, are hermitian and anticommute with all other γ matrices. The following 16 matrices are linearly independent and form a basis of all 4×4 matrices:

$$\mathbb{I}, \gamma^{\mu}, \gamma^5, \gamma^{\mu}\gamma^5 \text{ and } \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}] \tag{6.20}$$

In other words, a generic 4×4 matrix with complex entries (group called $GL(4,\mathbb{C})$) is a linear combination with complex coefficients of these 16 matrices. The anticommutation relation for the γ matrices enables us to find several properties of interest. In particular:

$$[\sigma^{\mu\nu}, \gamma^{\rho}] = 2i(\gamma^{\mu}\eta^{\nu\rho} - \gamma^{\nu}\eta^{\mu\rho})$$

It is very convenient to choose the γ matrices such that they fulfill the very useful property

$$(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0 \tag{6.22}$$

Note that the γ matrices in both the Dirac and the chiral basis fulfill this property. This enables to prove properties that will come handy later on:

$$\gamma_5^{\dagger} = \gamma_0\gamma_5\gamma_0, \qquad (\sigma^{\mu\nu})^{\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0 \tag{6.23}$$

[2] We won't discuss a third important representation called the Majorana representation.

Dirac bilinears

We here briefly mention the properties of Dirac field bilinears under Lorentz and parity transformations. The aim is to prepare building blocks in order to construct Lagrangians.

Show that $\bar{\psi}\psi$ is a true 4-scalar and that $\bar{\psi}\gamma^5\psi$ is a pseudo 4-scalar. That $\bar{\psi}\gamma^\mu\psi$ is a true 4-vector and that $\bar{\psi}\gamma^\mu\gamma^5\psi$ is a pseudo 4-vector. And that $\bar{\psi}\sigma^{\mu\nu}\psi$ is an anti-symmetric (rank 2) tensor, where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

In an exercise sheet, you will see that a Dirac spinor transforms as $\psi \to \psi' = S(\Lambda)\psi = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi$ under a Lorentz transformation Λ, with $S^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2}$. This last object contains the spin operator S^i and the boost generator K^i. The spin operator is $S^i = \frac{1}{2}\epsilon_{ijk}\frac{\sigma^{jk}}{2} = \frac{1}{2}\Sigma^i$. From there, we can show that $\vec{\Sigma} = \gamma^5\gamma^0\vec{\gamma}$, which in both the standard and the chiral representations gives $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.

Parity, chirality and helicity

These three operators with ± 1 eigenvalues should not be confused:
- The parity operator γ^0 realizes a space inversion P on a Dirac spinor − note that γ^0 is only the action of space inversion onto the internal degrees of freedom of the Dirac spinor field, but in addition parity P does $x^\mu = (t, \vec{x}) \to (t, -\vec{x})$. More generally, a parity transformation realizes a mirror reflection $x^\mu = (x^0, x^1, x^2, ...) \to (x^0, -x^1, x^2, ...)$ and is such that $\det P = -1$. Only when space has an odd number of dimensions can it be defined as inverting all spatial coordinates $x^\mu = (x^0, x^i) \to (x^0, -x^i)$. Therefore in 3+1 or in 1+1, parity can be considered to be equivalent to space inversion. Whereas in 2+1, parity $(x, y \to -x, y)$ is not the same transformation as space inversion ($x, y \to -x, -y$, which is equivalent to a π rotation).
- The chirality operator is γ^5, which by definition is the matrix that anticommutes with all other γ matrices. It allows one to distinguish between left and right Weyl spinors (to avoid confusion we should say left-chiral and right-chiral). The latter behave identically under a rotation, differently under a boost and are exchanged by a parity transformation ($[\gamma^5, \gamma^0] \neq 0$). Chirality is therefore related to the parity transformation (for example, it enters into the definition of a pseudo-scalar $\bar{\psi}\gamma^5\psi$ or of a pseudo-vector $\bar{\psi}\gamma^5\gamma^\mu\psi$ made of two Dirac fields). Note that γ^5 and γ^0 are essentially "rotations" of one another. Chirality and parity operators exchange their matrix representation when going from the chiral to the standard basis. Chirality only exists for even space-time dimensions (see below). Chirality is also related to the weak charge (only left spinors carry a weak charge, not right spinors; and the weak interaction therefore maximally violates parity). Chirality is a Lorentz-invariant quantity: it therefore has an absolute meaning independent of the observer.
- The helicity operator is essentially[3] the projection of the spin operator onto the direction of motion $h = \vec{\Sigma} \cdot \hat{p}$, where $\vec{p}$ is the momentum operator, $\hat{p} \equiv \vec{p}/|\vec{p}|$ and $\frac{1}{2}\vec{\Sigma}$ is the spin operator. Note that helicity is not a Lorentz-invariant quantity: it has no absolute meaning and depends on the observer. However, it is a conserved quantity (in the sense that it does not depend on time).

For a massless particle, helicity and chirality turn out to be the same. Indeed, if $m = 0$, then chirality (γ^5 with eigenvalues ± 1) and helicity ($\vec{\Sigma} \cdot \hat{p}$ with eigenvalues ± 1) are identical. Using $\vec{\Sigma} = \gamma^5\gamma^0\vec{\gamma}$, show it. Therefore, physically, chirality is similar to helicity, which is less abstract. Note however, the subtle differences between the two (which are important for massive particles). The helicity operator is not Lorentz invariant, whereas γ^0 and γ^5 are. Therefore the chirality is an intrinsic (absolute) property, whereas helicity is not (it is relative, it depends on the frame). But the helicity is conserved (the helicity operator commutes with the Dirac Hamiltonian $H_D = \gamma^0\vec{\gamma} \cdot \vec{p} + m\gamma^0$, see below), whereas the chirality is not (γ^5 does not commute with H_D). Show that $[\vec{\Sigma} \cdot \hat{p}, H_D] = 0$ and $[\gamma^5, H_D] = 2m\gamma^5\gamma^0$.

In order to clearly distinguish chirality and helicity, one should say left-chiral/right-chiral and left-helical/right-helical for the eigenstates of eigenvalue $-1/+1$ of γ^5 and $h = \vec{\Sigma} \cdot \hat{p}$.

In summary, for a massless Dirac particle, helicity = chirality, which has an absolute meaning and is conserved. Therefore it makes sense to say that a massless Dirac particle is left-handed. For a massive particle, helicity $\neq$ chirality: helicity is conserved but has no absolute meaning, whereas chirality has an

[3]Here, for the clarity of the present discussion, we did not include $S = \frac{1}{2}$ into the definition of the helicity $\frac{1}{2}\vec{\Sigma} \cdot \hat{p}$.

absolute meaning but evolves in time (i.e. is not conserved). Therefore it does not make sense to say that a massive Dirac particle is left-handed. See P.B. Pal, Am. J. Phys. 79, 485 (2011).

6.2.5 General solution of the Dirac equation: mode expansion

see Ryder [3] pages 51-54

This is a technical section included in preparation for the quantization of the Dirac field to be done in the next chapter. Consider the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ with a finite mass term $m \neq 0$. We want to find a general plane wave solution. The strategy is to work in the standard (Dirac) basis for γ matrices (as it is well adapted to the non-relativistic limit) and first to obtain a plane wave solution in the rest frame (which only exists for a finite mass). And then to boost this solution to another frame to obtain an arbitrary plane wave.

A plane wave is written $\psi(x) = \psi_k e^{-ik\cdot x}$ where ψ_k is a Dirac spinor (but no longer a field) and $k \cdot x = k_0 t - \vec{k}\cdot\vec{x} = \omega t - \vec{k}\cdot\vec{x}$.

We first obtain the dispersion relation from the "squared Dirac equation" which is the massive Klein-Gordon equation $(\Box + m^2)\psi(x) = 0$. On the plane wave ansatz, it gives $(k^2+m^2)\psi_k = 0$ and as $\psi_k \neq 0$ we find that $\omega^2 = \vec{k}^2 + m^2$. We define $\omega_k \equiv \sqrt{\vec{k}^2+m^2}$ such that $\omega = \pm\omega_k$. This means that for each wavevector $\vec{k}$, there are two solutions $\omega = \pm\omega_k$ corresponding to positive and negative energies[4]. In the following we choose to write positive energy solutions as

$$\psi(x) = u_{\vec{k}} e^{-ik\cdot x} \text{ with } k\cdot x = \omega_k t - \vec{k}\cdot\vec{x}$$

which (despite the notation) only depends on $\vec{k}$ (as $\omega = \omega_k = \sqrt{\vec{k}^2+m^2}$ is not independent of $\vec{k}$). It is a plane wave with energy ω_k and momentum $\vec{k}$, and $u_{\vec{k}}$ is a Dirac spinor. Negative energy solutions are written as

$$\psi(x) = v_{\vec{k}} e^{ik\cdot x} \text{ with } k\cdot x = \omega_k t - \vec{k}\cdot\vec{x}$$

which actually corresponds to a plane wave with energy $-\omega_k$ and momentum $-\vec{k}$, and $v_{\vec{k}}$ is a Dirac spinor.

The Dirac equation $(i\not{\partial} - m)\psi(x) = 0$ actually contains more information than the mere dispersion relation $k^2 = m^2$ obtained from the Klein-Gordon equation. When applied on the two above positive and negative energy solutions it gives

$$(\not{k} - m)u_{\vec{k}} = 0 \text{ and } (\not{k} + m)v_{\vec{k}} = 0$$

In the following, we will see that these are actually projection equations getting rid of two out of the four components of a Dirac spinor (which makes sense if we remember that the electron has only two internal degrees of freedom and not four).

Let us specialize first to the rest frame where $k^\mu = (m, \vec{0})$ so that $\not{k} = k_\mu\gamma^\mu = m\gamma^0$ and the projection equations become

$$(\gamma^0 - \mathbb{I})u_{\vec{0}} = 0 \text{ and } (\gamma^0 + \mathbb{I})v_{\vec{0}} = 0$$

as $m \neq 0$. In the standard basis $\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$, so that $\frac{\mathbb{I}+\gamma^0}{2} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}$ and $\frac{\mathbb{I}-\gamma^0}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$ act as projectors onto positive or negative energy states in the rest frame. Then $u_{\vec{0}} = (\neq 0, \neq 0, 0, 0)$ and $v_{\vec{0}} = (0, 0, \neq 0, \neq 0)$. We therefore choose the following basis of solutions:

$$u_{\vec{0}}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ u_{\vec{0}}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},\ v_{\vec{0}}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ v_{\vec{0}}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

[4]Indeed, as seen in exercise session, the energy/Hamiltonian associated to the Dirac Lagrangian – and obtained from Noether's theorem via the energy-momentum tensor – is $H = \int d^3x \psi^\dagger(x) i\partial_t \psi(x)$ which equals ω when computed on a normalized plane wave $\psi(x) = \psi(\vec{x})e^{-i\omega t}$ such that $\int d^3x\psi^\dagger(x)\psi(x) = 1$.

To summarize, for $\vec{k}=0$, we have found four different solutions of the Dirac equation: two with positive energy $u_{\vec{0}}^{(\alpha)}e^{-imt}$ and two with negative energy $v_{\vec{0}}^{(\alpha)}e^{imt}$ (with $\alpha=1,2$).

To find an arbitrary state of motion, we perform a boost Λ to a moving frame such that the 4-momentum is transformed as $(m,\vec{0})\to k^\mu=(\omega,\vec{k})$ with $\omega^2-\vec{k}^2=m^2$. Introducing $\hat{n}$ as the space direction of $\vec{k}$ (such that $\vec{k}=k\hat{n}$ with $k>0$), we easily find that the rapidity $\phi\hat{n}$ characterizing the Lorentz transformation fulfills $\sinh\phi=-k/m$ and $\cosh\phi=\omega_k/m$. Such a boost on a Dirac spinor is easily performed in the chiral representation (CR) in which $\psi'_{CR}=S(\Lambda)\psi_{CR}=\begin{pmatrix}\Lambda_L & 0\\ 0 & \Lambda_R\end{pmatrix}\psi_{CR}$ with $\Lambda_L=e^{\vec{\sigma}\cdot\vec{\phi}/2}$ and $\Lambda_R=e^{-\vec{\sigma}\cdot\vec{\phi}/2}$. These formulae involve the hyperbolic functions of argument $\phi/2$ which can be deduced by using standard hyperbolic trigonometry relations[5] we therefore find that $\cosh(\phi/2)=\sqrt{(\omega_k+m)/(2m)}$, $\sinh(\phi/2)=-\sqrt{(\omega_k-m)/(2m)}$. We also know how to go from the chiral to the standard representation by a unitary transformation $\psi_{SR}=U\psi_{CR}$ with $U=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbb{I} & \mathbb{I}\\ -\mathbb{I} & \mathbb{I}\end{pmatrix}$. Therefore, under a Lorentz transformation Λ, a Dirac spinor in the standard representation (SR) transforms as $\psi_{SR}\to\psi'_{SR}=U\psi'_{CR}=US(\Lambda)\psi_{CR}=US(\Lambda)U^\dagger\psi_{SR}$. Calculating the product of three matrices $US(\Lambda)U^\dagger$, we obtain the Lorentz transformation in the SR as:

$$\frac{1}{2}\begin{pmatrix}\Lambda_R+\Lambda_L & \Lambda_R-\Lambda_L\\ \Lambda_R-\Lambda_L & \Lambda_R+\Lambda_L\end{pmatrix}=\begin{pmatrix}\cosh(\phi/2) & \vec{n}\cdot\vec{\sigma}\sinh(\phi/2)\\ \vec{n}\cdot\vec{\sigma}\sinh(\phi/2) & \cosh(\phi/2)\end{pmatrix}=\sqrt{\frac{\omega_k+m}{2m}}\begin{pmatrix}\mathbb{I} & \frac{\vec{k}\cdot\vec{\sigma}}{\omega_k+m}\\ \frac{\vec{k}\cdot\vec{\sigma}}{\omega_k+m} & \mathbb{I}\end{pmatrix}$$

We now apply $US(\Lambda)U^\dagger$ to the Dirac spinors $u_{\vec{0}}^{(\alpha)}$ and $v_{\vec{0}}^{(\alpha)}$ to obtain $u_{\vec{k}}^{(\alpha)}$ and $v_{\vec{k}}^{(\alpha)}$ (we also know that $e^{\mp imt}$ become $e^{\mp ik\cdot x}$ with $\omega=\omega_k$). We find that the plane wave solutions are $u_{\vec{k}}^{(\alpha)}e^{-ik\cdot x}$ and $v_{\vec{k}}^{(\alpha)}e^{ik\cdot x}$ with

$$u_{\vec{k}}^{(1)}=\sqrt{\frac{\omega_k+m}{2m}}\begin{pmatrix}1\\0\\\frac{k_z}{\omega_k+m}\\\frac{k_-}{\omega_k+m}\end{pmatrix},\ u_{\vec{k}}^{(2)}=\sqrt{\frac{\omega_k+m}{2m}}\begin{pmatrix}0\\1\\\frac{k_+}{\omega_k+m}\\-\frac{k_z}{\omega_k+m}\end{pmatrix}$$

and

$$v_{\vec{k}}^{(1)}=\sqrt{\frac{\omega_k+m}{2m}}\begin{pmatrix}\frac{k_z}{\omega_k+m}\\\frac{k_-}{\omega_k+m}\\1\\0\end{pmatrix},\ v_{\vec{k}}^{(2)}=\sqrt{\frac{\omega_k+m}{2m}}\begin{pmatrix}\frac{k_+}{\omega_k+m}\\-\frac{k_z}{\omega_k+m}\\0\\1\end{pmatrix}$$

where $k_\pm\equiv k_x\pm ik_y$. One may check the following basis independent relations:

$$u_{\vec{k}}^{(\alpha)}=\frac{\not{k}+m}{\sqrt{2m(m+\omega_k)}}u_{\vec{0}}^{(\alpha)},\ v_{\vec{k}}^{(\alpha)}=\frac{-\not{k}+m}{\sqrt{2m(m+\omega_k)}}v_{\vec{0}}^{(\alpha)}$$

The Dirac spinors $u_{\vec{k}}^{(\alpha)}$ and $v_{\vec{k}}^{(\alpha)}$ are normalized as follows:

$$u_{\vec{k}}^{(\alpha)\dagger}u_{\vec{k}}^{(\beta)}=\frac{\omega_k}{m}\delta^{\alpha\beta}=v_{\vec{k}}^{(\alpha)\dagger}v_{\vec{k}}^{(\beta)}\text{ and }u_{\vec{k}}^{(\alpha)\dagger}v_{-\vec{k}}^{(\beta)}=0=v_{\vec{k}}^{(\alpha)\dagger}u_{-\vec{k}}^{(\beta)}$$

$$\bar{u}_{\vec{k}}^{(\alpha)}u_{\vec{k}}^{(\beta)}=\delta^{\alpha\beta}=-\bar{v}_{\vec{k}}^{(\alpha)}v_{\vec{k}}^{(\beta)}\text{ and }\bar{u}_{\vec{k}}^{(\alpha)}v_{\vec{k}}^{(\beta)}=0=\bar{v}_{\vec{k}}^{(\alpha)}u_{\vec{k}}^{(\beta)}$$

We also define the following projectors on positive and negative energy states

$$P_+(\vec{k})\equiv\sum_{\alpha=1}^{2}u_{\vec{k}}^{(\alpha)}\bar{u}_{\vec{k}}^{(\alpha)}\text{ and }P_-(\vec{k})\equiv-\sum_{\alpha=1}^{2}v_{\vec{k}}^{(\alpha)}\bar{v}_{\vec{k}}^{(\alpha)}$$

[5] Such as $\cosh(\phi/2)=\sqrt{(1+\cosh\phi)/2}$.

Exercise: show that $P_+^2 = P_+$ and similarly for P_-. Assume that $P_+(\vec{k}) = a\mathbb{I} + b\not{k}$ and find a and b to obtain that:

$$P_+(\vec{k}) = \frac{\not{k} + m}{2m} \text{ and } P_-(\vec{k}) = \frac{-\not{k} + m}{2m}$$

Remark: $(\not{k} - m)u_{\vec{k}} = 0$ and $(\not{k} + m)v_{\vec{k}} = 0$

A general solution of the Dirac equation can be written as a linear combinaison of the planes wave solutions that we found

$$\boxed{\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{m}{\omega_k} \sum_\alpha \left(b_\alpha(k) u_{\vec{k}}^{(\alpha)} e^{-ik\cdot x} + d_\alpha^*(k) v_{\vec{k}}^{(\alpha)} e^{ik\cdot x} \right)} \tag{6.24}$$

where b and d's are the name of the expansion coefficients. This expression will be useful when quantizing the Dirac field.

6.2.6 Conserved currents and charges

see also exercise sheet

In this section, we apply the results of the Noether theorem to obtain three conserved currents: the energy-momentum tensor, the vector current and the axial current. Starting from the Dirac Lagrangian $\mathcal{L} = \frac{i}{2}[\bar{\psi}\gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi] - m\bar{\psi}\psi$ and using the EL equations, we obtain the equations of motion for ψ and for $\bar{\psi}$: $(i\gamma^\mu \partial_\mu - m)\psi = 0$ and $i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0$.

The energy-momentum tensor is the Noether current associated to the symmetry under space-time translation. It reads[6]

$$\theta^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi + \partial_\nu \bar{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \delta^\mu_\nu \mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu \partial_\nu \psi - \frac{i}{2}(\partial_\nu \bar{\psi})\gamma^\mu \psi - \delta^\mu_\nu \mathcal{L} \tag{6.25}$$

Therefore $\theta^{00} = \Pi_\psi \partial_0 \psi + \Pi_{\bar{\psi}} \partial_0 \bar{\psi} - \mathcal{L}$, which we recognize as the Hamiltonian density $\mathcal{H}$, with $\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\dot{\psi})} = \frac{i}{2}\bar{\psi}\gamma^0 = \frac{i}{2}\psi^\dagger$ and $\Pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\dot{\bar{\psi}})} = -\frac{i}{2}\gamma^0 \psi = \frac{i}{2}\psi^\dagger$. After integration by part and using the equations of motion, we find that the Hamiltonian is:

$$H = \int d^3x \theta^{00} = \int d^3x \psi^\dagger (i\partial_t)\psi \tag{6.26}$$

Similarly $\theta^{0i} = \frac{i}{2}\bar{\psi}\gamma^0 \partial^i \psi - \frac{i}{2}(\partial^i \bar{\psi})\gamma^0 \psi$ and the corresponding Noether charge (the momentum) is:

$$\vec{P} = \int d^3x \theta^{0i} = \int d^3x \psi^\dagger (-i\vec{\nabla})\psi \tag{6.27}$$

Both equations can be summarized by saying that the 4-momentum $P^\mu = \int d^3x \psi^\dagger (i\partial^\mu)\psi$ is a conserved charge, in which we recognize the 4-momentum operator $i\partial^\mu$.

There is also a so-called vector current that is conserved as a result of global $U(1)$ phase invariance. Indeed when $\psi \to e^{i\alpha}\psi$ with $\alpha(x) = \alpha =$ constant, $\mathcal{L}$ is invariant. The Noether theorem implies that $J_V^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} i\psi + i\bar{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}$ should be conserved (the V index is for Vector in order to distinguish it from another conserved current called Axial, see below. This current is also often simply called J^μ.). We find that

$$J^\mu = J_V^\mu = \bar{\psi}\gamma^\mu \psi. \tag{6.28}$$

We know that this is a true 4-vector (see the section on Dirac bilinears). Let's check that it is divergenceless: $\partial_\mu(\bar{\psi}\gamma^\mu \psi) = (\partial_\mu \bar{\psi})\gamma^\mu \psi + \bar{\psi}\gamma^\mu(\partial_\mu \psi) = -\frac{m}{i}\bar{\psi}\psi + \bar{\psi}\frac{m}{i}\psi = 0$ upon using the equations of motion. In non-covariant notation $\partial_t(\psi^\dagger \psi) + \vec{\nabla}\cdot(\psi^\dagger \vec{\alpha}\psi) = 0$ where $\vec{\alpha} \equiv \gamma^0 \vec{\gamma}$ is the velocity operator, $\psi^\dagger \psi$ is the particle

[6] Observe that this tensor is not symmetric under permutation of the indices. There is a general procedure, due to Belinfante, to obtain a symmetric tensor. Since the tensor is not symmetric in this form, it is important to keep in mind that the two indices have a different status. The second one characterizes which symmetry transformation we are considering and the first one describes the components of the associated conserved current. As a consequence, $\partial_\mu \theta^{\mu\nu} = 0$, as you can check for yourself.

density and $\psi^\dagger\vec{\alpha}\psi$ is the particle current density (here we anticipate in invoking "particles"). The conserved charge is $Q = Q_V = \int d^3x J_V^0 = \int d^3x \psi^\dagger\psi = \int d^3x(\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L)$ such that $\frac{dQ_V}{dt} = 0$. To clearly understand the nature of this conservation law, one would need to couple the electronic field to the electromagnetic field (in order to see the electric charge as a coupling strength) and to quantize the fields (in order to really have particles and give a meaning to the "number of particles").

Consider now the massless Dirac Lagrangian. In the chiral basis, it reads $\mathcal{L} = i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R$. On top of the $U(1)$ invariance discussed above, there is an extra phase symmetry. Indeed, one can transform the phase of the left and right Weyl spinors separately. The usual way to define this transformation is $\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$, which means that $\psi_L \to e^{-i\alpha}\psi_L$ and $\psi_R \to e^{i\alpha}\psi_R$. Under such a transformation the mass term $\bar{\psi}\psi$ is not left invariant, which explains why we consider only the massless Dirac Lagrangian. According to Noether's theorem the conserved current is

$$J_5^\mu = J_A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi. \tag{6.29}$$

It is called the axial current (it is also often called J_5^μ) and is a pseudo 4-vector. Check that it is indeed conserved if and only if $m = 0$. The Noether charge is $Q_5 = Q_A = \int d^3x\bar{\psi}\gamma^5\gamma^0\psi = \int d^3x(\psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L)$, which is a pseudo 4-scalar. The conservation law is that of the difference in number of right and left particles. When $m = 0$ both the vector and the axial current are conserved, which means that the number of left particles is separately conserved and the number of right particles as well.

Remark:

- The phase transformation of a Weyl spinor is usually known as a chiral transformation. When applied on a Dirac bispinor, it decomposes into a vector transformation (acting similarly on left and right fields) and an axial transformation (acting in opposite ways on the right and left fields). Indeed if $\psi_L \to e^{i\alpha_L}\psi_L$ and $\psi_R \to e^{i\alpha_R}\psi_R$, one can define $\alpha_V = (\alpha_R + \alpha_L)/2$ and $\alpha_A = (\alpha_R - \alpha_L)/2$ such that $\psi_L \to e^{i\alpha_V}e^{-i\alpha_A}\psi_L$ and $\psi_R \to e^{i\alpha_V}e^{i\alpha_A}\psi_R$ or, in other words, $\psi \to e^{i\alpha_V}e^{i\alpha_A\gamma^5}\psi$. Here α_V is the overall global phase and α_A is the relative phase between the two Weyl spinors.
- The axial symmetry of the massless spinor field is quite interesting. It is a classical symmetry that does not survive upon quantizing the theory, a fact known as an *anomaly* in quantum field theory. In the present case, it is called the axial or chiral or triangle or Adler-Jackiw-Bell anomaly. For a first introduction to anomalies see Zee [4] pages 243-254. The anomaly manifests itself as a non-conservation of the quantum current in the presence of an electromagnetic field. See also the exam 2016-2017.

6.3 Quantization

[see Ryder [3], pages 139-143]

6.3.1 Anticommutators: spinor fields are weird

We start from the mode expansion of a classical Dirac field obeying the equation $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$. Remember, the four plane wave solutions: $u_{\vec{k}}^{(\alpha)}e^{-ik\cdot x}$ (positive energy) and $v_{\vec{k}}^{(\alpha)}e^{ik\cdot x}$ (negative energy) with $\alpha = 1, 2$ (α is something like a spin index, labelling the two spin projections). The u and v's are Dirac spinors (not fields). The information they carry is about the internal polarization (the spin). A general mode expansion is

$$\psi(x) = \int_k \sum_\alpha \left(b_\alpha(k)u_{\vec{k}}^{(\alpha)}e^{-ik\cdot x} + d_\alpha^*(k)v_{\vec{k}}^{(\alpha)}e^{ik\cdot x}\right) \tag{6.30}$$

where $k^\mu = (\omega_k, \vec{k})$ as usual and here $\int_k \equiv \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_k}$ is different then in the case of the scalar fields[7] (but it is also Lorentz invariant). This is conventional and is related to the way the u and v's are normalized.

[7]For scalar fields the convention was that $\int_k \equiv \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}$.

Let ψ become an operator. Its mode expansion is

$$\psi(x) = \int_k \sum_\alpha \left(b_\alpha(k) u_{\vec{k}}^{(\alpha)} e^{-ik\cdot x} + d_\alpha^\dagger(k) v_{\vec{k}}^{(\alpha)} e^{ik\cdot x} \right) \tag{6.31}$$

as $b_\alpha(k)$ and $d_\alpha(k)$ are now also operators. And the Dirac conjugate field operator $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ is

$$\bar{\psi}(x) = \int_k \sum_\alpha \left(b_\alpha^\dagger(k) \bar{u}_{\vec{k}}^{(\alpha)} e^{ik\cdot x} + d_\alpha(k) \bar{v}_{\vec{k}}^{(\alpha)} e^{-ik\cdot x} \right) \tag{6.32}$$

where $\bar{u} \equiv u^\dagger \gamma^0$ and similarly for $\bar{v}$.

From the Lagrangian $\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$, we obtain the conjugate field $\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger$ and the Hamiltonian $H = \int d^3x(\Pi\dot{\psi} - \mathcal{L}) = \int d^3x \bar{\psi}(-i\gamma^j \partial_j + m)\psi = \int d^3x \psi^\dagger i\partial_0 \psi$ (in the last step, we used the Dirac equation)[8]. Inserting the mode expansion and using the normalization conditions on the u and v's, we find that (check it, it is not that obvious)

$$H = \int_k \sum_\alpha \omega_k \left(b_\alpha^\dagger(k) b_\alpha(k) - d_\alpha(k) d_\alpha^\dagger(k) \right) \tag{6.33}$$

with $k_0 = \omega_k$. Note that we have not performed normal-ordering for the time being and remark the important minus sign in the above expression.

We now impose commutation relations (as for the scalar fields):

$$[b_\alpha(k), b_{\alpha'}^\dagger(k')] = (2\pi)^3 \frac{\omega_k}{m} \delta_{\alpha,\alpha'} \delta(\vec{k} - \vec{k}') = [d_\alpha(k), d_{\alpha'}^\dagger(k')] \tag{6.34}$$

and all other commutators (with b and b, $b^\dagger$ and $b^\dagger$, b and d, etc.) vanish. The strange factors $(2\pi)^3 \frac{\omega_k}{m}$ come from our normalization convention for the u and v's (nothing profound). From these relations, we find that the Hamiltonian is

$$H = \int_k \sum_\alpha \omega_k \left(b_\alpha^\dagger(k) b_\alpha(k) - d_\alpha^\dagger(k) d_\alpha(k) \right) - \sum_\alpha \int d^3k \delta^{(3)}(\vec{0}) \omega_k \tag{6.35}$$

A first issue is the diverging vacuum energy $-\sum_\alpha \int d^3k \delta^{(3)}(\vec{0})\omega_k = -\sum_{\alpha,\vec{k}} \omega_k = -4\sum_{\vec{k}} \frac{\omega_k}{2}$ (we used that $\int d^3k \delta^{(3)}(\vec{k}) \to \sum_{\vec{k}} \delta_{\vec{0},\vec{0}} = \sum_{\vec{k}}$). It is negative, it has a fourfold degeneracy (later to be interpreted as spin up/spin down and particle/anti-particle) and comes from the zero-point motion. This could be taken care of by redefining the zero of energy (e.g. via normal ordering) $H \equiv H - \langle vac|H|vac\rangle$. The second problem is much more serious, it is related to $\int_k \sum_\alpha \omega_k \left(b_\alpha^\dagger(k) b_\alpha(k) - d_\alpha^\dagger(k) d_\alpha(k) \right)$ being unbounded from below because of the minus sign in front of $d^\dagger d$. There is no lower bound to the energy! This means that we can not define a stable vacuum or groundstate, because we could always lower the energy (and hence it would not be a groundstate) by adding more d-type particles. Therefore, we step back and give up imposing commutation relations.

To overcome this problem, we follow Jordan and Wigner (1928) and take a radical step by postulating that <u>anticommutation</u> relations should be used instead:

$$\{b_\alpha(k), b_{\alpha'}^\dagger(k')\} = (2\pi)^3 \frac{\omega_k}{m} \delta_{\alpha,\alpha'} \delta(\vec{k} - \vec{k}') = \{d_\alpha(k), d_{\alpha'}^\dagger(k')\} \tag{6.36}$$

We also explicitly write the anticommutators that vanish because they are weird:

$$\{b, b\} = 0 = \{b^\dagger, b^\dagger\} = \{d, d\} = \{d^\dagger, d^\dagger\} \text{ i.e. } (b^\dagger)^2 = 0 \text{ e.g.} \tag{6.37}$$

[8]Note that we have a "single-particle" Hamiltonian $h_D = -i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0 = i\partial_t$ (this is the historical Dirac Hamiltonian). We also have a Hamiltonian for the field (or a "many-body" Hamiltonian) $H = \int d^3x \mathcal{H}$ given in terms of the Hamiltonian density $\mathcal{H} = \psi^\dagger(x) h_D \psi(x)$.

and even weirder

$$\{b,d\}=0=\{b^\dagger,d\}=\{b,d^\dagger\}=\{b^\dagger,d^\dagger\}\text{ i.e. } b^\dagger d^\dagger=-d^\dagger b^\dagger \text{ e.g.} \tag{6.38}$$

Contrary to the case of the scalar field, which was quantized using commutation relations, here all these equations are "quantum" (not just the one explicitly containing $\delta(\vec{k}-\vec{k}')$) in the sense that an anticommutator can never be equal to zero in a classical theory (unless one introduces anticommuting numbers, known as Grassmann numbers). Note that b and d-type particles are all excitation quanta of the same field $\psi(x)$ and are not like two species of particles (as e.g. electrons and protons). With these anticommutation relations, the Hamiltonian (with the vacuum energy removed) becomes

$$H\equiv H-\langle vac|H|vac\rangle=\int_k\sum_\alpha \omega_k\left(b^\dagger_\alpha(k)b_\alpha(k)+d^\dagger_\alpha(k)d_\alpha(k)\right)\,, \tag{6.39}$$

which is now positive definite (it has a lower bound).

Anticommutation relations are required by the positivity of energy H. We will see another motivation for anticommutation relations below. It is related to the excitation quanta (the particles) being fermions, which we admit for the moment.

It is also possible to define a normal ordering procedure for fermions. It moves all creation operators to the left of annihilation operators (while preserving the order among either creation operators and also among annihilation operators) and gives a minus sign for each exchange of two (creation or annihilation) operators in the process. For example

$$:b(k_1)b^\dagger(k_2):=-b^\dagger(k_2)b(k_1)\text{ and } :b(k_1)b(k_2)b^\dagger(k_3):=(-1)^2b^\dagger(k_3)b(k_1)b(k_2)=b^\dagger(k_3)b(k_1)b(k_2) \tag{6.40}$$

Then the procedure of canonical quantization for spin 1/2 fields is: use anticommutators (instead of commutators) to quantize the fields and normal-order (with the specific prescription) observables (such as the Hamiltonian, 3-momentum, angular momentum, etc) which are constructed from bilinears in the field operators. For example, the 4-momentum is

$$P^\mu=\int d^3x:\psi^\dagger i\partial^\mu\psi:=\int_k\sum_\alpha k^\mu:\left(b^\dagger_\alpha(k)b_\alpha(k)-d_\alpha(k)d^\dagger_\alpha(k)\right):=\int_k\sum_\alpha k^\mu\left(b^\dagger_\alpha(k)b_\alpha(k)+d^\dagger_\alpha(k)d_\alpha(k)\right) \tag{6.41}$$

with $k^\mu=(\omega_k,\vec{k})$.

As an exercise (see Zee [4], pages 106-107), show the following equal-time anticommutation relations (ETAR):

$$\{\psi_i(t,\vec{x}),\psi^\dagger_j(t,\vec{x}')\}=\hbar\delta_{ij}\delta(\vec{x}-\vec{x}') \tag{6.42}$$

where $i,j=1,2,3,4$ here label the quadruplet of components of a Dirac bispinor (not to be confused with $i=1,2,3=x,y,z$). Note that $\Pi(x)=i\psi(x)^\dagger=i\bar{\psi}(x)\gamma^0$ is the conjugate field and that indeed $\{\psi_i(t,\vec{x}),\Pi_j(t,\vec{x}')\}=i\hbar\delta_{ij}\delta(\vec{x}-\vec{x}')$. Other anticommutators are[9]

$$\{\psi_i(t,\vec{x}),\psi_j(t,\vec{x}')\}=0=\{\psi^\dagger_i(t,\vec{x}),\psi^\dagger_j(t,\vec{x}')\} \tag{6.43}$$

Hints: use the mode expansion, the anticommutation relation between b and $b^\dagger$ and also between b and d and then the relation $\sum_\alpha u^{(\alpha)}(k)u^{(\alpha)\dagger}(k)=P_+(k)\gamma^0=\frac{\not{k}+m}{2m}\gamma^0$ and similarly for the v's.

Question: obtain the equation of motion for the Dirac field $\psi(x)$ in several different ways. First from the EL equation. Then from the Hamilton equation $\dot{\psi}=\frac{\delta H}{\delta\Pi}$. And as a third way, from the Heisenberg equation of motion $\dot{\psi}=-i[\psi,H]$. And now a tricky question: what about $-i\{\psi,H\}$?

[9] Think how weird fermions are: at equal-time, the Dirac field should anticommute at all distances! Doesn't that violate relativity? No. Actually fermionic fields are not observables. Observables are constructed from field bilinears and this makes a huge difference. The field bilinears can be shown to respect the requirements of special relativity even if the fields themselves seem not to. In addition, upon taking $\hbar\to 0$, only the first anticommutator is modified to $\{\psi_i(t,\vec{x}),\psi^\dagger_j(t,\vec{x}')\}=0$. Is this the classical limit of fermionic fields? All equal-time anticommutators (not commutators!) vanish. These are no longer operators but are non commuting numbers. They are anticommuting numbers. The latter are usually called Grassmann numbers or variables, i.e. anticommuting c-numbers.

6.3.2 Fock space, Fermi-Dirac statistics and the spin-statistics relation

The construction of Fock space starts by defining the vacuum state $|vac\rangle$ as being annihilated by all $b_\alpha(k)$ and $d_\alpha(k)$ operators. So that $H|vac\rangle = 0$ after normal ordering. There are four types of single particle states $b^\dagger_\alpha(k)|vac\rangle$ and $d^\dagger_\alpha(k)|vac\rangle$ (these will later be interpreted as spin up/down and particle/anti-particle). However, because of the anticommutation relation $\{b^\dagger_\alpha(k), b^\dagger_\alpha(k)\} = 0$, one hase $[b^\dagger_\alpha(k)]^2 = 0$, which means that it is impossible to have two identical (b_α-type here) particles in the same mode. This is reminiscent of the Pauli exclusion principle. Note that Fermi-Dirac statistics is more than the mere exclusion principle. It requires having many-body wavefunctions that are antisymmetric under exchange. But the anticommutation relation $\{b^\dagger_\alpha(k), b^\dagger_{\alpha'}(k')\} = 0$ implies that the two-particle state $b^\dagger_\alpha(k)b^\dagger_{\alpha'}(k')|vac\rangle = -b^\dagger_{\alpha'}(k')b^\dagger_\alpha(k)|vac\rangle$ is indeed antisymmetric in the exchange of the two particles. This is indeed Fermi-Dirac statistics.

We here glimpse at a particular case of a general relation: half-integer spin fields are quantized using anticommutation relations, which leads to Fermi-Dirac statistics. Whereas integer spin fields are quantized with commutation relations, leading to Bose-Einstein statistics. This is the famous spin-statistics theorem, which we only state here. It was proven in the frame of relativistics quantum field theory by Pauli, Fierz, Lüders, Zumino and others.

6.3.3 $U(1)$ charge and anti-particles

From the invariance of the Dirac action under a global internal $U(1)$ transformation, we found a conserved vector current $J^\mu_V = \bar\psi\gamma^\mu\psi$ such that $\partial_\mu J^\mu_V = 0$. Upon gauging the symmetry, we understood that this is actually the conservation of electric charge. The conserved charge being $Q = \int d^3x J^0_V = \int d^3x \psi^\dagger(x)\psi(x)$. Now, when quantizing the Dirac field, the conserved charge becomes

$$Q = \int d^3x : \psi^\dagger(x)\psi(x) := \int_k \sum_\alpha : \left(b^\dagger_\alpha(k)b_\alpha(k) + d_\alpha(k)d^\dagger_\alpha(k)\right) := \int_k \sum_\alpha \left(b^\dagger_\alpha(k)b_\alpha(k) - d^\dagger_\alpha(k)d_\alpha(k)\right) \quad (6.44)$$

The vacuum is uncharged (thanks to normal ordering) $Q|vac\rangle = 0$. The charge is quantized $Q \in \mathbb{Z}$ and counts the number of particles (i.e. b-type particles, carrying a +1 charge) minus the number of anti-particles (i.e. d-type particle, carrying a -1 charge). Actually the electric charge is $-eQ$ where $e = 1.6 \times 10^{-19}$ C is the electric charge unit that plays the role of a coupling strength.

6.3.4 Charge conjugation

[see Zee [4], pages 97-98]

Charge conjugation is a discrete transformation that exchanges particles and anti-particles. The Dirac equation in a $U(1)$ gauge field is $(i\gamma^\mu D_\mu - m)\psi = 0$ with the covariant derivative $D_\mu = \partial_\mu + iqA_\mu$, where $q = -e < 0$ is a property of the Dirac field that measures the coupling strength between the Dirac and the gauge field (it is called the electric charge of the matter field). We now check that if ψ satisfies the Dirac equation

$$[i\gamma^\mu(\partial_\mu + iqA_\mu) - m]\psi = 0, \quad (6.45)$$

we can find a transformed field ψ^c that satisfies the Dirac equation with the opposite charge. In either the chiral or the standard representation (but not in a general representation), the transformation is

$$\psi \to \psi^c = -i\gamma^2\psi^* \quad (6.46)$$

(the phase $-i$ is conventional). It is a transformation that involves complex conjugation (as time reversal).

As $(\gamma^2)^2 = -1$, we have that $(\psi^c)^c = -i\gamma^2(-i\gamma^2\psi^*)^* = -i\gamma^2 i(-\gamma^2)\psi = (-i\gamma^2)^2\psi = \psi$, as expected. Note that $(\gamma^2)^* = -\gamma^2$ because $\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$ contains the purely imaginary Pauli matrix $\sigma^2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Second, we take the complex conjugation of the above Dirac equation to find

$$[-i(\gamma^\mu)^*(\partial_\mu - iqA_\mu) - m]\psi^* = 0 \tag{6.47}$$

Then we multiply to the left by γ^2 and insert the identity in the form $(\gamma^2)^2$ between [...] and ψ^* to get

$$[-i\gamma^2(\gamma^\mu)^*\gamma^2(\partial_\mu - iqA_\mu) - m]\gamma^2\psi^* = 0 \tag{6.48}$$

Using that $\gamma^2(\gamma^\mu)^*\gamma^2 = -\gamma^\mu$, we finally obtain:

$$[\gamma^\mu(\partial_\mu - iqA_\mu) - m]\gamma^2\psi^* = 0 \tag{6.49}$$

This shows the desired property that $[i\gamma^\mu(\partial_\mu - iqA_\mu) - m]\psi^c = 0$ i.e. $(i\gamma^\mu D^*_\mu - m)\psi^c = 0$. The charge conjugate field ψ^c satisfies the same Dirac equation (same mass) but with an opposite electric charge.

6.3.5 Motivation for anticommutation

[see Zee [4], pages 103-108]

Here, we would like to show that imposing Fermi-Dirac statistics leads to anticommutators (instead of the Jordan-Wigner way: assuming anticommutators leading to FD statistics).

Let $b^\dagger_\alpha|vac\rangle$ be a single fermion state (a mode) with quantum number α (here α is not the same thing as the index $\alpha = 1, 2$ of the section on the Dirac equation in which it serves to number the spinor components. Here it serves as a short hand notation for something like $\vec{k}$.). In first quantization this state would be called $|1 : \alpha\rangle$. Now add another fermion in mode $\beta \neq \alpha$: $b^\dagger_\beta b^\dagger_\alpha|vac\rangle$. This is a two particle state.

1) For two electrons, which are fermions, we would like the wavefunction to be antisymmetric under exchange. Therefore $b^\dagger_\beta b^\dagger_\alpha|vac\rangle = -b^\dagger_\alpha b^\dagger_\beta|vac\rangle$. Actually, we would like this to hold for an arbitrary state $|\chi\rangle$ and not only for the vacuum state. We want $b^\dagger_\beta b^\dagger_\alpha|\chi\rangle = -b^\dagger_\alpha b^\dagger_\beta|\chi\rangle$ i.e. $\{b^\dagger_\alpha, b^\dagger_\beta\} = 0$ and by taking the adjoint we find $\{b_\alpha, b_\beta\} = 0$ when $\alpha \neq \beta$.

As a subcase, we get the Pauli exclusion principle. Indeed, when $\beta = \alpha$, there is no such two-particle state, i.e. $(b^\dagger_\alpha)^2|vac\rangle = 0$ (and also $(b^\dagger_\alpha)^2|\chi\rangle = 0$ for an arbitrary state $|\chi\rangle$ in Fock space). Therefore $\{b^\dagger_\alpha, b^\dagger_\alpha\} = 0$. By taking the adjoint, we also have $\{b_\alpha, b_\alpha\} = 0$.

2) At this point, we already have $\{b^\dagger_\alpha, b^\dagger_\beta\} = 0 = \{b_\alpha, b_\beta\}$ for all α and β modes. But we still need to obtain $\{b_\alpha, b^\dagger_\beta\} \neq 0$. We now ask that $N \equiv \sum_\alpha b^\dagger_\alpha b_\alpha$ be the number operator, which means that $[N, b^\dagger_\alpha] = b^\dagger_\alpha$. Indeed, the whole construction of Fock space such as $|N_\alpha = 1\rangle = b^\dagger_\alpha|0\rangle$ proceeds from this relation. For bosons ($[b_\alpha, b^\dagger_\beta] = \delta_{\alpha,\beta}$ and $[b_\alpha, b_\beta] = 0$), the relation follows from $[N, b^\dagger_\beta] = \sum_\alpha [b^\dagger_\alpha b_\alpha, b^\dagger_\beta] = \sum_\alpha \left(b^\dagger_\alpha[b_\alpha, b^\dagger_\beta] + [b^\dagger_\alpha, b^\dagger_\beta]b_\alpha\right) = b^\dagger_\beta$ upon using $[AB, C] = A[B, C] + [A, C]B$. Let's try to repeat that for fermions: $[N, b^\dagger_\beta] = \sum_\alpha [b^\dagger_\alpha b_\alpha, b^\dagger_\beta] = \sum_\alpha \left(b^\dagger_\alpha\{b_\alpha, b^\dagger_\beta\} - \{b^\dagger_\alpha, b^\dagger_\beta\}b_\alpha\right)$ upon using $[AB, C] = A\{B, C\} - \{A, C\}B$. We already have $\{b^\dagger_\alpha, b^\dagger_\beta\} = 0$ and therefore $[N, b^\dagger_\beta] = \sum_\alpha b^\dagger_\alpha\{b_\alpha, b^\dagger_\beta\}$. To obtain the desired relation $[N, b^\dagger_\alpha] = b^\dagger_\alpha$, we therefore require that $\{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha,\beta}$, which is what we wanted to show.

In the end, we obtain

$$\boxed{\{b_\alpha, b_\beta\} = 0 = \{b^\dagger_\alpha, b^\dagger_\beta\} \text{ and } \{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha,\beta}} \tag{6.50}$$

as a consequence of asking for Fermi-Dirac statistics and the corresponding Fock space.

6.3.6 Spin

The spin of a particle is an observable associated with the generator of rotations. It is therefore necessary to derive the Noether current associated with the infinitesimal Lorentz transformations. A lengthy calculation shows that the conserved currents are

$$M^{\rho,\mu\nu} = x^\mu\theta^{\rho,\nu} - x^\nu\theta^{\rho,\mu} + \frac{1}{4}\bar{\psi}\{\gamma^\rho, \sigma^{\mu\nu}\}\psi \tag{6.51}$$

which satisfies $\partial_\rho M^{\rho,\mu\nu} = 0$ when the equations of motion are imposed.

The conserved charge associated with the rotation in the plane (ij) is the integral over space of $M^{0,ij}$. It contains an orbital part and a spin part. To simplify the discussion, we will consider a massive particle at rest, described by the ket $b_\alpha(0)|vac\rangle$. In this state, only the spin part contributes. In order to characterize the spin of this state, we look at the action of this ket on the operator $\int d^3x\, M^{0,ij}$. We will eventually show that this ket is an eigenstate of the operator with eigenvalues $\pm\frac{1}{2}$. To do so, we will use the fact that the vacuum is invariant under rotation: $\int d^3x\, M^{0,ij}|vac\rangle$, which enables us to write:

$$\int d^3x\, M^{0,\mu\nu} b_s^\dagger(0)|vac\rangle = \int d^3x\, \left[M^{\rho,\mu\nu}, b_s^\dagger(0)\right]|vac\rangle \tag{6.52}$$

we can then use the anticommutation relations to compute this commutator by writing $[ab,c] = a\{b,c\} - \{a,c\}b$, which eventually leads to

$$\int d^3x\, M^{0,ij} b_\alpha^\dagger(0)|vac\rangle = \frac{1}{2} u_{\vec{0}}^{(\beta)\dagger} \sigma^{ij} u_{\vec{0}}^{(\alpha)\dagger} b_\beta^\dagger(0)|vac\rangle \tag{6.53}$$

We are now done! if we choose to study the spin in the z direction (that is $(ij) = (12)$, we find two eigenstates with eigenvalues $\pm\frac{1}{2}$. This proves that indeed the Dirac field describes a spin 1/2 particle.

6.3.7 Causality and locality

A consequence of the anticommutation relations that we impose is that the commutator of the fields at points separated by a spacelike interval do not commute. This implies that measuring a field at one event in general influences the measure of the field in a presumably causally independent region. Is this a disaster, as we may expect

The answer to this paradox comes from the observation that the fermionic field itself in not an observable. A true observable is always quadratic in the field (think for instance of the energy density, etc. If we compute the commutator of these quadratic operators at causally independent events, we retrieve that they indeed commute, as they should.

Chapter 7

Electromagnetism

7.1 Electromagnetic field and the Maxwell equation

The electromagnetic (or Maxwell) field is a real massless 4-vector field. We will see that it is actually more than that: it is also a gauge field. We start by discussing the covariant formulation of the Maxwell equations.

7.1.1 Covariant form of the Maxwell equations

(see also exercise sheet #1)
Here we change perspective and do not start by constructing a 4-vector field theory just from symmetries but suppose that we already know the Maxwell equations describing the electromagnetic field. In Heaviside-Lorentz rationalized units (and with $c = 1$)[1], these equations are

(a) $\vec{\nabla} \cdot \vec{B} = 0$ (magnetic monopoles (or magnetic charges) do not exist)

(b) $\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0$ (Faraday: time dependent B field produces an E field)

(c) $\vec{\nabla} \cdot \vec{E} = \rho$ (Gauss: electric charges exist and are sources of E field)

(d) $\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{j}$ (Ampère + Maxwell: electric currents and time-dependent E field produce B field)

when written in terms of 3-vectors and 3-scalars. The two first (a and b) are the homogeneous Maxwell equations (no sources in the right hand side). The two last (c and d) are the inhomogeneous Maxwell equations (sources in the right hand side, i.e. ρ and $\vec{j}$). These equations are Lorentz covariant (the Lorentz transformation was actually discovered from them). But they are not manifestly covariant as they are written in terms of irreps of the rotation group, whereas they should be written in terms of irreps of the Lorentz group.

We start by introducing the electromagnetic field strength $F^{\mu\nu}$, which is just a smart way of writing the electric and magnetic fields together in a single object. It is an anti-symmetric rank 2 tensor, whose 6 independent components are $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B^k$. Or the other way around: $E^i = -F^{0i}$ and $B^i = -\epsilon^{ijk} F^{jk}/2$. Written in matrix form, the field strength is:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{7.1}$$

[1] In this footnote, we explicitly restore the units of c to show the Maxwell equations in Heaviside-Lorentz rationalized units:

$$\vec{\nabla} \cdot \vec{B} = 0\,, \vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0\,, \vec{\nabla} \cdot \vec{E} = \rho \text{ and } \vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{1}{c}\vec{j}$$

For more details on the manipulations needed to get rid of μ_0 and ϵ_0 and have only c to appear in the Maxwell equations, see [15].

We now rewrite the Maxwell equations in covariant form using the field strength tensor and start with the homogeneous ones (a) and (b). Usually the potential vector $\vec{A}$ is introduced so that $\vec{B} = \vec{\nabla} \times \vec{A}$ is defined as a 3-curl in order that $\vec{\nabla} \cdot \vec{B} = 0$ is automatically satisfied. Likewise, the scalar potential A_0 is introduced such that $\vec{E} = -\vec{\nabla} A_0 - \partial_t \vec{A}$ in order that $\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0$ is automatically satisfied. Note that $\partial_t \vec{A} + \vec{\nabla} A_0 = -\vec{E}$ together with $\vec{\nabla} \times \vec{A} = \vec{B}$ actually defines a 4-curl i.e. $\partial^\mu A^\nu - \partial^\nu A^\mu$. Here, similarly, if we say that there exist a real 4-vector $A^\mu = (A^0, A^i) = (A_0, \vec{A})$ such that the field strength is defined as a 4-curl $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, then (a) and (b) are automatically satisfied. Indeed $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ implies that $\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0$ known as a Bianchi or Jacobi identity (check that it is equivalent to (a) and (b)). The field A^μ is known as the 4-vector potential (or the gauge potential or the gauge field). Still another way of writing the homogeneous Maxwell equations, that does not require introducing a gauge potential A^μ, is by introducing the dual field tensor $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ and noticing that $\partial_\mu \tilde{F}^{\mu\nu} = 0$ is equivalent to (a) and (b). Indeed

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} \tag{7.2}$$

It is called the dual field strength because it is obtained from $F^{\mu\nu}$ by the replacement $(\vec{E}, \vec{B}) \to (\vec{B}, -\vec{E})$. Check that $\partial_\mu \tilde{F}^{\mu\nu} = 0$ gives (a) $\partial_i \tilde{F}^{i0} = 0 = \vec{\nabla} \cdot \vec{B}$ and (b) $\partial_\mu \tilde{F}^{\mu i} = 0 = -\partial_t \vec{B} - \vec{\nabla} \times \vec{E}$.

Consider what happens to the four Maxwell equations (a-d) in the absence of sources ($\rho = 0$ and $\vec{j} = 0$) under this replacement. Would $(\vec{E}, \vec{B}) \to (-\vec{B}, \vec{E})$ also work? And $(\vec{E}, \vec{B}) \to (\vec{B}, \vec{E})$? This property is called duality. It would be worth studying in more depth.

Let's turn our attention to the other two Maxwell equations (c) and (d). We define a 4-vector $j^\mu \equiv (\rho, \vec{j})$ called the 4-current in order to rewrite the inhomogeneous Maxwell equations as $\partial_\mu F^{\mu\nu} = j^\nu$. Indeed $\partial_i F^{i0} = j^0$ is (c) $\partial_i E^i = \rho$ and $\partial_\mu F^{\mu i} = j^i$ is (d) $\partial_0 F^{0i} + \partial_j F^{ji} = -\partial_t E^i + \epsilon^{ijk}\partial_j B^k = -\partial_t \vec{E} + \vec{\nabla} \times \vec{B} = j^i = \vec{j}$.

In summary, we defined the field strength tensor, its dual and the 4-current

$$\boxed{F^{\mu\nu} \equiv \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \ \tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \text{ and } j^\nu \equiv (\rho, \vec{j})} \tag{7.3}$$

such that the Maxwell equations are compactly and covariantly written

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0 \text{ and } \partial_\mu F^{\mu\nu} = j^\nu} \tag{7.4}$$

or

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \text{ and } \partial_\mu F^{\mu\nu} = j^\nu \tag{7.5}$$

or

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \text{ and } \partial_\mu F^{\mu\nu} = j^\nu \,. \tag{7.6}$$

7.1.2 Free electromagnetic field and gauge invariance

In this section, we restrict to the free electromagnetic field, i.e. the field in absence of the sources $j^\mu = 0$. The field equations are then $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $\partial_\mu F^{\mu\nu} = j^\nu$, which, upon inserting the first equation into the second, gives

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \Box A^\nu - \partial^\nu (\partial \cdot A) = 0 \tag{7.7}$$

The second term $-\partial^\nu(\partial \cdot A)$ is weird for the moment and is discussed below: we will show that it actually disappears upon making a certain choice. For the moment, we just neglect it. The first term looks like d'Alembert's wave equation for a 4-vector field $\Box A^\nu = 0$. Note the absence of a "mass term" $\Box A^\nu + m^2 A^\nu = 0$

(we don't obtain the Klein-Gordon equation for a 4-vector field). This is in agreement with the dispersion relation expected for light $\omega = \pm|\vec{k}|$.

The electromagnetic field can be described either in terms of the field strength $F^{\mu\nu}$ (i.e. $\vec{E}$ and $\vec{B}$) or in terms of the vector potential A^μ (i.e. A_0 and $\vec{A}$) [2]. However the description in terms of A^μ is not unique. In other words, there is a redundancy or freedom: several choices of A^μ lead to the same equations (a-d). The transformation

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \partial^\mu\theta(x)\,, \tag{7.8}$$

where $\theta(x)$ is any differentiable function (field), is called a gauge transformation. It is an internal and continuous transformation. It leaves the field strength invariant as $F^{\mu\nu} \to F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu\partial^\nu\theta - \partial^\nu\partial^\mu\theta = F^{\mu\nu}$. And therefore the free field equations $\partial_\mu\tilde{F}^{\mu\nu} = 0$ and $\partial_\mu F^{\mu\nu} = 0$ are also invariant. We postpone the discussion of what happens to the 4-current j^μ under a gauge transformation.

The Lagrangian for the free electromagnetic field is

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{\vec{E}^2 - \vec{B}^2}{2}} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu\partial^\nu A^\mu \tag{7.9}$$

It is a Lorentz scalar. Note that (apart for the sign) the first term looks similar to the Lagrangian of a massless real scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. The above Lagrangian is also gauge-invariant as it only depends on the field strength. Therefore the gauge transformation is a symmetry (called gauge invariance). Note that it is a weird symmetry: it does not act on $F^{\mu\nu}$ directly but only on A^μ. It is an internal symmetry (not a spacetime one) but it is a local internal symmetry as $\theta(x)$ depends on the spacetime point. The factor $-1/4$ is here to make the kinetic energy positive and have the familiar $1/2$ factor for a real field. Indeed $\mathcal{L} = \frac{1}{2}(\partial_t\vec{A})^2 - \frac{1}{2}(\partial_i\vec{A})^2 + 0\times(\partial_t A_0)^2 + \frac{1}{2}(\partial_i A_0)^2 + \partial_t\vec{A}\cdot\vec{\nabla}A_0 + \frac{1}{2}\partial_i\vec{A}\cdot\vec{\nabla}A^i$ (one also notices the absence of a kinetic energy term for A_0 – it has therefore no dynamics – and the surprising sign of the elastic energy term for A_0). The Lagrangian is quadratic in A^μ (also in $\vec{E}$ and $\vec{B}$): it is a free field theory (light does not scatter light). A term like $A^\mu A_\mu$ is also quadratic in the field and could be present in a free field theory. But it is actually absent. Its presence would spoil the gauge invariance. Such a term would correspond to a "mass term" $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu$ [3]. Note also that from our knowledge of the real scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$, we would have probably guessed that $\mathcal{L} = \frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu - \frac{m^2}{2}A_\mu A^\mu$ instead of (7.9). Think of the different problems related to such a Lagrangian.

We check that the Lagrangian is correct by re-obtaining the equations of motion from the EL equation[4] $\frac{\partial\mathcal{L}}{\partial A_\nu} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}$ giving:

$$0 = -\partial_\mu\partial^\mu A^\nu + \partial_\mu\partial^\nu A^\mu = -\Box A^\nu + \partial^\nu(\partial\cdot A) \tag{7.10}$$

qed.

By integration by part, the Lagrangian can also be written as

$$\mathcal{L} = \frac{1}{2}A^\mu\left(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)A^\nu \tag{7.11}$$

The main message of this subsection is that the electromagnetic field is not only a 4-vector field $A^\mu(x)$, but it is a gauge field, i.e. a vector field that has a gauge invariance. Gauge invariance is not truly a symmetry (although it is often called "gauge symmetry"). The gauge freedom is only a recognition of the fact that we are not able to construct a unique covariant description of the electromagnetic field. Remember that when

[2] Actually, this is a subtle point. In classical physics, the electric and magnetic fields seem fundamental (they are also gauge-independent) and the scalar and vector potentials appear as a mathematical construction lacking physical reality (they are also gauge-dependent). In quantum physics, the situation is partially reversed. There are actually good reasons – see the Aharonov-Bohm effect, the Dirac monopole, etc. – to believe that the 4-vector potential is fundamental and physical (in the sense that it couples locally to other fields), despite its being gauge-dependent and therefore not directly measurable. We will come back later on that issue.

[3] This is known as the Proca Lagrangian. See for example Ryder [3] pages 69-70.

[4] First show that $\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = F^{\alpha\beta}$.

we discussed rotation symmetry, we insisted on the difference between a transformation and a symmetry – that one could rotate a system, even if it did not possess rotational symmetry. However, electromagnetism always has gauge invariance. Gauge invariance is just a redundancy in our description of the electromagnetic field. It is therefore important to have in mind the difference between a vector field and a gauge field. A gauge field is a vector field that has a gauge invariance (i.e. the action should be invariant under the transformation $A^\mu \to A^\mu + \partial^\mu\theta$ for any $\theta(x)$). The Maxwell (or electromagnetic) field is a gauge field (it is actually the simplest example of a gauge field). But it is also possible to construct a theory for a vector field that has no gauge invariance (see, for example, the Proca field theory for a massive vector field with Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu$). In this case, the vector field A^μ should not be called a gauge field.

7.1.3 Gauge choices

There is something puzzling about the above construction. We know (experimentally) that the electromagnetic field has two internal degrees of freedom (two linear polarizations or two circular polarizations[5]). But the 4-vector field A^μ has 4 internal components. There are too many degrees of freedom. All of them can not be physical. There is a redundancy. We have already noticed that the component A_0 has no dynamics (no kinetic energy in the Lagrangian) and can therefore not represent a physical degree of freedom.

Getting rid of the redundancy can be done by fixing a gauge. There are several usual gauge choices. Below, we discuss the Lorenz gauge condition and the radiation gauge. Gauge freedom allows one to impose

$$\partial_\mu A^\mu = 0 \tag{7.12}$$

(known as the Lorenz – and not Lorentz, who is a different physicist – gauge condition) by choosing $\theta(x)$. Imagine that A^μ is given. Let $A'^\mu = A^\mu + \partial^\mu\theta$. We want to find $\theta(x)$ such that $\partial_\mu A'^\mu = 0 = \partial_\mu A^\mu + \Box\theta$ we therefore only have to find a solution to the equation $\Box\theta = -\partial_\mu A^\mu$. However, this does not fully fix θ, so that there remains some partial gauge freedom[6]. The Lorenz gauge condition is Lorentz covariant. But it does not fully fix the gauge freedom. Several gauge choices satisfy the Lorenz gauge condition (see below). The latter fixes one component of A^μ among four but three remain, whereas only two are physical. To see this, we note that, within the Lorenz gauge condition, the equation of motion for the gauge field becomes

$$\Box A^\nu = 0 \tag{7.13}$$

This is a d'Alembert wave equation for each component of the 4-vector field and corresponds to a dispersion relation $\omega = \pm|\vec{k}|$ as expected for light in the vacuum. Consider a plane wave $A^\mu(x) = A^\mu e^{-ik\cdot x}$ with $k^\mu = (|\vec{k}|, \vec{k})$. The Lorenz condition reads $k^\mu A_\mu = 0 = |\vec{k}|A_0 - \vec{k}\cdot\vec{A}$. Therefore $A_0 = \hat{k}\cdot\vec{A}$, which shows that the time-component A_0 is not independent of the spatial components A^i. This proves that the Lorenz condition reduces the number of independent components of the gauge field from 4 to 3. The advantage of the Lorenz gauge condition is that it removes part of the redundancy without sacrifying manifest Lorentz covariance. The drawback is that we still have to handle some redundancy and manipulate non-physical components.

A choice that fully fixes the gauge freedom is the so-called radiation gauge. It is such that $A_0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$ (so that $\partial_\mu A^\mu = \partial_t A_0 + \vec{\nabla}\cdot\vec{A} = 0$). These conditions are obviously not Lorentz covariant because under a Lorentz transformation A_0 mixes with the other components of the 4-vector field whereas 0 is a 4-scalar. The advantage of the radiation gauge is that only physical degrees of freedom appear (there is no more redundancy). The drawback is that the theory is no longer manifestly Lorentz covariant. Let's construct the radiation gauge choice step by step following Maggiore [2] pages 66-67. Starting from a given A^μ, we perform a first gauge transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu\theta$ with $\theta(x) = -\int^t dt' A_0(t, \vec{x})$, such that $\partial_0\theta(x) = -A_0(x)$. Therefore $A'_0 = 0$, which shows that we can get rid of a first component. Next, we perform a second gauge

[5] Anticipating, we also know that the photon exists in two helicities ± 1. It carries a spin 1 but it is massless and therefore the longitudinal component ($S_z = 0$) is not possible and only the two transverse ones ($S_z = \pm 1$) are.

[6] If $\theta(x)$ satisfies $\Box\theta = -\partial_\mu A^\mu$ then $\theta'(x) = \theta(x) + f(t - \vec{n}\cdot\vec{x})$ also satisfies $\Box\theta' = -\partial_\mu A^\mu$ for any smooth function f and any unit vector $\vec{n}$. Indeed, show that $\phi(x) = f(t - \vec{n}\cdot\vec{x})$ is a general solution of the d'Alembert equation $\Box\phi(x) = 0$.

transformation $A'^\mu \to A''^\mu = A'^\mu + \partial^\mu\theta'$ with $\theta'(x) = \int d^3y \frac{1}{4\pi|\vec{x}-\vec{y}|}\partial_i A'^i(t,\vec{y})$. Actually $\theta'(x)$ does not depend on time. The electric field $E^i = -\partial_0 A'^i$ (because $A'^0 = 0$) and the Gauss equation (in the absence of a source) $\partial_i E^i = 0$ shows that $\partial_0\partial_i A'^i = 0$. Therefore $\partial_0\theta'(x) = \int d^3y \frac{1}{4\pi|\vec{x}-\vec{y}|}\partial_0\partial_i A'^i(t,\vec{y}) = 0$, which shows that $\theta'(x) = \theta'(\vec{x})$. As a consequence, $A''_0 = A'_0 + \partial_0\theta' = A'_0 = 0$. In addition $\partial_j A''^j(x) = \partial_j A'^j(x) + \partial_j\partial^j\theta'(x) = \vec{\nabla}\cdot\vec{A}'(x) + \int d^3y \vec{\nabla}^2\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right)\vec{\nabla}\cdot\vec{A}'(t,\vec{y})$. With the help of the identity $\vec{\nabla}^2\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right) = -\delta^{(3)}(\vec{x}-\vec{y})$ (this is the Green's function of the Laplacian in 3 spatial dimensions), we see that $\vec{\nabla}\cdot\vec{A}'' = \vec{\nabla}\cdot\vec{A}' - \vec{\nabla}\cdot\vec{A}' = 0$. In summary, starting from a given gauge field, we have found a gauge choice that allows one to have $A''_0 = 0$ and $\vec{\nabla}\cdot\vec{A}'' = 0$. Going to Fourier space, we have $A''_0(k) = 0$ and $\vec{k}\cdot\vec{A}''(k) = 0$. The last equation means that for a plane wave propagating in the z direction, we have $A''_z(k) = 0$ in addition to $A''_0(k) = 0$ and only two non-zero components A''_x and A''_y.

In conclusion, quoting Ryder [3] pages 143-144, "The origin of the difficulty is that the elecromagnetic field, like any massless field, possesses only two independent components, but is covariantly described by a 4-vector A_μ. In choosing two of these components as the physical ones, [...], we lose manifest covariance. Alternatively, if we wish to keep covariance, we have two redundant components."

7.1.4 Energy-momentum tensor

(see exercise sheet)

7.1.5 Coupling to matter and electric charge conservation as a consequence of gauge invariance

We now include the sources in the inhomogeneous Maxwell equations. From $\partial_\mu\tilde{F}^{\mu\nu} = 0$ and $\partial_\mu F^{\mu\nu} = j^\nu$ we get the equation of motion:

$$\boxed{\Box A^\nu - \partial^\nu(\partial\cdot A) = j^\nu} \tag{7.14}$$

Taking the 4-divergence of the preceding equation, it follows that $\partial_\nu j^\nu = \Box(\partial_\nu A^\nu) - \partial_\nu\partial^\nu(\partial\cdot A) = 0$, which means that there is a divergenceless current. This is nothing but the familiar local form of the conservation of electric charge. Let us show that it is possible to derive this conservation law using Noether's theorem for a specific symmetry (namely gauge symmetry). The Lagrangian for the electromagnetic field in presence of a source is[7]

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu}, \tag{7.15}$$

which obviously gives back the correct equations of motion. Is this Lagrangian invariant under a gauge transformation $A^\mu \to A^\mu - \partial^\mu\theta$? We know that the field strength is invariant and we further assume that the 4-current j^μ is also invariant[8]. Then $\mathcal{L} \to \mathcal{L} + j^\mu\partial_\mu\theta$ is not invariant. But the action $S = \int d^4x\mathcal{L} \to S + \int d^4x j^\mu\partial_\mu\theta = S - \int d^4x\theta\partial_\mu j^\mu$ (using an integration by part in the last step). Therefore if $\partial_\mu j^\mu = 0$ then the action is gauge invariant and there is a gauge symmetry. Reciprocally, if there is a gauge symmetry, i.e. if the action is gauge invariant, then $\int d^4x\theta\partial_\mu j^\mu = 0$ for any $\theta(x)$ and therefore $\partial_\mu j^\mu = 0$.

In summary, gauge invariance of the action is equivalent to $\partial_\mu j^\mu = 0$. So that the conservation of electric charge can be seen as a consequence of a gauge symmetry. This is not yet entirely satisfying because of at least two things: (i) the conservation law is for charged matter but we derived it from a symmetry pertaining

[7]For a single particle of mass m and charge q, the Lagrangian $L(\vec{x},\dot{\vec{x}}) = \frac{m\dot{\vec{x}}^2}{2} - V(\vec{x})$ becomes $L = \frac{m\dot{\vec{x}}^2}{2} - V(\vec{x}) - qA_0 + q\dot{\vec{x}}\cdot\vec{A}$ in the presence of an electro-magnetic field described by a potential $A^\mu = (A_0,\vec{A})$. Note that $-qA_0 + q\dot{\vec{x}}\cdot\vec{A}$ is similar to $-j_\mu A^\mu = -\rho A_0 + \vec{j}\cdot\vec{A}$, with $\rho \to q$ and $\vec{j} \to q\dot{\vec{x}}$. The canonical momentum is $\vec{p} = \frac{\partial L}{\partial\dot{\vec{x}}} = m\dot{\vec{x}} + q\vec{A} = m\vec{v} + q\vec{A}$ so that the Hamiltonian is $H(\vec{x},\vec{p}) = \frac{(\vec{p}-q\vec{A})^2}{2m} + V(\vec{x}) + qA_0$. This has a familiar form of the sum of kinetic energy $\frac{(\vec{p}-q\vec{A})^2}{2m} = \frac{m\vec{v}^2}{2}$ and potential energy $V(\vec{x}) + qA_0$, where the last term is the electric potential energy. Note also that the canonical momentum $\vec{p}$ is gauge-dependent, while the mechanical momentum $m\vec{v}$ is gauge-invariant.

[8]Actually, we know that the Maxwell equations (a) to (d) only involve gauge-independent quantities, when written in terms of E and B fields. Therefore, we know that the current and density are gauge invariant. So that j^μ is also gauge invariant.

mainly to the neutral gauge field (the charged matter field being hidden in the 4-current j^μ); (ii) the gauge symmetry is local whereas Noether's theorem was proven for the case of a global symmetry (of course a global symmetry can be seen as a particular case of a local symmetry[9]).

7.1.6 Gauging a symmetry

We now come to a very interesting point that goes back to the work of Hermann Weyl (in 1918 and 1929). We present his general method for writing a gauge invariant action for a matter field. It is called "gauging of an internal symmetry" (see pages 69-72 in [2] and pages 93-100 in [3]). It goes in four steps:

1) Start from a complex scalar (matter) field

$$\mathcal{L}_1 = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \,. \tag{7.16}$$

This Lagrangian has a global $U(1)$ symmetry. It is an internal and continuous symmetry. It is known as a phase symmetry. The Lagrangian is invariant under the transformation $\phi(x) \to e^{i\theta}\phi(x)$ with $\theta =$ constant ($e^{i\theta} \in U(1)$). From Noether's theorem, there is a conserved current

$$J^\mu = i\phi^* \partial^\mu \phi - i\phi \partial^\mu \phi^*, \partial_\mu J^\mu = 0 \text{ if } (\Box + m^2)\phi = 0. \tag{7.17}$$

2) The "gauge principle" is the idea that an exact symmetry of Nature can not be global but has to be local (otherwise it would have the flavor of action at a distance). But it is an unproven assumption, it is a principle. It is inspired by Einstein's step of going from special to general relativity by asking that the invariance under change of frame be local rather than global (an idea out of which popped the theory of gravitation)[10].

Let us therefore force the $U(1)$ phase symmetry to be local (this is called "gauging the symmetry"):

$$\begin{aligned} \phi(x) &\quad\to\quad \phi'(x) = e^{i\theta(x)}\phi(x) \\ \partial_\mu \phi(x) &\quad\to\quad e^{i\theta(x)}[\partial_\mu \phi + \phi i \partial_\mu \theta] \end{aligned} \tag{7.18}$$

The last equation implies that $(\partial_\mu \phi)^* \partial^\mu \phi$ is not invariant under the local $U(1)$ transformation, so that neither $\mathcal{L}_1$ nor the corresponding action S_1 are. The idea is to introduce a real field $A^\mu(x)$ such that $A^\mu(x) \to A^\mu(x) - \partial^\mu \theta(x)$ under the local $U(1)$ transformation. The purpose of this new field is solely to make the Lagrangian invariant under the local $U(1)$ transformation by changing the differential ∂_μ. Indeed

$$\begin{aligned} \phi(x) &\quad\to\quad \phi'(x) = e^{i\theta(x)}\phi(x) \\ (\partial_\mu + iA_\mu(x))\phi(x) &\quad\to\quad e^{i\theta}[\partial_\mu \phi + \phi i \partial_\mu \theta] + i(A_\mu - \partial_\mu \theta)e^{i\theta}\phi = e^{i\theta(x)}(\partial_\mu + iA_\mu)\phi \end{aligned} \tag{7.19}$$

By changing ∂_μ into $\partial_\mu + iA_\mu$, we now have the property that the field ϕ and its "gradient" $(\partial_\mu + iA_\mu)\phi$ transform similarly under a local $U(1)$ phase transformation.

The new field is called a gauge field and the local $U(1)$ phase transformation law for the matter field corresponds to a gauge transformation for the gauge field A^μ. We define

$$\boxed{D_\mu \equiv \partial_\mu + iA_\mu} \tag{7.20}$$

[9] From the point of view of the 4-vector potential $A^\mu(x)$, a gauge transformation is an internal and continuous transformation. It can be made infinitesimal and rendered global by choosing a specific transformation. Indeed $x^\mu \to x'^\mu = x^\mu$ and $A^\mu(x) \to A^\mu(x') = A^\mu(x) - \partial^\mu\theta \approx A^\mu(x) + \epsilon a^\mu$ for a specific field $\theta(x) = -\epsilon a^\nu x_\nu$ with $\epsilon \to 0$ and a^ν is a 4-vector.

[10] Weyl was inspired by Einstein's theory of general relativity. He first hoped to derive electromagnetism from a deeper symmetry – just as Einstein had done with gravitation and the symmetry under general coordinate change – related to changing the scale length or changing locally the ruler or the gauge (in the sense of a metal bar) used to measure distances. This is now called dilatation transformation and conformal symmetry. This was not the correct symmetry for electromagnetism. But the name "gauge symmetry" sticked. Weyl later – in 1929 after the advent of quantum mechanics and complex wavefunctions – understood what the correct symmetry was: namely a $U(1)$ phase symmetry of the matter field.

called the covariant[11] derivative so that $(D_\mu\phi)^*(D^\mu\phi)$ is invariant under the local $U(1)$ transformation. We can therefore postulate a new Lagrangian

$$\mathcal{L}_2 = (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi = (\partial_\mu\phi^* - iA_\mu\phi^*)(\partial^\mu\phi + iA^\mu\phi) - m^2\phi^*\phi \tag{7.21}$$

The latter now includes coupling between the matter field and the gauge field. To see it clearly, we rewrite the Lagrangian as:

$$\begin{aligned}\mathcal{L}_2 &= (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi - A_\mu(i\phi^*\partial^\mu\phi - i\phi\partial^\mu\phi^*) + \phi^*\phi A_\mu A^\mu \\ &= \mathcal{L}_1 - A_\mu J^\mu + \phi^*\phi A^2\end{aligned} \tag{7.22}$$

This coupling (between the matter field ϕ and the gauge field A_μ) is called minimal coupling and is usually written as $\partial_\mu \to D_\mu = \partial_\mu + iA_\mu$ or $i\partial_\mu \to iD_\mu = i\partial_\mu - A_\mu$, where $i\partial_\mu$ is the momentum operator (impulsion) and $i\partial_\mu - A_\mu$ is the gauge-invariant or mechanical momentum (quantité de mouvement). In the above equation, we recognize the free matter field Lagrangian $\mathcal{L}_1$, the coupling between the gauge field and the current $J^\mu = i\phi^*\partial^\mu\phi - i\phi\partial^\mu\phi^*$ of the matter field (similar to the one in eq. (7.15)) and there is a third term that could look like a mass term for the gauge field (it is proportional to $A_\mu A^\mu$). This last term plays an important role when discussing the Higgs mechanism.

We also understand why a local $U(1)$ symmetry is called a gauge symmetry. At first it was a global phase symmetry for the complex matter field. We turned it into a local phase transformation and were forced to introduce a gauge field if we wanted to maintain invariance under the local phase transformation. Then we realized that for the gauge field, the local phase transformation is actually a gauge transformation (in the sense of the familiar gauge transformation in the Maxwell equations). So that from now on, a gauge transformation actually means the following combined transformation

$$\begin{aligned}\phi(x) &\to e^{i\theta(x)}\phi(x) \\ A^\mu(x) &\to A^\mu(x) - \partial^\mu\theta(x)\end{aligned} \tag{7.23}$$

namely, a local $U(1)$ phase transformation on the matter field, together with a gauge transformation (in the old acceptation of classical electromagnetism) on the 4-vector potential (now called a gauge field).

3) The gauge field A^μ that we introduced does not have a dynamic yet. There is no kinetic energy or elastic energy for this field in $\mathcal{L}_2$. But we already know how to write a free field Lagrangian for a gauge-invariant real 4-vector field. It is $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. A term such as $m_A^2 A_\mu A^\mu$ is forbidden by gauge invariance and therefore the gauge field has to be massless. We therefore arrive at the following Lagrangian:

$$\mathcal{L}_3 = (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7.24}$$

This is actually a baby version of quantum electrodynamics (QED) called scalar QED. It is a theory for electrons and photons except that the electron is here spinless (scalar field instead of spinor field).

4) Gauge invariance of the Lagrangian $\mathcal{L}_3$ is now different from gauge invariance of the Maxwell action $S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu)$. We therefore revisit Noether's theorem in this new context. The gauge invariance of $\mathcal{L}_3$ implies global $U(1)$ invariance and therefore a conserved current via Noether. However this current is not necessarily $J^\mu = i\phi^*\partial^\mu\phi - i\phi\partial^\mu\phi^*$ as we show below. From $\mathcal{L}_3$ and the EL equations,we obtain the equations of motion for the matter field

$$(D^\mu D_\mu + m^2)\phi = 0 \tag{7.25}$$

and the gauge field

$$\partial_\mu F^{\mu\nu} = J^\nu - 2\phi^*\phi A^\nu = i\phi^*\partial^\nu\phi - i\phi\partial^\nu\phi^* - 2\phi^*\phi A^\nu \tag{7.26}$$

where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength associated to the gauge field A^μ. These equations are coupled: the evolution of the matter field depends on the gauge field and vice versa. From $\partial_\nu\partial_\mu F^{\mu\nu} = 0$

[11] Here covariant means covariant with respect to the gauge transformation.

(contraction of a symmetric tensor with an anti-symmetric one), we find that $\partial_\nu(J^\nu - 2\phi^*\phi A^\nu) = 0$, which allows us to identify the conserved current as

$$j^\nu \equiv J^\nu - 2\phi^*\phi A^\nu = i\phi^* D^\mu\phi - i\phi(D^\mu\phi)^* \text{ and } \partial_\nu j^\nu = 0 \tag{7.27}$$

The equation of motion for the gauge field is:

$$\Box A^\nu - \partial^\nu(\partial\cdot A) = j^\nu \text{ i.e. } (\Box + 2\phi^*\phi)A^\nu - \partial^\nu(\partial\cdot A) = J^\nu = i\phi^*\partial^\nu\phi - i\phi\partial^\nu\phi^* \tag{7.28}$$

We can also check the current we found directly from the explicit form of the general Noether current. For a global continuous internal symmetry, the Noether current is $j^\mu = -\frac{\partial\mathcal{L}_3}{\partial(\partial_\mu\phi)}F_\phi - \frac{\partial\mathcal{L}_3}{\partial(\partial_\mu\phi^*)}F_{\phi^*}$. The infinitesimal transformations are $\phi \to \phi + \theta i\phi$ and $\phi^* \to \phi^* + \theta(-i\phi^*)$ so that $F_\phi = i\phi$ and $F_{\phi^*} = -i\phi^*$. Therefore the Noether current is $j^\mu = -(D^\mu\phi)^* i\phi + (D^\mu\phi)i\phi^*$, which is indeed correct.

We rewrite the Lagrangian to identify three terms:

$$\begin{aligned}\mathcal{L}_3 &= \left[(\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu + \phi^*\phi A_\mu A^\mu \\ &= \mathcal{L}_{\text{matter field}} + \mathcal{L}_{\text{gauge field}} + \mathcal{L}_{\text{interaction}}\end{aligned} \tag{7.29}$$

The last term $\mathcal{L}_{\text{interaction}} = -A_\mu J^\mu(\phi) + \phi^*\phi A_\mu A^\mu$ can also be written $\mathcal{L}_{\text{interaction}} = -A_\mu j^\mu(\phi, A) - \phi^*\phi A_\mu A^\mu$, where the notation $j^\mu(\phi, A)$ emphasizes that, in the present case, the gauge-invariant current j^μ depends both on the matter and on the gauge field. Whereas J^μ is the gauge-dependent current.

The construction of Weyl (inspired by the general relativity construction of Einstein) is marvellous! (cf. G. t'Hooft "Under the spell of the gauge principle"). It means that each time a new conservation law is found to exist in Nature and which appears to be exact, it should correspond to a local internal symmetry of the matter fields and to a new gauge field corresponding to an interaction. But remember it is a principle (why do exact symmetries have to be local?).

7.1.7 Coupling of the Dirac and the electromagnetic fields

see Maggiore [2] pages 69-72

This section is essentially a repetition of the section on the gauging of a global internal $U(1)$ symmetry, expect that it will now be performed on a Dirac field rather than on a complex scalar field. We start by making two remarks:

- $\mathcal{L}_D = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi$ is a parity invariant free theory for a spin 1/2 complex and massive field (the Dirac field, which is a matter field). It is invariant under a global $U(1)$ phase transformation $\psi(x) \to e^{i\theta}\psi(x)$.
- $\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is a free theory for a massless spin 1 and real field (the gauge field A^μ). It is invariant under the (local) gauge transformation $A^\mu(x) \to A^\mu(x) - \partial^\mu\theta(x)$ with $\theta(x)$.

Starting from the Dirac Lagrangian, we apply the gauge principle and decide to make the $U(1)$ phase symmetry local: $\psi(x) \to e^{i\theta(x)}\psi(x)$. Then the derivative $\partial_\mu\psi \to e^{i\theta(x)}[\partial_\mu\psi + \psi i\partial_\mu\theta]$ is not covariant (i.e. it does not transform as the field ψ) under the local phase transformation. We therefore introduce the covariant derivative $D_\mu \equiv \partial_\mu + iA_\mu$ where A_μ is an auxiliary field such that $A_\mu \to A_\mu - \partial_\mu\theta$ and $D_\mu\psi \to e^{i\theta(x)}D_\mu\psi$. Therefore we consider the modified Dirac Lagrangian $\mathcal{L} = i\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi$. We also need to add a dynamics for the field A^μ and therefore arrive at

$$\mathcal{L} = i\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu\bar\psi\gamma^\mu\psi = \mathcal{L}_D + \mathcal{L}_M - A_\mu J_V^\mu \tag{7.30}$$

where $\mathcal{L}_{\text{int}} = -A_\mu J_V^\mu$ is the Lagrangian describing the minimal coupling between the Dirac field and the gauge field and $J_V^\mu = \bar\psi\gamma^\mu\psi$ is the vector current. Note that the vector current $\bar\psi\gamma^\mu\psi$ is invariant under a local phase transformation (unlike the current $i\phi^*\partial^\mu\phi$+c.c. obtained in the complex scalar case). The reason why the dynamics for the gauge field has this form is that it has to be quadratic, contain the correct kinetic energy and can not have a mass term that would spoil the invariance under the local phase transformation.

Note that in the above construction, we did not introduce a knob to vary the strength for the coupling between the matter (Dirac) field and the gauge field. In other words, we would like to have $\mathcal{L}_{\text{int}} = -qA_\mu\bar\psi\gamma^\mu\psi$

instead of $\mathcal{L}_{\text{int}} = -A_\mu J_V^\mu$ with q a real number expressing the strength of the coupling between the two fields. The way to do it is simply to realize that when introducing the Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, there is the freedom to introduce an arbitrary constant $1/g > 0$ [12] such that

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi + \frac{1}{g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{7.31}$$

This Lagrangian is as valid as the one with $g = 1$ in order to describe a Dirac field with a local phase symmetry. But it has one additional parameter. This parameter g controls the relative weight between the two Lagrangians for the fields ψ and A^μ. If one wants to recover the traditional form of the Lagrangian, this parameter can be absorbed in a redefinition of the gauge field $A^\mu \to A^\mu\sqrt{g}$. Then it means that the gauge transformation of the field A^μ is unchanged but that the phase transformation of ψ becomes $\psi \to e^{iq\theta(x)}\psi$ with $q \equiv \sqrt{g}$ and the covariant derivative is now $D_\mu = \partial_\mu + iqA_\mu$. Therefore q can be considered as a characteristic property of the matter field (it will later be interpreted as the electric charge of an electron[13]). The Lagrangian becomes

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu q\bar{\psi}\gamma^\mu\psi \tag{7.32}$$

which is the QED Lagrangian describing electrons, photons and their interaction. The minimal coupling is now $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ i.e. $p^\mu = i\partial^\mu \to iD^\mu = i\partial^\mu - qA^\mu$.

The EL equation for the Dirac field gives

$$(i\gamma^\mu D_\mu - m)\psi = (i\gamma^\mu\partial_\mu - q\gamma^\mu A_\mu - m)\psi = 0 \tag{7.33}$$

and for the gauge field:

$$\partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\nu\psi = j^\nu \tag{7.34}$$

From the antisymmetry of $F^{\mu\nu}$ we obtain $\partial_\nu\partial_\mu F^{\mu\nu} = \partial_\nu j^\nu = 0$, in other words, we have a conserved current $j^\nu = q\bar{\psi}\gamma^\nu\psi$ and a conserved charge $Q = \int d^3x j^0 = q\int d^3x\bar{\psi}\gamma^0\psi = q\int d^3x\psi^\dagger\psi$. Here $j^\nu = qJ_V^\nu$: this should be contrasted to the complex scalar field case for which $j^\nu = i\phi^* D^\nu\phi + c.c. \neq J^\nu = i\phi^*\partial^\nu\phi + c.c.$. Upon quantizing the Dirac field, we will see that q is actually the electric charge carried by a single particle (an electron). Here we see the role of the electric charge as the coupling strength of the matter field to the gauge field.

Remark: Note that

$$J_V^\mu = -\frac{\partial\mathcal{L}}{\partial A_\mu} = -\frac{\delta S}{\delta A_\mu}, \tag{7.35}$$

which is often taken as a definition of the current of a matter field that is coupled to a gauge field.

[12] It is conventional to call this parameter $1/g$. The reason is that g plays the role of a dimensionless coupling constant. It is actually $\sim$ the fine structure constant.

[13] And then indeed $g = q^2$ which is units such that $\hbar = 1$ and $c = 1$ is essentially the fine structure constant $\alpha = \frac{q^2}{\hbar c}$ with q the charge of a single electron.

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