First steps towards a non-perturbative Renormalization Group approach for spin glasses in finite dimensions

Marco Tarzia

LPTMC, Université Pierre et Marie Curie

Beyond Mean Field Theory: Renormalisation Group and non perturbative approaches in disordered and glassy systems

Rome, January 3-5, 2018

In collaboration with

Giulio Biroli, Gilles Tarjus and *<u>Charlotte Rulquin</u>*

Plan of the talk

1. Introduction and open problems

- Spin glasses: quenched disorder and frustration
- A new kind of criticality
- Mean-field theory and Droplet theory
- Nature of the low temperature phase?
- Transition in presence of a magnetic field?

2. The non-perturbative RG approach

- Exact flow equation for the effective average action
- Approximations: derivative expansion, truncations, …

3. The NPRG for the critical behavior of spin glasses

- Expansion in invariants around the minimum of the potential
- Definition of the coupling constants
- Truncations at the cubic order
- Truncations at the quartic order

4. Future perspectives and conclusions

- Develop a functional NPRG approximation scheme
- Transition in presence of an external magnetic field

Spin glasses

Random ferromagnetic and antiferromagnetic interactions due to **quenched** magnetic impurities

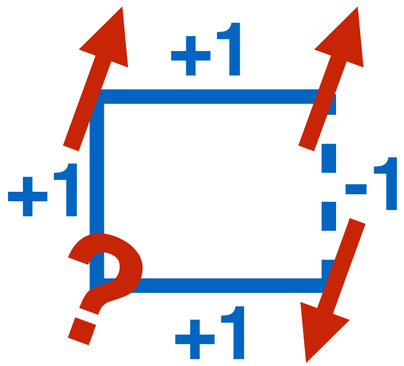
Edwards-Anderson model
[Edwards & Anderson ('75)]

$$\boxed{\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - h \sum_i S_i}$$

J_{ij} i.i.d random variables (Gaussian or binary)

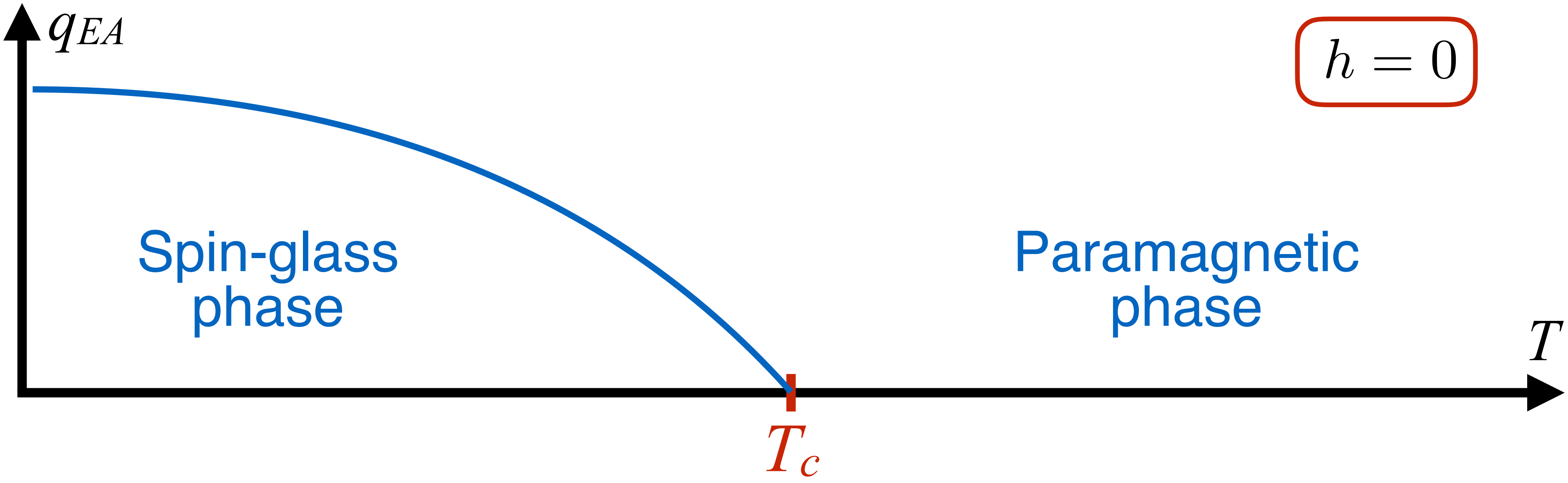
Impossibility of satisfying all local interactions

Frustration



- Degenerescence of the low energy states
- No conventional long range order

An unconventional phase transition



Spins point in random but well defined directions

$$m = \frac{1}{N}\sum_i \overline{\langle S_i \rangle} = 0$$

Overlap order parameter

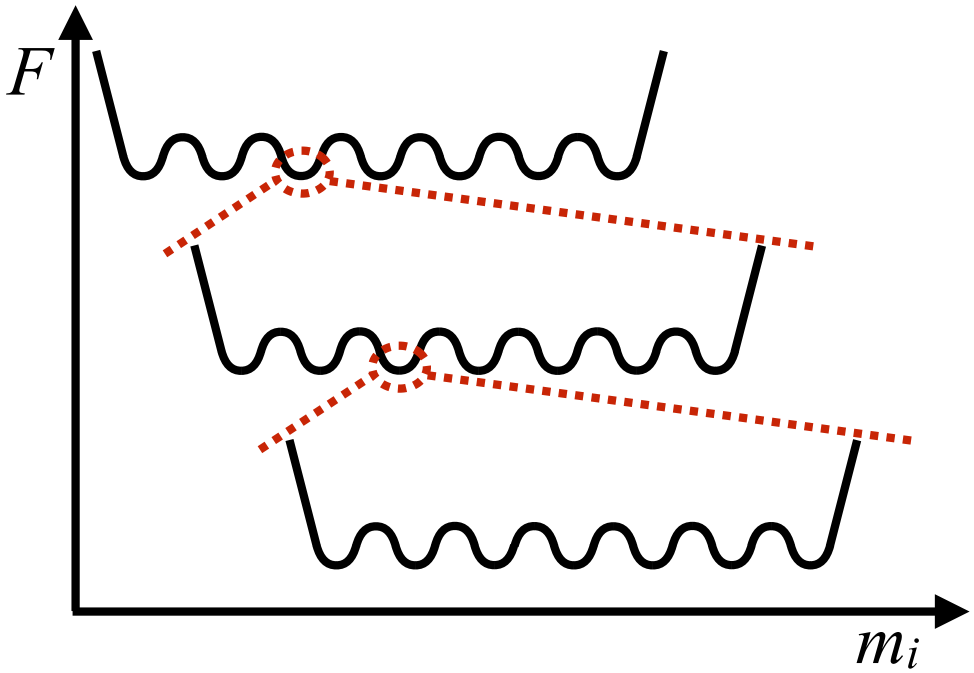
$$\boxed{q_{EA} = \frac{1}{N}\sum_i \overline{\langle S_i \rangle^2}}$$

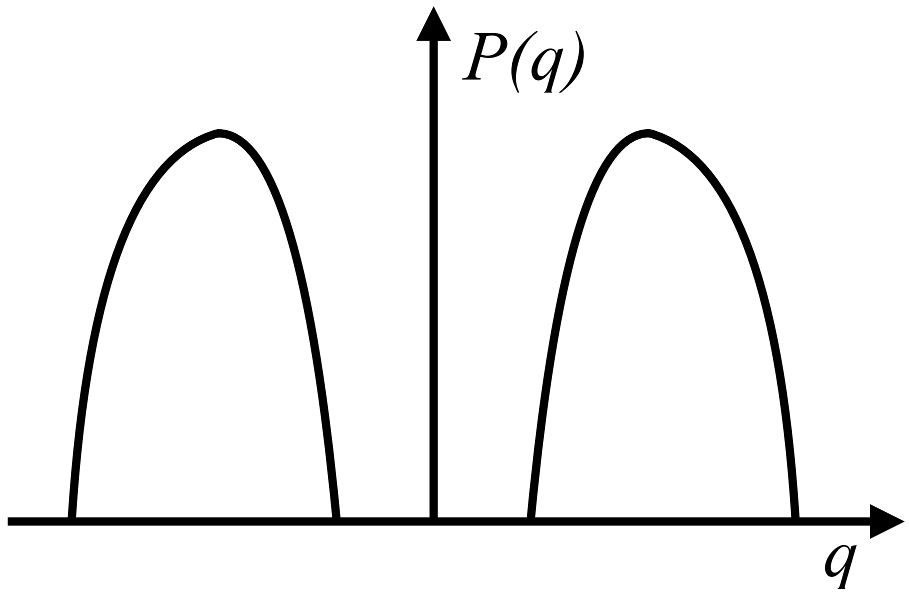
- Large scale numerical simulations in $3d$ [Janus collaboration ('10–)] and perturbative RG calculations in $\epsilon = 6 - d$ [Bray & Moore ('84, '85)]
- Lower critical dimension $d_L \approx 2.5$ [Bray & Moore ('84); Franz & al ('94)]

Theoretical approaches: Replicas

★ **Mean-field theory** [Paris ('79, '80)]

- Full replica-symmetry breaking in the spin glass phase
- Complex free-energy landscape (infinite hierarchy of minima, ultrametric structure) [Mézard & al ('84)]
- Spin glass transition in presence of an external field (dAT line) [de Almeida & Thouless ('02, '03)]
- Exact in infinite dimensions [Guerra, Toninelli, Talagrand ('02, '03)]

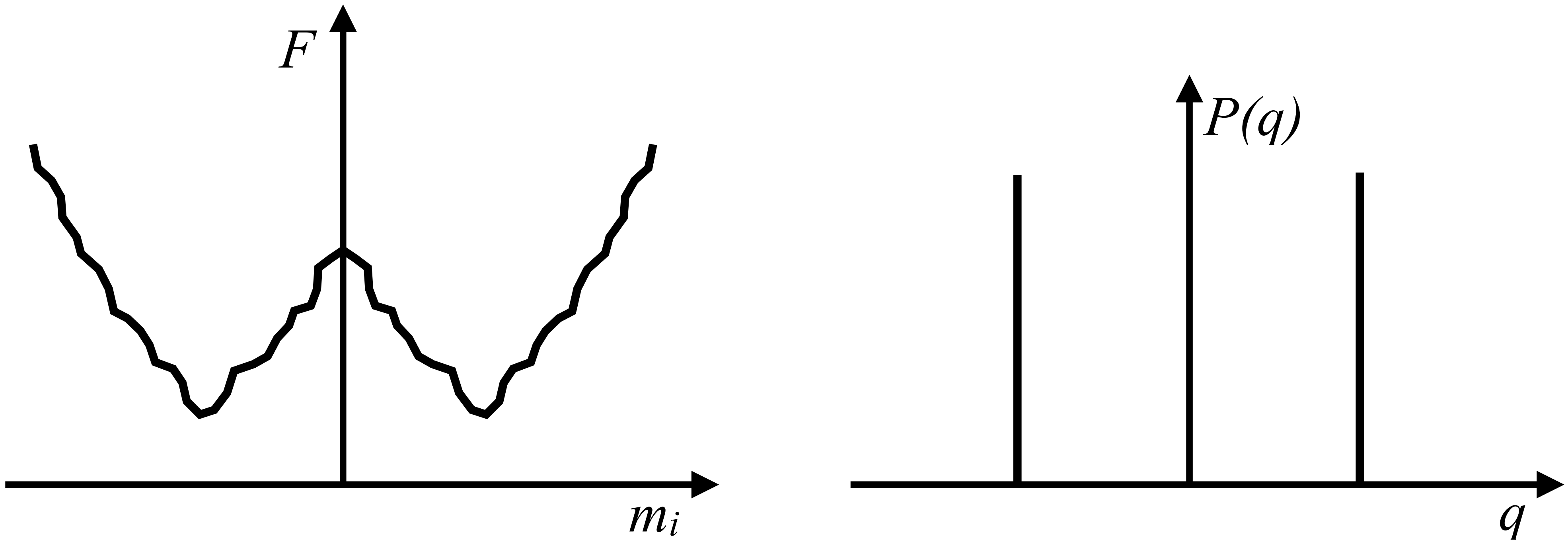




Theoretical approaches: Droplets

★ **Droplet theory** [Bray & Moore ('85); Fisher & Huse ('85)]

- Based on scaling arguments in finite dimensions
- Only two pure states
- Supported by real space RG calculations [Bray & Moore ('85)]
- No transition in presence of an external field



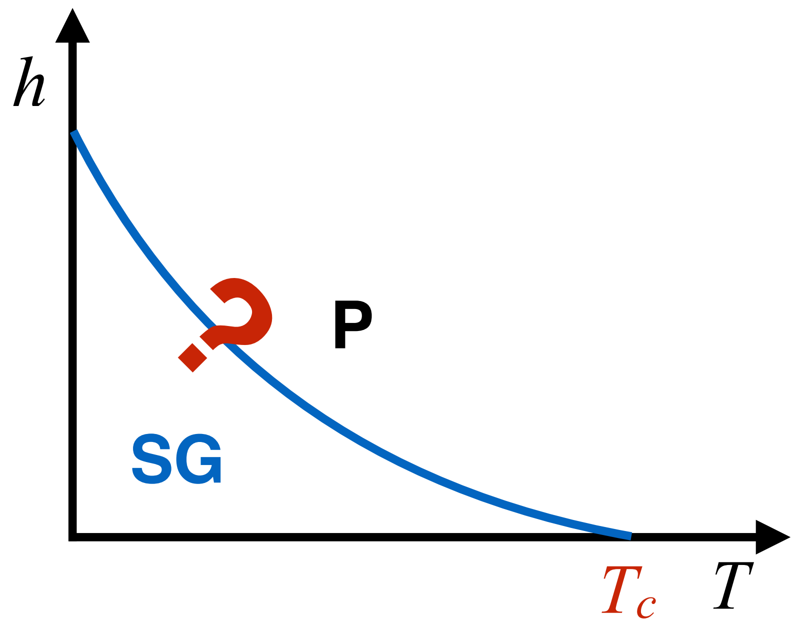
Open problems

1. Nature of the low temperature phase?

- The same as the one predicted by mean-field? Full RSB? Infinite number of pure states?
- Finite-dimensional fluctuations should anyway destroy metastability and alter the MF scenario

2. Transition in presence of an external uniform (or random) magnetic field?

- The existence of the dAT line in $3d$ is still strongly debated [Aspelmeier & al ('16); Jorg & al ('07)]
- The basin of attraction of the Gaussian fixed point is finite even above the upper critical dimension and shrinks to zero as $\sqrt{d-6}$ [Moore & Bray ('11)] [Charbonneau & Yaida ('17)]



A non-perturbative RG approach

★ Practical issues

- Very hard to settle these questions from the results of experiments and simulations (e.g., one can only access the critical point from above, as the relaxation time is infinite in the whole spin glass phase)

➡ The standard theoretical framework to include fluctuations beyond mean-field is the Renormalization Group approach

★ Technical and conceptual difficulties

- Intricate nature of the order parameter
- Interplay of disorder and metastability (complex free-energy landscape, new kind of low-T fixed points?)

➡ A novel approach: The non-perturbative RG [Wetterich ('93)]

First step: critical properties of the spin glass transition in zero field using a NPRG formalism

Perturbative RG has been extended up to 3 loops
[Harris & Lubensky ('76); Green ('85); Yeo & al ('05)]

The NPRG: Basic ideas

★ **The "effective average action" (Gibbs free-energy)**

Legendre transform of the free-energy

$$Z[J] = \int \mathcal{D}\varphi \, \exp\left(- S[\varphi] + \int_x J_x \, \varphi_x \right)$$

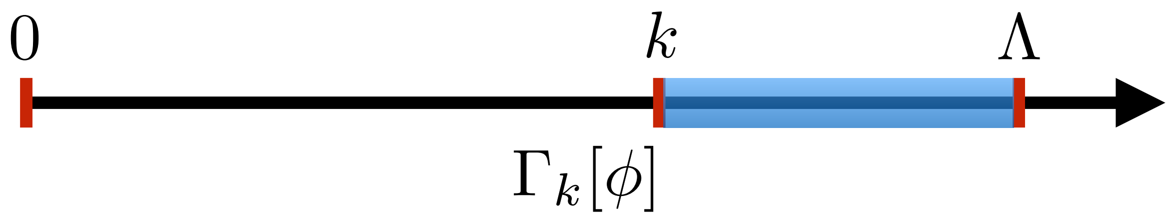
$$\Gamma[\phi] = - \log Z[J] + \int_x J_x \, \phi_x \qquad\qquad \phi_x = \langle \varphi_x \rangle = \frac{\delta \log Z[J]}{\delta J_x}$$

★ **Physical information about the equilibrium properties**

- $\Gamma[\phi_x = \phi] = \mathcal{U}(\phi)$: Local potential $\longrightarrow$ Thermodynamics
- $\Gamma^{(2)}_{x,y} = \dfrac{\delta^2 \Gamma[\phi]}{\delta\phi_x \, \delta\phi_y}$: Inverse of the propagator $\longrightarrow$ Correlation function

★ **Integrate progressively short wave-length fluctuations**

RG "à la Wilson"

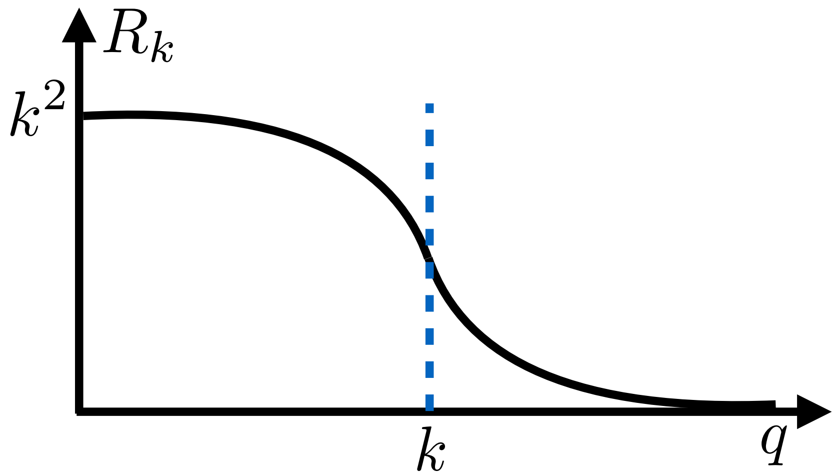


The NPRG: Exact flow equation

★ Add a "regulator" (or cutoff) to the microscopic action

$$S[\varphi] \longrightarrow S_k[\varphi] = S[\varphi] + \Delta S_k[\varphi] \qquad \Delta S_k[\varphi] = \frac{1}{2}\int_q R_k(q^2)\varphi(q)\varphi(-q)$$

Mass term that "freezes"
the low momenta modes



$$\Gamma_k[\phi] = -\log Z_k[J] + \int_x J_x\,\phi_x - \Delta S_k[\phi]$$

- $k = \Lambda \longrightarrow$ Mean-field (no fluctuations) $\longrightarrow \Gamma_{k=\Lambda} = S$
- $k = 0 \longrightarrow$ Exact Gibbs free-energy $\longrightarrow \Gamma_{k=0} = \Gamma$

★ Exact flow equation
[Wetterich ('93)]

$$\boxed{\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\partial_k R_k \left(\Gamma^{(2)}[\phi] + R_k\right)^{-1}\right\}}$$

★ Scale invariance at the critical point: Fixed point of the (dimensionless) flow equation after coarse graining and rescaling

The NPRG: Approximations & Truncations

★ Long-distance physics ➡ Derivative expansion (LPA')

$$\Gamma_k[\phi] = \int_x \left\{ \mathcal{U}_k(\phi) + \frac{\mathcal{Z}_k(\phi)}{2} (\partial_x \phi)^2 + \cdots \right\}$$

★ Expansion in powers of the field

Describe the properties of the system around a constant nontrivial profile

$$\mathcal{U}_k(\phi) = \sum_n \frac{U_k^{(n)}}{n!} (\phi - \phi_{0,k})^n \qquad \mathcal{Z}_k(\phi) = \mathcal{Z}_k(\phi_{0,k})$$

Approximations and truncations must preserve the symmetries of the theory (expansion in invariants)

Very successful in describing the critical physics (as well as non-universal properties) of a broad variety of models [Berges & al ('02); Delamotte ('12)]: e.g., **RFIM** [Tissier & Tarjus ('06)], **Turbulence** [Canet & al ('16)], **KT transition** [Jakubczyk & al ('14)], **KPZ equation** [Kloss & al ('14)], …

The critical behavior of spin glasses (h=0)

★ Cubic Ginzburg-Landau microscopic (bare) action (at zero field)

$q_{ab}(x) \longrightarrow n \times n$ overlap matrix $(n \to 0)$

[Harris & Lubensky ('76); Bray & Moore ('79); Temesvári & al ('02)]

$$S[\{q_{ab}\}] = \int_x \Big\{ \frac{1}{2} \sum_{a<b} \big(\partial_x q_{ab}(x)\big)^2 + \frac{m_1}{2} \sum_{a<b} q_{ab}^2(x) + w_1 \sum_{a<b<c} q_{ab}(x) q_{bc}(x) q_{ca}(x) + \ldots \Big\}$$

★ $\mathbb{Z}_2$ symmetry $S_i^a \to -S_i^a$: $S[\{q_{ab}\}]$ is left invariant if $q_{ab}(x) \to -q_{ab}(x)\ \forall b$ (every replica index must appear an even number of times)

$$\Delta S_k[\{q_{ab}\}] = \frac{1}{2} \int_q R_k(q^2) q_{ab}(q) q_{ab}(-q) \qquad \mathcal{Z}_{k;ab,cd} = \mathcal{Z}_k \delta_{ac} \delta_{bd}$$

★ The NPRG flow equation

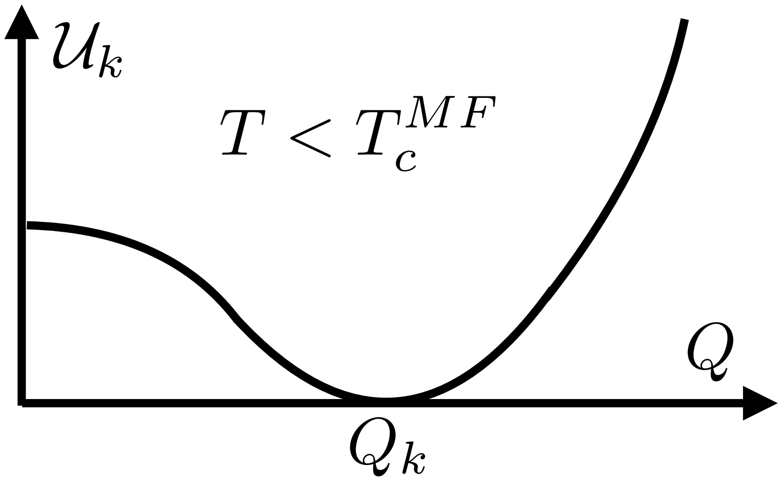
$$\partial_k \Gamma_k[\{Q_{ab}\}] = \frac{1}{2} \sum_{a<b} \int_q \partial_k R_k(q^2) \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \right]_{q,-q}^{ab,ab}$$

The inversion is not possible for generic configurations in replica space

Expansion in invariants of the potential

★ Expand $\mathcal{U}_k(\{Q_{ab}\})$ around the nontrivial replica symmetric (running) minimum

$$\boxed{\left.\frac{\delta \mathcal{U}_k(\{Q_{ab}\})}{\delta Q_{cd}}\right|_{\{Q_{ab}\}=Q_k} = 0}$$



Akin to: $\frac{r_k}{2}\phi^2 + \frac{g_k}{4!}\phi^4 \longrightarrow \frac{\lambda_k}{4}(\phi^2 - \phi_{0,k}^2)^2$

★ Preserve the symmetry of the theory ➡ Expansions in invariants

Combinations of the overlap field that satisfy the symmetry and vanish at the minimum and whose first derivatives also vanish at Q_k

$$O(Q^2) \longrightarrow \quad \rho_{ab}^{(2)} = \frac{1}{2}(Q_{ab}^2 - Q_k^2)$$

$$O(Q^3) \longrightarrow \quad \mu_{abc}^{(3)} = Q_{ab}Q_{bc}Q_{ca} - Q_k^3 - Q_k(\rho_{ab}^{(2)} + \rho_{bc}^{(2)} + \rho_{ca}^{(2)})$$

$$O(Q^4) \longrightarrow \mu_{abcd}^{(4)} = Q_{ab}Q_{bc}Q_{cd}Q_{da} - Q_k^4 - Q_k^2(\rho_{ab}^{(2)} + \rho_{bc}^{(2)} + \rho_{cd}^{(2)} + \rho_{da}^{(2)})$$

$$\rho_{ab}^{(4)} = \rho_{ab}^{(2)}\rho_{ab}^{(2)} \qquad \lambda_{abc}^{(4)} = \rho_{ab}^{(2)}\rho_{bc}^{(2)} \qquad \sigma_{abcd}^{(4)} = \rho_{ab}^{(2)}\rho_{cd}^{(2)}$$

Definition of the coupling constants

$$\mathcal{U}_k(\{Q_{ab}\}) = W_k \sum_{a<b<c} \mu^{(3)}_{abc} + U_{k,1} \sum_{a<b<c<d} \mu^{(4)}_{abcd} + U_{k,2} \sum_{a<b} \rho^{(4)}_{ab} + U_{k,3} \sum_{a<b<c} \lambda^{(4)}_{abc} + U_{k,4} \sum_{a<b<c<d} \sigma^{(4)}_{abcd} + \dots$$

★ Coupling constants defined through derivatives evaluated at Q_k, whose expressions must not depend on the order of the truncation

- Additional reorganization of the invariants (implemented order by order explicitly)
- Number of derivatives grows more rapidly that the number of invariants (under-constrained system of equations)
- Always possible to find a solution but the solution is not unique

◇ $\left.\dfrac{\delta^3 \mathcal{U}_k}{\delta Q_{ab}\delta Q_{bc}\delta Q_{ca}}\right|_{Q_k} = W_k$

◇ $\left.\dfrac{\delta^2 \mathcal{U}_k}{\delta Q_{ab}^2}\right|_{Q_k} = f_1(Q_k, W_k, U_{k,1}, U_{k,2}; n)$

◇ $\left.\dfrac{\delta^2 \mathcal{U}_k}{\delta Q_{ab}\delta Q_{bc}}\right|_{Q_k} = f_2(Q_k, W_k, U_{k,1}, U_{k,3}; n)$

◇ $\left.\dfrac{\delta^2 \mathcal{U}_k}{\delta Q_{ab}\delta Q_{cd}}\right|_{Q_k} = Q_k^2(2U_{k,1} + U_{k,4})$

◇ $\left.\dfrac{\delta^3 \mathcal{U}_k}{\delta Q_{ab}^2\delta Q_{cd}}\right|_{Q_k} = Q_k U_{k,4}$ ← or → ◇ $\left.\dfrac{\delta^3 \mathcal{U}_k}{\delta Q_{ab}\delta Q_{bc}\delta Q_{cd}}\right|_{Q_k} = Q_k U_{k,1}$

Lowest order truncation

$$\Gamma_k[\{Q_{ab}\}] = \int_x \left\{ \frac{\mathcal{Z}_k}{2} \sum_{a<b} \left(\partial_x Q_{ab}(x)\right)^2 + W_k \sum_{a<b<c} \mu_{abc}^{(3)} \right\}$$

★RG flow of three parameters by differentiating the exact RG equation
(and evaluating the expressions for uniform configurations $Q_{ab} = Q_k$)

$$\diamond\ \frac{\delta \Gamma_k}{\delta Q_{ab}}\bigg|_{Q_k} = 0\,; \quad \diamond\ W_k = \frac{\delta^3 \Gamma_k}{\delta Q_{ab}\delta Q_{bc}\delta Q_{ca}}\bigg|_{Q_k}; \quad \diamond\ \mathcal{Z}_k = \frac{(2\pi)^d}{V}\frac{\mathrm{d}}{\mathrm{d}q^2}\left(\frac{\delta^2 \Gamma_k}{\delta Q_{ab}(q)\delta Q_{ab}(-q)}\bigg|_{Q_k}\right)_{q^2=0}$$

Litim regulator $\longrightarrow$ $R_k(q^2) = \mathcal{Z}_k(k^2 - q^2)\theta(k^2 - q^2)$

★Fixed point ➡ Recast the RG equations in a dimensionless form
Introduce scaling dimensions and dimensionless quantities

$\diamond\ Q_k = k^{d/2-1}\mathcal{Z}_k^{-1/2}\mathfrak{q}_k$ $\quad\quad$ $\diamond\ W_k = k^{3-d/2}\mathcal{Z}_k^{3/2}\mathfrak{w}_k$

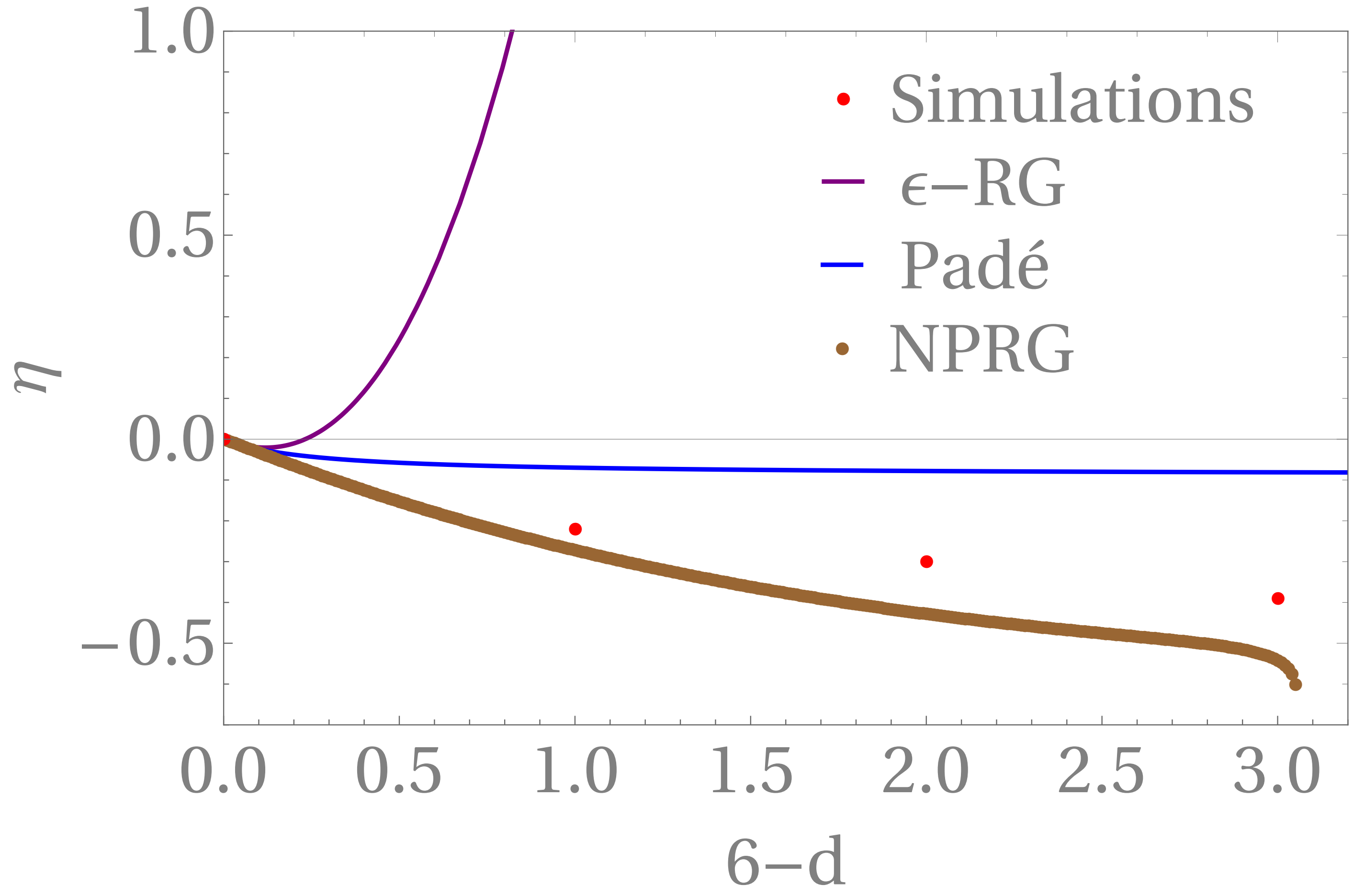
$\diamond\ q = k\hat{q}$ $\quad\quad$ $\diamond\ R_k(q^2) = k^2\mathcal{Z}_k\mathfrak{r}_k(\hat{q}^2)$

$$\diamond\ P_{k;ab,cd}(q^2) = \left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\right]^{ab,cd}_{q,-q} = k^{-2}\mathcal{Z}_k^{-1}\mathfrak{p}_k(\hat{q}^2)$$

★Running anomalous dimension: $\mathcal{Z}_{k\to 0} \sim \left(\dfrac{k}{\Lambda}\right)^{-\eta} \longrightarrow \eta_k = -k\,\partial_k \log \mathcal{Z}_k$

Results: anomalous dimension

- Nontrivial fixed point $\{\mathfrak{q}^\star, \mathfrak{w}^\star\}$ below six dimension (in addition to the Gaussian fixed point $\{\mathfrak{q}_g^\star = 0, \mathfrak{w}_g^\star = 0\}$)

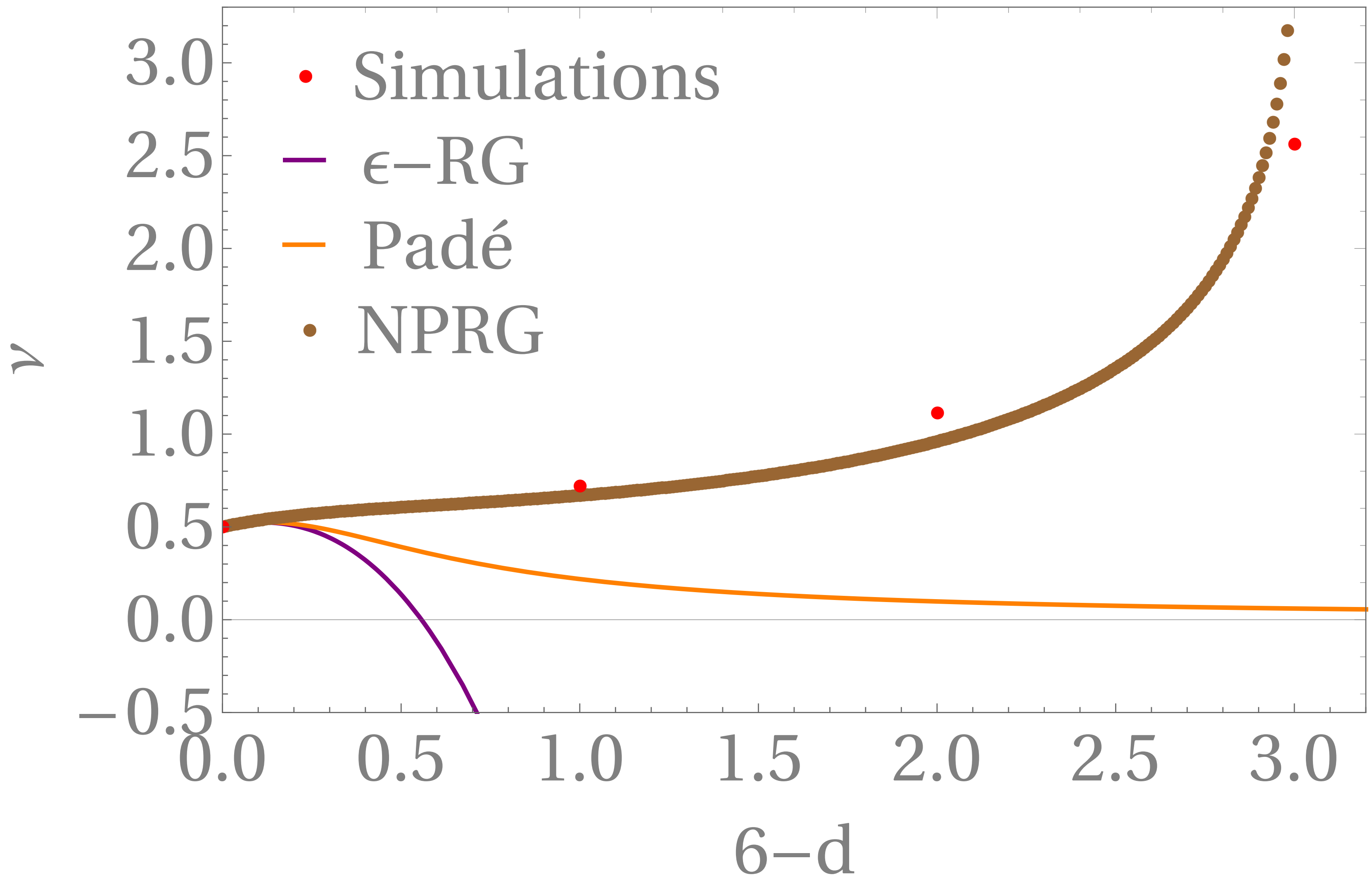


- The critical fixed point is found from d_U=6 down to d_L≈2.97
- Drastic improvement over the perturbative RG
- Near d=6 the NPRG exactly reproduces the 1-loop perturbative RG calculation [Harris & Lubensky ('76)]

Results: correlation length exponent

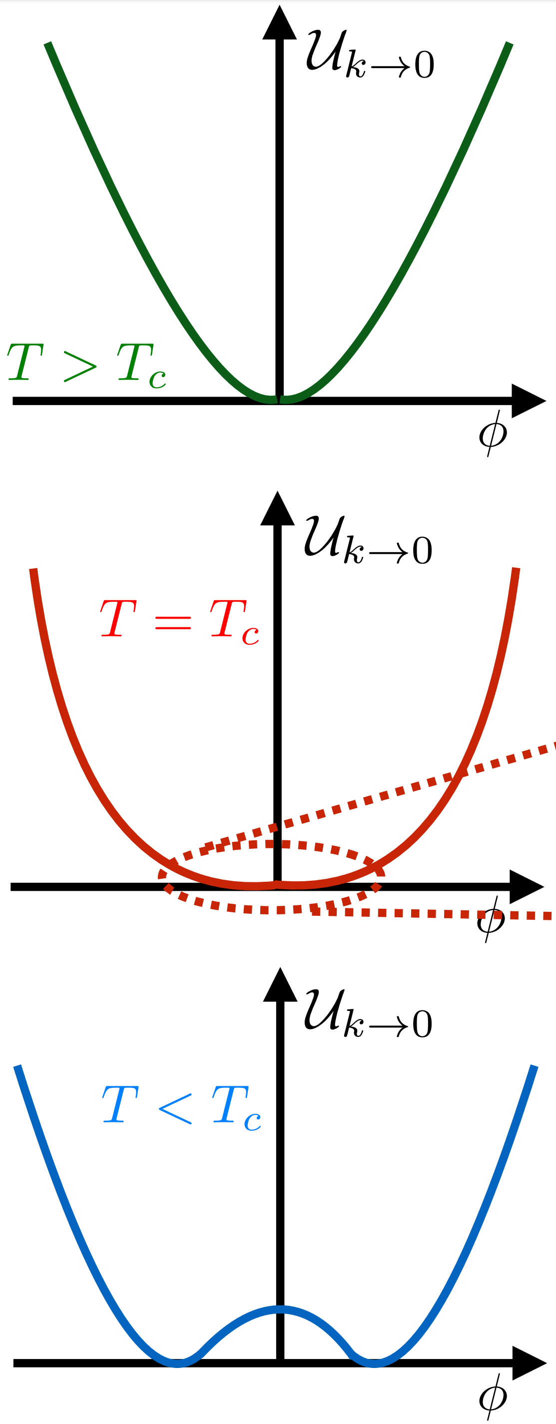
- Linear stability analysis of the RG flow equations around the critical FP

$$\nu = \frac{1}{\lambda_+}$$



Dimensionless potential at the critical FP

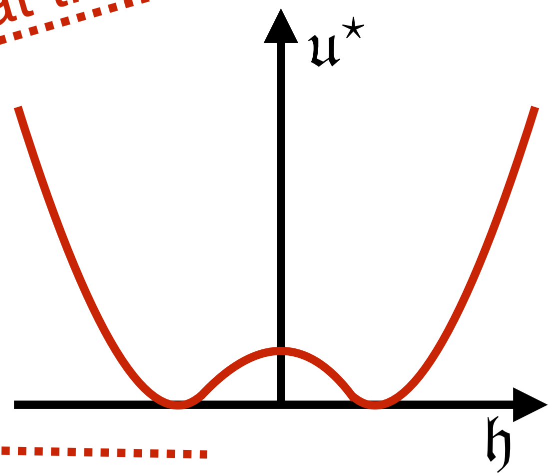
Dimensionful potential



e.g., Ising model: The *dimensionless* potential at the FP has a shape that resembles the shape of the *dimensionful* potential in the symmetry broken phase of a finite-size system

dimensionless potential at the FP

- $\phi = k^{d/2-1} \mathcal{Z}_k^{-1/2} \mathfrak{h}_k$
- $\mathcal{U}_k = k^d \mathfrak{u}_k$



$$\mathrm{P}\left(\frac{\phi}{L^{d/2-1+\eta/2}}\right) = \exp\left[-L^d\,\mathcal{U}\left(\frac{\phi}{L^{d/2-1+\eta/2}}\right)\right]$$

Indication of the symmetry-breaking scenario!

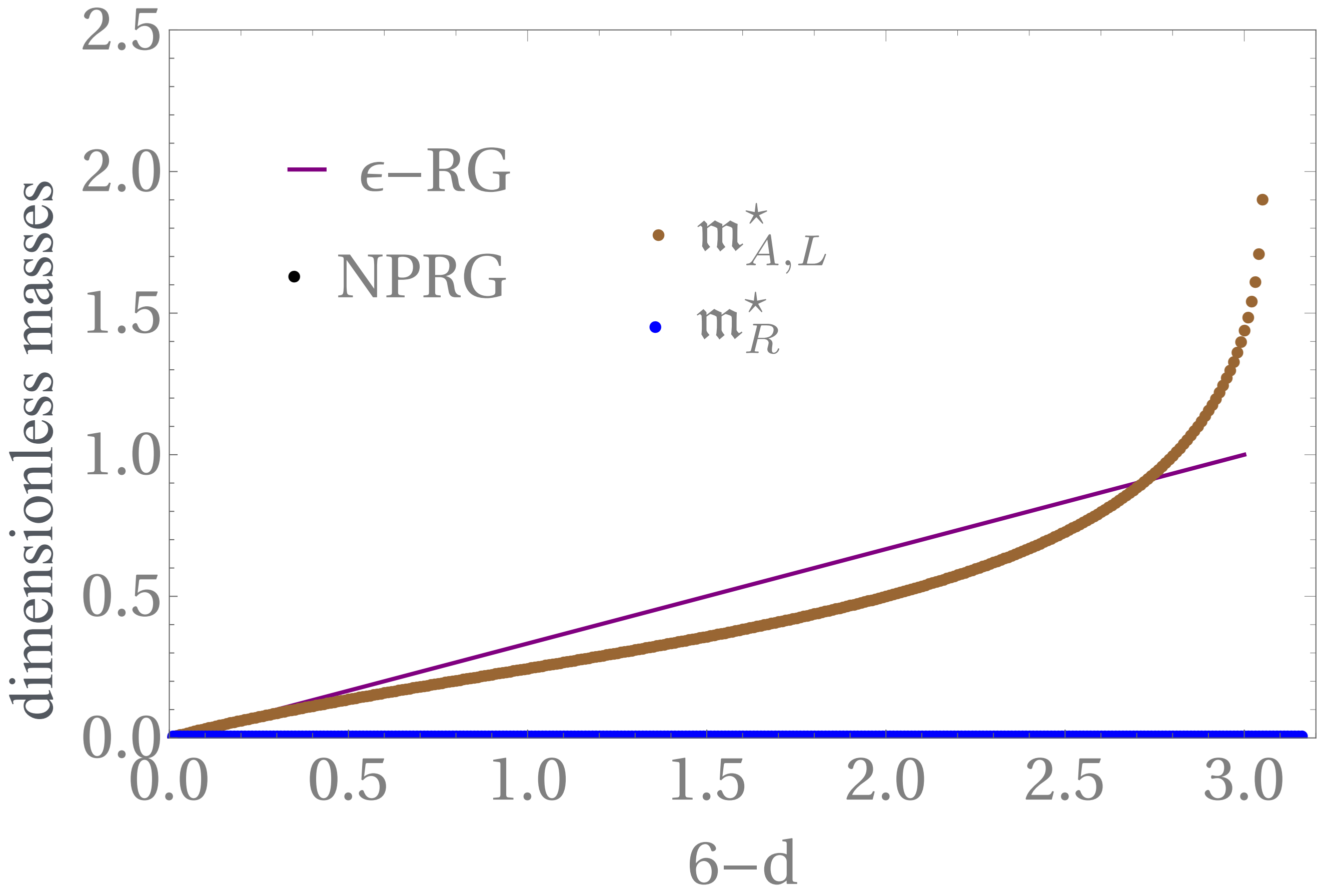
Results: dimensionless masses

$$\left.\frac{\delta^2 \mathcal{U}_k(\{Q_{ab}\})}{\delta Q_{bc}\delta Q_{ef}}\right|_{Q_k} \longrightarrow \frac{n(n-1)}{2} \times \frac{n(n-1)}{2} \quad \text{mass matrix}$$

1 "longitudinal" mode M_L
n-1 "anomalous" modes M_A
n(*n*-3)/2 "replicon" modes M_R

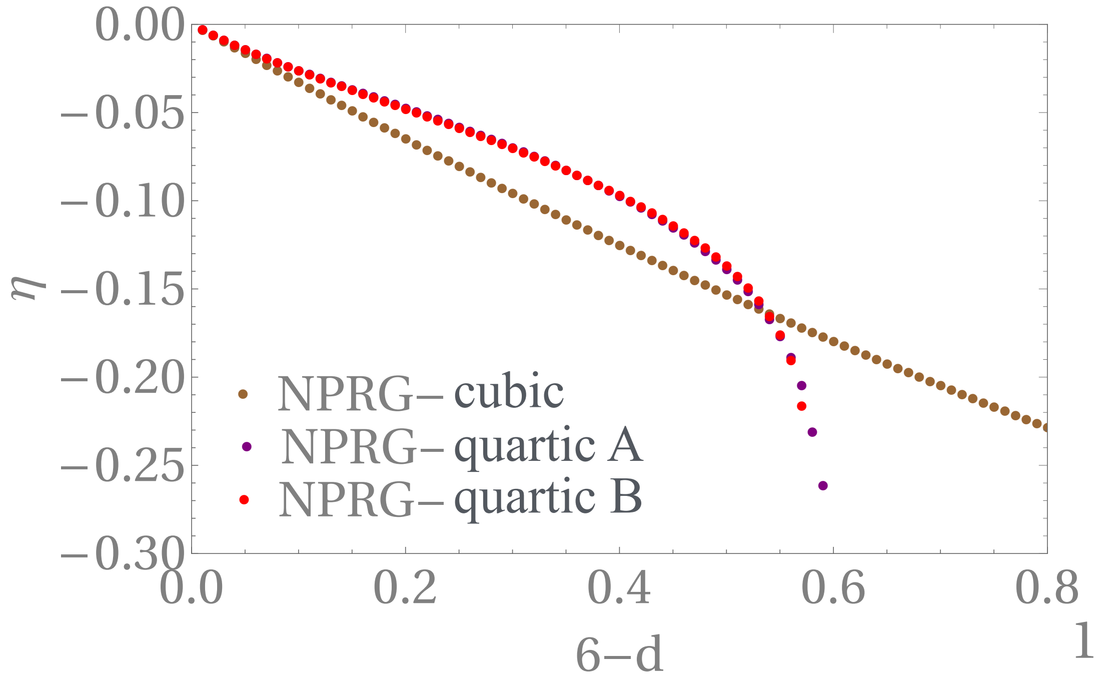
$$\mathfrak{m}^\star_{A,L} = -2\,\mathfrak{q}^\star \mathfrak{w}^\star > 0$$

$$\mathfrak{m}^\star_R = 0$$ (by construction at this order of the truncation)



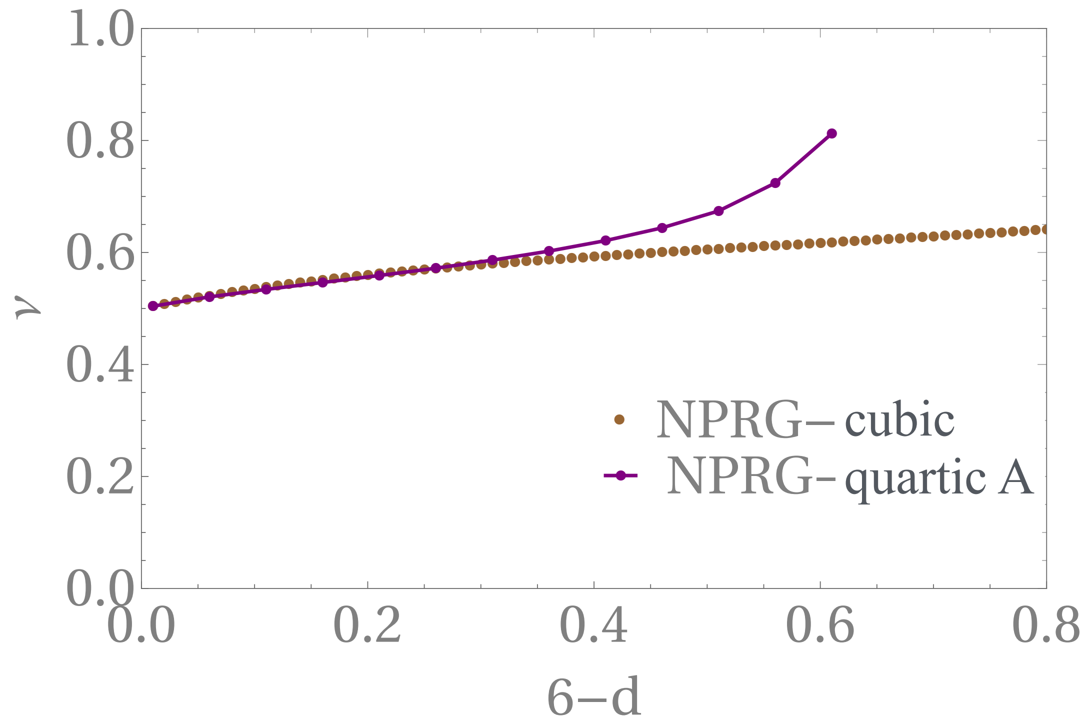
Quartic order truncation

7 independent flowing parameters $\longrightarrow$ $\mathcal{Z}_k, Q_k, W_k, U_{k,1}, U_{k,2}, U_{k,2}, U_{k,4}$

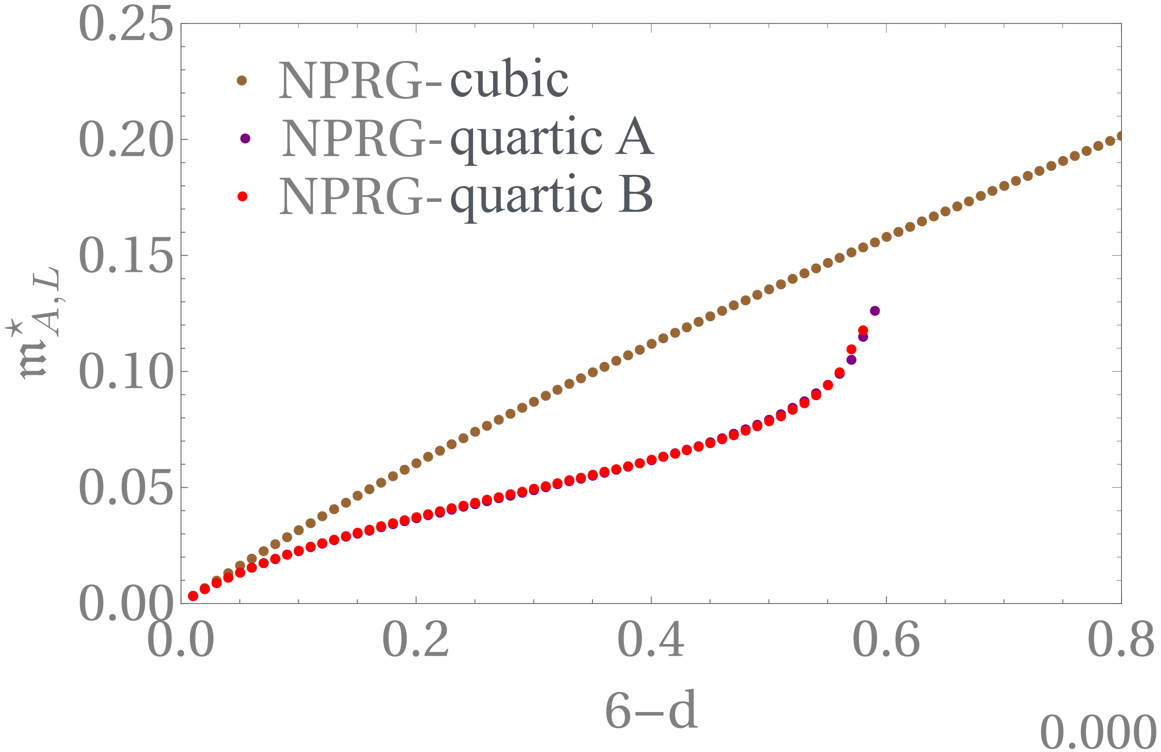


The critical FP no longer extends down to a dimension between 2 and 3 (it disappears around $d \approx 5.4$)

Typical behavior of an accidental collapse with spurious fixed points (also observed in the *O(N)* model studied with the NPRG with similar truncation schemes [Delamotte])

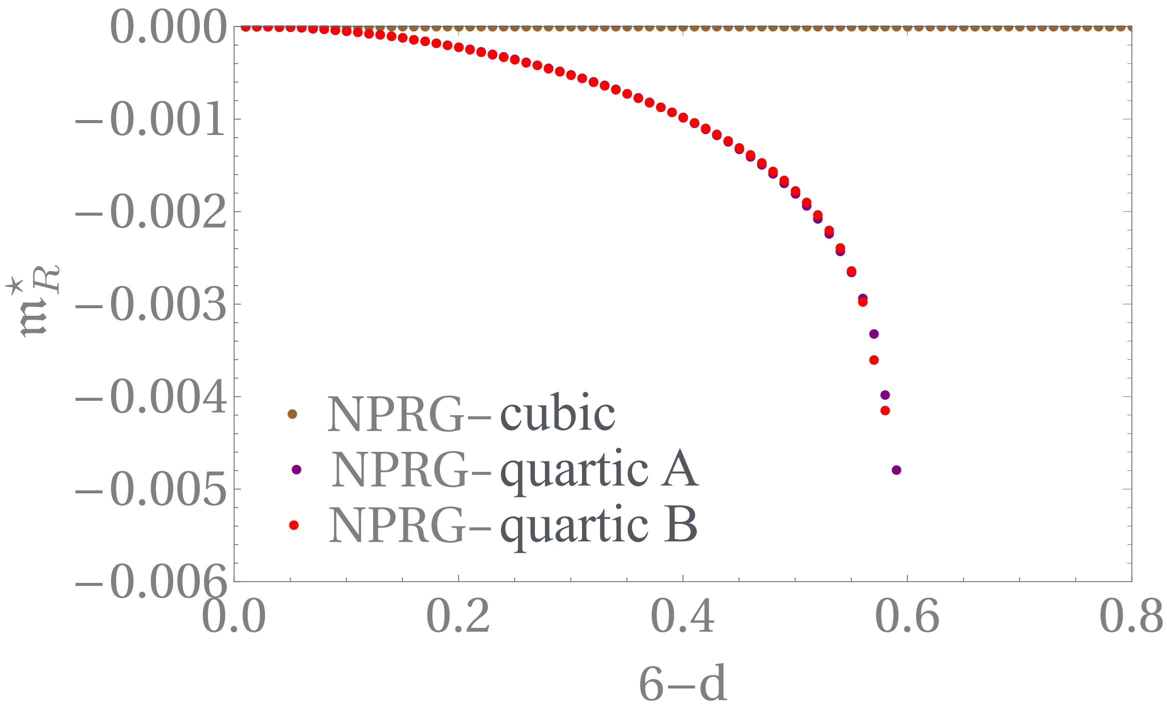


Negative dimensionless replicon mass



Over the range of dimensions where the FP exists the dimensionless replicon mass is negative

This might suggest that replica symmetry should be broken in the spin glass phase for all finite-size systems



Summary and perspectives

★ **NPRG approach for the critical behavior of spin glasses at zero magnetic field**

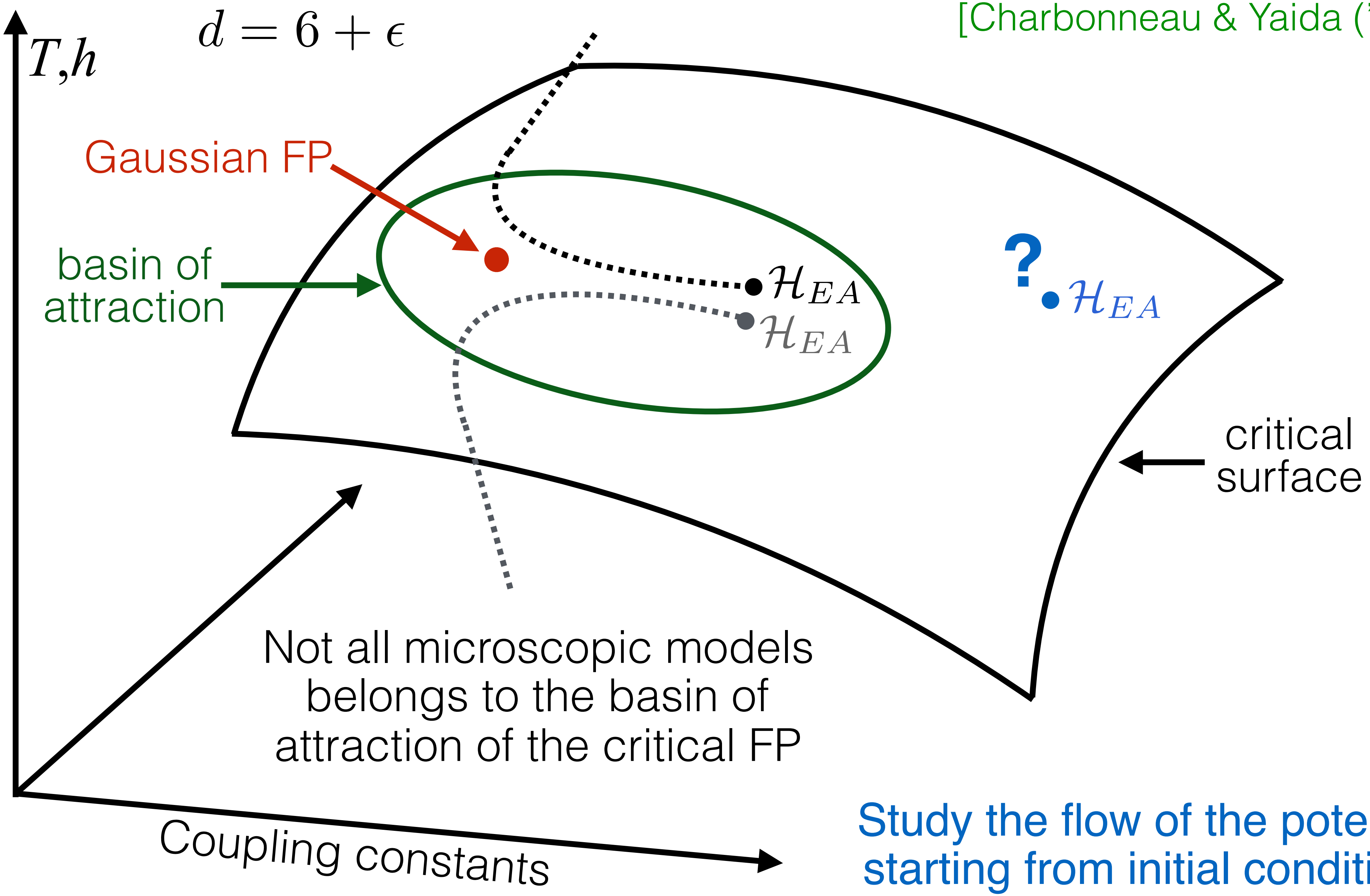
- First implementation of the NPRG for glassy systems characterized by an overlap field
- Approximation scheme that preserves explicitly the symmetry of the theory at each order of the truncation
- Good quantitative and qualitative description down to $d=3$
- Spurious fixed points when higher order truncations are considered
- The dimensionless replicon mass is negative at the FP

★ **Future work & perspectives**

- Study different kinds regulators (in progress)
- Expand around a full-RSB minimum [De Dominicis & al ('97)]
- Develop a functional NPRG approximation scheme to avoid field expansions

Transition in presence of an external field

[Charbonneau & Yaida ('17)]



Study the flow of the potential starting from initial conditions corresponding to the microscopic EA Hamiltonian

NPRG vs perturbative RG

★ Advantages

- More intuitive physical interpretation
- Technically easier (all computations are performed at the level of 1-loop)
- It allows (in principle) to find non-perturbative fixed points
- It allows to recover exactly the d_U-ϵ, d_L+ϵ and $1/N$ expansions at first order within a unique framework

★ Disadvantages

- The accuracy and reliability of the results depend on the quality of the ansatz for the Gibbs free-energy. Based on the physical intuition. One needs to individuate (and preserve) all the symmetries of the theory
- There is a certain degree of arbitrariety (e.g., choice of the regulator, definition of the coupling constants, …)
- No systematic way to improve the approximations