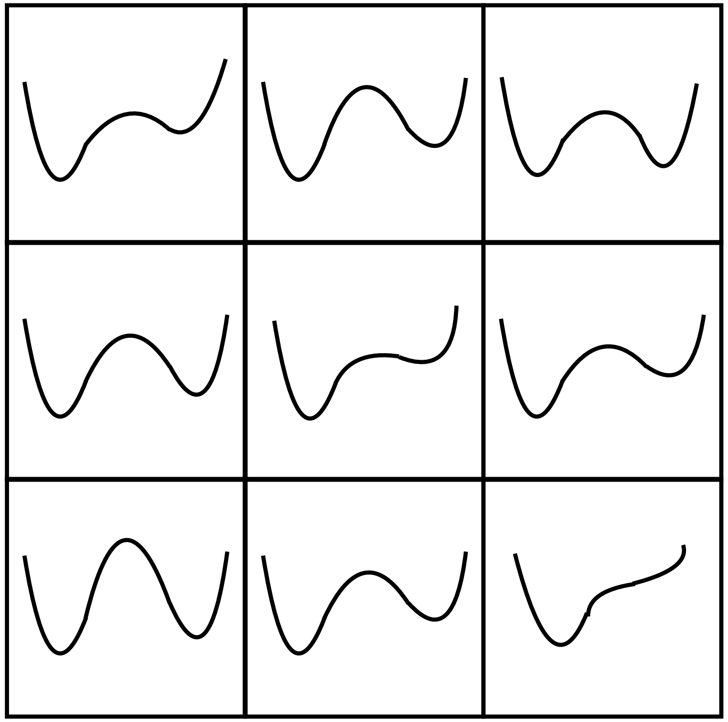
RFIM-like criticality: Intuitive argument

The equilibrium reference configuration acts as a quenched disorder

[Franz & al '11]



Local fluctuations of the reference configuration

Local fluctuations of the shape of the Franz-Parisi potential

Overlap $p(\mathbf{x})$
Configurational entropy S_c
Barrier Υ

Magnetization $m(\mathbf{x})$
Magnetic field h
Ferromagnetic coupling J

Random-field random-bond Ising model

What do we need to compute?

- Choose an equilibrium reference configuration at random

$$\mathcal{P}(\mathcal{C}_{\rm eq}) = e^{-\beta\mathcal{H}(\mathcal{C}_{\rm eq})}/Z$$

- Probability that the system has an overlap profile $p(\mathbf{x})$ with the reference configuration

$$e^{-\mathcal{S}[p(\mathbf{x})|\mathcal{C}_{\rm eq}]} = \frac{1}{Z}\sum_{\mathcal{C}} e^{-\beta\mathcal{H}(\mathcal{C})}\,\delta\left[p(\mathbf{x}) - Q_{\mathbf{x}}(\mathcal{C},\mathcal{C}_{\rm eq})\right]$$

- Introduce $n+1$ replicas ($n\to 0$) to average over $\mathcal{C}_{\rm eq}$

$$e^{-\mathcal{S}_{\rm rep}[\{p_a(\mathbf{x})\}]} \propto \sum_{\mathcal{C}_a,\mathcal{C}_{\rm eq}} e^{-\beta\mathcal{H}(\mathcal{C}_{\rm eq})}\, e^{-\beta\sum_a \mathcal{H}(\mathcal{C}_a)} \prod_a \delta\left[p_a(\mathbf{x}) - Q_{\mathbf{x}}(\mathcal{C}_a,\mathcal{C}_{\rm eq})\right]$$

- The cumulants of *S* can be computed through an expansion in increasing number of free replica sums [LeDoussal '03; Tarjus & Tissier '04]

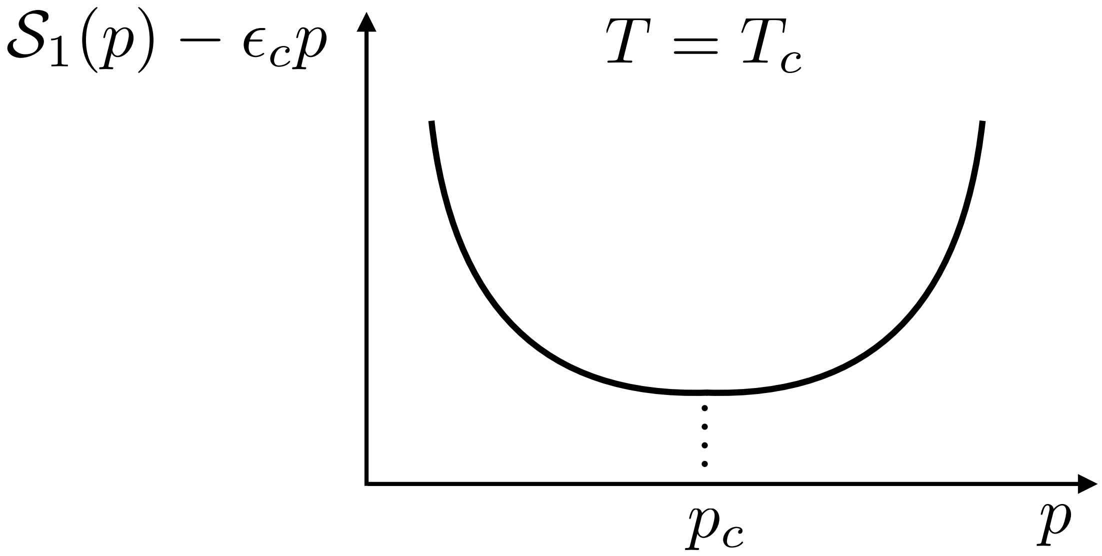
$$\mathcal{S}_{\rm rep}[\{p_a(\mathbf{x})\}] = \sum_{a=1}^{n} \mathcal{S}_1[p_a(\mathbf{x})] - \frac{1}{2}\sum_{a,b=1}^{n} \mathcal{S}_2[p_a(\mathbf{x}),p_b(\mathbf{x})] + \cdots$$

$$\mathcal{S}_1[p(\mathbf{x})] = \overline{\mathcal{S}[p(\mathbf{x})|\mathcal{C}_{\rm eq}]} \qquad \mathcal{S}_2[p_1(\mathbf{x}),p_2(\mathbf{x})] = \overline{\mathcal{S}[p_1(\mathbf{x})|\mathcal{C}_{\rm eq}]\mathcal{S}[p_2(\mathbf{x})|\mathcal{C}_{\rm eq}]}^{\,c}$$

The terminal critical point

[Biroli & al '14]

Study the long wave-length fluctuations of the overlap by integrating out the other degrees of freedom for fixed profiles of $p_a(\mathbf{x})$



$$p_a(\mathbf{x}) = p_c + \varphi_a(\mathbf{x})$$

The overlap $q_{ab}(\mathbf{x})$ become "massive"

Saddle-point approximation + gradient expansion

Scalar field theory with random field and random couplings

$$\mathcal{S}_{\text{eff}}[\varphi(\mathbf{x})] = \int \mathrm{d}\mathbf{x} \left\{ c' \left[\nabla\varphi(\mathbf{x})\right]^2 + \frac{r_2(\mathbf{x})}{2}\varphi^2(\mathbf{x}) + \frac{r_3(\mathbf{x})}{3!}\varphi^3(\mathbf{x}) + \frac{r_4}{4!}\varphi^4(\mathbf{x}) - h(\mathbf{x})\varphi(\mathbf{x}) \right\}$$

Higher order terms are expected to be irrelevant at criticality [Balog & al '14]

Consequences of the mapping

The most relevant term at criticality is the random field → RFIM universality class controlled by a zero temperature fixed point

Does the transition exists when all fluctuations beyond mean-field theory are taken into account?

Variance of the random field

RFIM on a cubic lattice
[Middleton & Fisher ’02]

$$\frac{\sqrt{\sigma_h^2}}{J} \leq 2.3$$

Surface tension
[Dzero & al ’05]
[Franz & Montanari ‘07]

We found values compatible with the existence of a transition for the Ginzburg-Landau functional, and too large for the existence of a transition in *d=3* for the Kač 3-spin

Deriving the parameters of the effective theory should be possible in numerical simulations of small-size systems
[Berthier & Jack ’15; Rilquin & al ’16]

Mapping in the vicinity of T_K: The REM

Much more challenging ← **Long-wavelength fluctuations associated with diverging point-to-set spatial correlations**

An illustrative exemple: The REM (The simplest model with a RFOT)

The energies $E(C)$ of the configurations $C=\{S_1,\dots,S_N\}$ are i.i.d. Gaussian random variables with zero mean and variance $N/2$ [Derrida '81]

- **States and configurations coincide**
- **The overlap is a binary variable,** $p_a=0,1$

$$\mathcal{S}_{\text{rep}}[\{p_a\}] = \left(N\ln 2 - \frac{\beta^2 N}{4}\right)\sum_{a=1}^{n} p_a - \frac{\beta^2 N}{4}\left(\sum_{a=1}^{n} p_a\right)^2 + \text{cst}$$

Replicated action for a Ising spin coupled to a disordered Gaussian magnetic field **(0-dimensional RFIM)** [Biroli & al '17]

$$\mathcal{H}_{\text{eff}} = (H + \delta h)\sigma$$

$$H = N(\ln 2 - \beta^2/4)/2 = N(\beta_K^2 - \beta^2)/8 \qquad \overline{\delta h^2} = \beta^2 N/8$$

Coupled REMs with a finite number of states

On each site *i* there are 2^M configurations, $\mathcal{C}_i = \{1, \ldots, 2^M\}$ [Franz & al '08]

$$\mathcal{H} = \frac{1}{2\sqrt{N}} \sum_{i \neq j} E_{ij}(\mathcal{C}_i, \mathcal{C}_j)$$

$$\overline{E_{ij}(\mathcal{C}_i, \mathcal{C}_j)} = 0$$

$$\overline{E_{ij}(\mathcal{C}_i, \mathcal{C}_j) E_{ij}(\mathcal{C}'_i, \mathcal{C}'_j)} = M \delta_{\mathcal{C}_i, \mathcal{C}'_i} \delta_{\mathcal{C}_j, \mathcal{C}'_j}$$

Average over the disorder and over the reference configuration

$$e^{-\mathcal{S}_{\mathrm{rep}}[\{p_a^i\}]} \propto \sum_{\{\mathcal{C}_i^\alpha\}} e^{\frac{\beta^2 M}{8N} \sum_{i \neq j} \sum_{\alpha, \beta} \delta_{\mathcal{C}_i^\alpha, \mathcal{C}_i^\beta} \delta_{\mathcal{C}_j^\alpha, \mathcal{C}_j^\beta}} \prod_{a,i} \delta_{p_a^i, \delta_{\mathcal{C}_i^0, \mathcal{C}_i^a}}$$

$$\delta_{\mathcal{C}_i^\alpha, \mathcal{C}_i^\beta} \delta_{\mathcal{C}_j^\alpha, \mathcal{C}_j^\beta} = 1 + n + 2 \sum_a p_a^i p_a^j + \sum_{a \neq b} q_{ab}^i q_{ab}^j$$

The only undetermined case corresponds to $p_a^i = 0, p_b^i = 0$

$$p_a^i = 1,\ p_b^i = 1 \Longrightarrow \mathcal{C}_i^a = \mathcal{C}_i^0,\ \mathcal{C}_i^b = \mathcal{C}_i^0 \Longrightarrow q_{ab}^i = 1$$

$$p_a^i = 1,\ p_b^i = 0 \Longrightarrow \mathcal{C}_i^a = \mathcal{C}_i^0,\ \mathcal{C}_i^b \neq \mathcal{C}_i^0 \Longrightarrow q_{ab}^i = 0$$

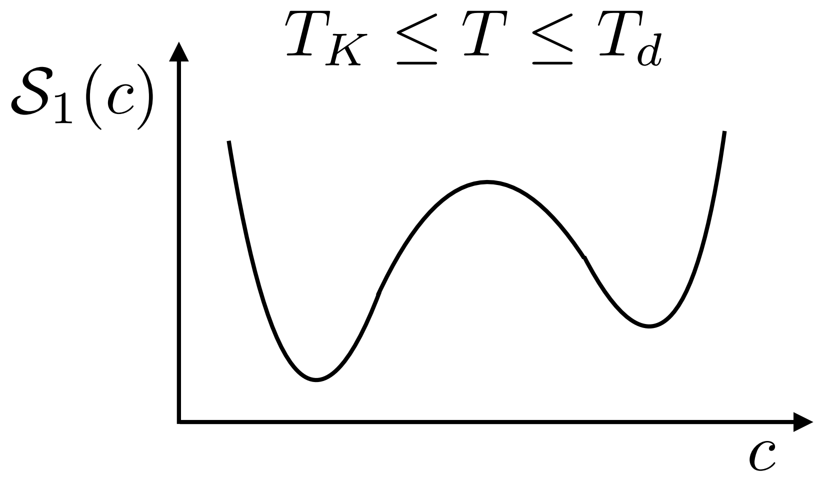
$$p_a^i = 0,\ p_b^i = 1 \Longrightarrow \mathcal{C}_i^a \neq \mathcal{C}_i^0,\ \mathcal{C}_i^b = \mathcal{C}_i^0 \Longrightarrow q_{ab}^i = 0$$

Average effective Hamiltonian

Set all replica fields equal ($p_a^i = p^i \ \forall \, a, i$) and keep only the terms of order *n* in the expression $\mathcal{S}_{\text{rep}}$

$\mathcal{S}_1[\{p^i\}]$ can only be a function of $c = \dfrac{1}{N}\sum_i p^i$ (global overlap with $\mathcal{C}_{\text{eq}}$)

Akin to the computation of the Franz-Parisi potential



$$\mathcal{S}_1(c) \approx K_0 + K_1 c + K_2 c^2 + K_3 c^3 + K_4 c^4 + \dots$$

- This naturally leads to an infinite number of multi-body couplings
- Exact expressions of the couplings for $M \gg 1$

Variance of the effective Hamiltonian

- Divide the n replicas in two groups of n_1 and n_2 replicas

$$p_a^i = p_1^i \quad \forall i, \quad \forall a = 1, \ldots, n_1$$
$$p_a^i = p_2^i \quad \forall i, \quad \forall a = n_1 + 1, \ldots, n$$

- Keep only the terms of order n_1 n_2 in the expression $\mathcal{S}_{\text{rep}}$

- Divide the N sites in four groups

Nc_1 $\quad$ Nc_2 $\quad$ Nc_{12} $\quad$ $N(1-c_1-c_2-c_{12})$

$p_1 = 1, p_2 = 0$ $\quad$ $p_1 = 0, p_2 = 1$ $\quad$ $p_1 = 1, p_2 = 1$ $\quad$ $p_1 = 0, p_2 = 0$

$\mathcal{S}_2[\{p_1^i, p_2^i\}]$ can only be a function of c_1, c_2, c_{12} :

$$c_1 = \frac{1}{N}\sum_i p_1^i(1-p_2^i)\,, \;\; c_2 = \frac{1}{N}\sum_i p_2^i(1-p_1^i)\,, \;\; c_{12} = \frac{1}{N}\sum_i p_1^i p_2^i$$

- Expand S_2 in powers of c_1, c_2, c_{12}

  Exact asymptotic expressions for $M \gg 1$

The effective Hamiltonian

[Biroli & al '17]

Random-field random-bond fully connected Ising model with multi-body interactions and higher order random terms ($\sigma^i = 2p^i - 1$)

$$\beta\mathcal{H}_{\text{eff}} = \sum_i (H + \delta h_i)\sigma^i - \sum_{i \neq j}\left(\frac{J_2}{N} + \frac{\delta J_{ij}}{\sqrt{N}}\right)\sigma^i\sigma^j + \frac{J_3}{N^2}\sum_{i,j,k\neq}\sigma^i\sigma^j\sigma^k + \frac{J_4}{N^3}\sum_{i,j,k,l\neq}\sigma^i\sigma^j\sigma^k\sigma^l + \cdots$$

Exact asymptotic expressions of the coupling constants for $M \gg 1$

$$H \approx \frac{M}{16}(\beta_K^2 - \beta^2) \qquad J_2 \approx \frac{M\beta^2}{32}$$

$$\overline{\delta h_i \delta h_j} \approx \frac{M\beta^2}{32}\delta_{ij}$$

$$\overline{\delta J_{ij}\delta J_{kl}} \approx \frac{M\beta^2}{128}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad \overline{\delta h_i \delta J_{jk}} \approx \frac{M\beta^2}{128}(\delta_{ij} + \delta_{ik})$$

Could be generalized to other MF models

(BUT in other MF models the overlap is not a 2-state variable!)

Supercooled liquids in finite dimensions

Two "low-T" variational approximations

1. Scanning all possible overlap profiles with the reference configuration is an impossible task.	→	We account for the effect of the correlations induced by amorphous order by **restricting to arbitrarily specific patterns**
2. Still the problem of tracing out the q_{ab} remains hardly tractable	→	**Variational approximation** for the q_{ab} in the form of 1-RSB matrices

Reproduce the results via a translationally invariant effective Hamiltonian:
Ising model with random-field and random-bond disorder and long-range competing multi-body interactions

$$\beta\mathcal{H}_{\mathrm{eff}} = -\sum_{\langle i,j\rangle}(J_2+\delta J_{ij})\sigma^i\sigma^j + \sum_i (H+\delta h_i)\,\sigma^i + \frac{1}{2}\sum_{i\neq j}\tilde{J}_2(|r_i-r_j|)\,\sigma^i\sigma^j + \dots$$

Long-range interactions are tracked back to the diverging (point-to-set) correlations associated with amorphous order induced by the specific configurations of the overlap patterns [Biroli & al '17]

Conclusions

- An effective description of the fluctuations of the overlap with a reference configuration in terms of RFIM-like theories emerges in a general, robust, natural and transparent way

- Multi-body (and possibly long-range) effective interactions are tightly related to the diverging (point-to-set) correlations close to T_K and to the complex structure of the Franz-Parisi potential

- Qualitative and semi-quantitative information on the existence of the Kauzmann transition

- Identify the mechanisms which might alter or destroy the RFOT transition in finite dimensions
    - Multi-body competing interactions and/or higher order random terms
    - If the random-bond fluctuations become larger than the effective ferromagnetic coupling a spin-glass physics might set in

[Moore & Yeo '06]

Perspectives & Open questions

- Generalize the mapping to other mean-field and Kač RFOT models
- Improve the variational approximation used for finite-dimensional systems
- Estimate the surface tension and of the fluctuations of the potential for more realistic models (e.g., hard spheres)
- Use the effective theory in **direct conjunction with numerical simulations of glass-former models of small size** [Rulquin & al '16]
  - Determination of the key effective coupling constants
  - Use the effective model as a benchmark for the original model
- No obvious mapping from the dynamics of the glass-forming liquid to that of the effective theory.
  - **A complete characterization of the dynamics of glassy systems is much more challenging:** this requires a full understanding of the activated dynamics of a system evolving in a rugged energy landscape
  - Bridge the gap between RFOT theory and Kinetically Constrained Models [Rulquin & al '16]

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