Counting lattice walks by winding angle using Jacobi theta functions

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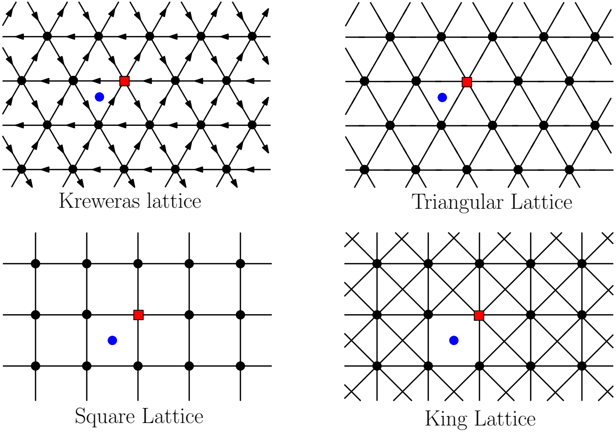
Lattice walks by winding angle

The model: count walks starting at the red square by end point and winding angle around blue point.

Cell-centred lattices:



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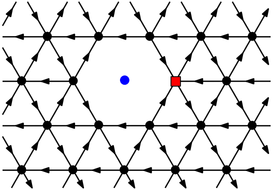
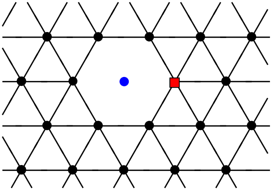
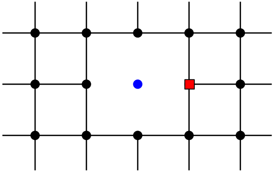
Vertex-centred lattices:



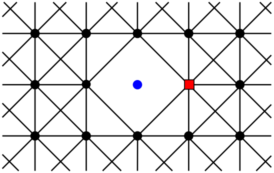
Kreweras lattice



Triangular Lattice



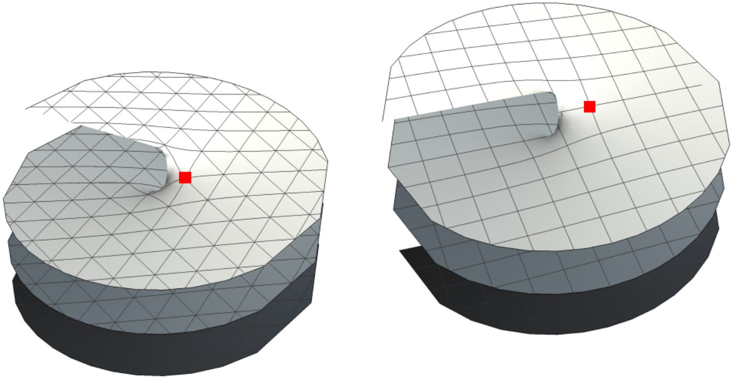
Square Lattice



King Lattice

LATTICE WALKS BY WINDING ANGLE

The model: count walks starting at the red square (by end point).



Left: Cell-centred triangular lattice
Right: Vertex-centred square lattice

Square lattice walks by winding angle

[Timothy Budd, 2017]: enumeration of square lattice walks (starting and ending on an axis or diagonal) by winding angle

- **Method:** Matrices counting paths, eigenvalue decomposition etc.
- **Solution:** Jacobi theta function expressions
- **Corollaries:**
  - Square lattice walks in cones (eg. Gessel walks)
  - Loops around the origin (without a fixed starting point)
  - Algebraicity results, asymptotic results, etc.

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This work:

- Completely different method
- Slightly different set of results
- Extension to three other lattices

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This talk: Kreweras lattice (mostly)

Jacobi theta function

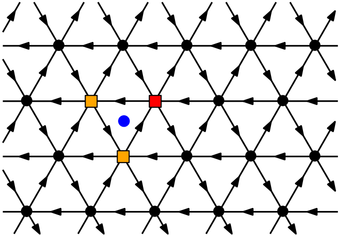
All results are in terms of the series:

$$T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1}-(-1)^k u^{-n})$$
$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

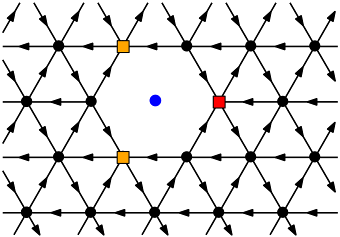
Related to Jacobi Theta function $\vartheta(z,\tau) \equiv \vartheta_{11}(z,\tau)$ by

$$\vartheta^{(k)}(z,\tau) \equiv \left(\frac{\partial}{\partial z}\right)^k \vartheta(z,\tau) = e^{\frac{(\pi\tau-2z)i}{2}} i^k T_k(e^{2iz}, e^{2i\pi\tau}).$$

PREVIEW: KREWERAS ALMOST-EXCURSIONS



Cell-centred Kreweras lattice



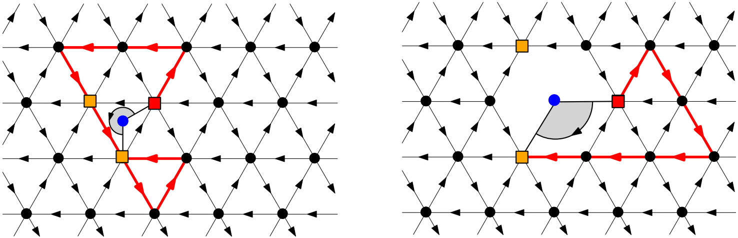
Vertex-centred Kreweras lattice

On each lattice: count walks from red square to a square (red or orange). Walks with length n and winding angle $\frac{2\pi k}{3}$ contribute $t^n s^k$.

Cell-centred: $E(t,s) = 1 + st + \left(s^2 + s^{-1}\right) t^2 + \dots$

Vertex-centred: $\tilde{E}(t,s) = 1 + \left(s^{-1} + 4 + s\right) t^3 + \dots$

Preview: Kreweras almost-excursions



Cell-centred Kreweras lattice   Vertex-centred Kreweras lattice

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Preview: Kreweras almost-excursions

Define $$T_k(u,q)=\sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1}-(-1)^k u^{-n})$$
$$=(u\pm 1)-3^k q(u^2\pm u^{-1})+5^k q^3(u^3\pm u^{-2})+O(q^6).$$

Let $q(t)\equiv q=t^3+15t^6+279t^9+\cdots$ satisfy

$$t=q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3)+6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

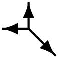
$$E(t,s)=\frac{s}{(1-s^3)t}\left(s-q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)}-q^{-1/3}\frac{T_0(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T_0(sq^{-2/3},q)}\right).$$

The gf for vertex-centred Kreweras-lattice almost-excursions is:

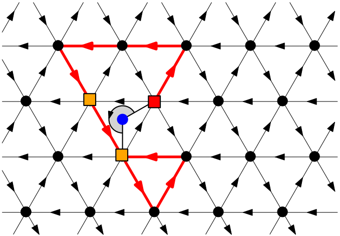
$$\tilde{E}(t,s)=\frac{s(1-s)q^{-\frac{2}{3}}}{t(1-s^3)}\frac{T_0(q,q^3)^2}{T_1(1,q^3)^2}\left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2}-\frac{T_2(q,q^3)}{T_0(q,q^3)}-\frac{T_2(s,q)}{2T_0(s,q)}+\frac{T_3(1,q)}{6T_1(1,q)}+\frac{T_3(1,q^3)}{3T_1(1,q^3)}\right).$$

Talk outline

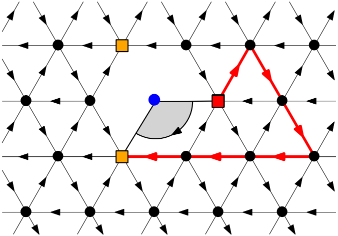
Focus: Kreweras lattice (for parts 1 to 4).

- **Part 1:** Decomposition of lattice $\rightarrow$ functional equations
- **Part 2:** Solving the functional equations (with theta functions!)
- **Part 3:** Corollaries: walks restricted to cones
  - **New result:** Excursions with step set  avoiding a quadrant
- **Part 4:** Analysing the solution
  - Algebraicity results using modular forms
  - Asymptotic results
- **Part 5:** Square, triangular and king lattices
- **Part 6:** Final comments and open problems

Part 1: Functional equations for Kreweras walks by winding angle



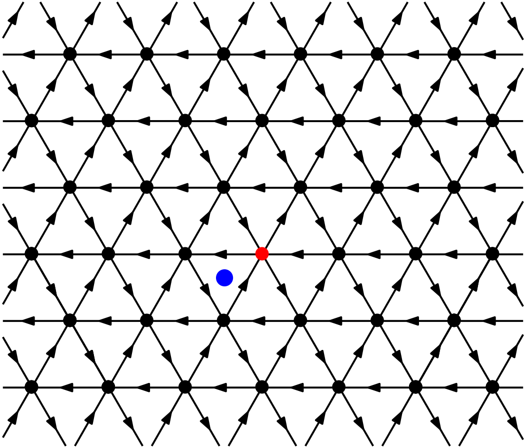
Cell-centred Kreweras lattice



Vertex-centred Kreweras lattice

Kreweras walks by winding number

The model: Count walks starting at the red point by end point and number of times winding around the blue point.



KREWERAS WALKS BY WINDING NUMBER

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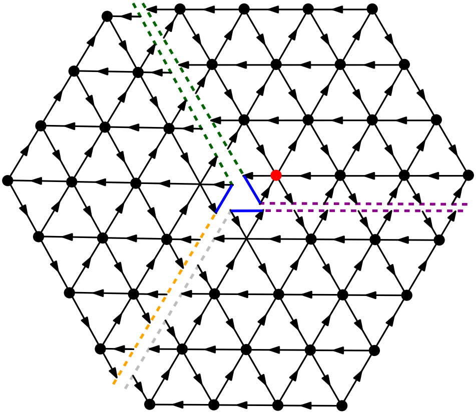

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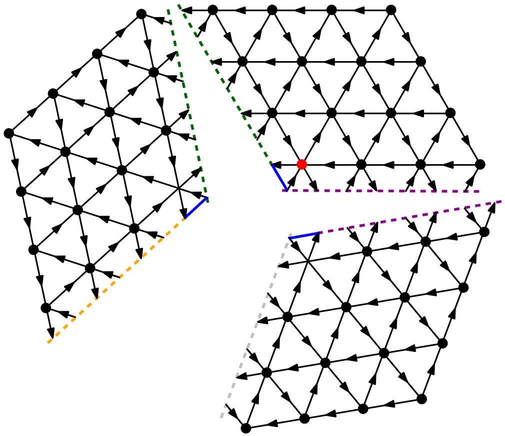

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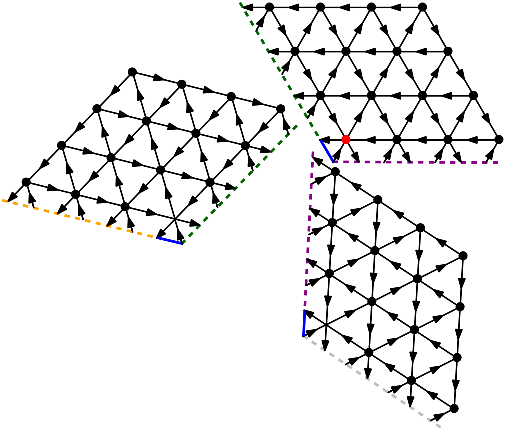

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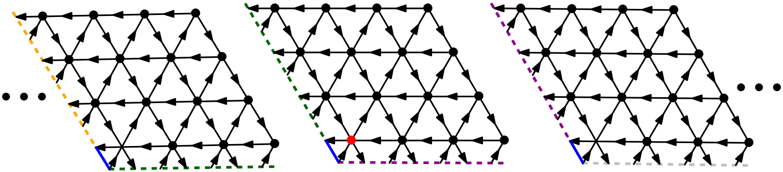

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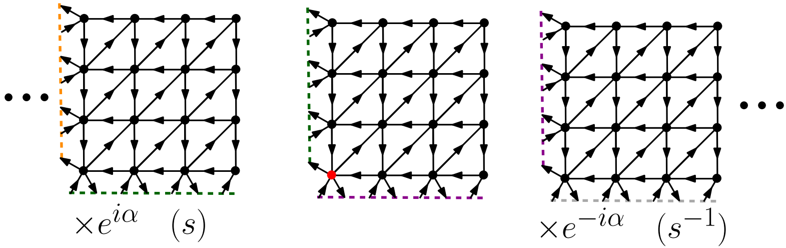

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t, e^{i\alpha})$

KREWERAS WALKS BY WINDING NUMBER

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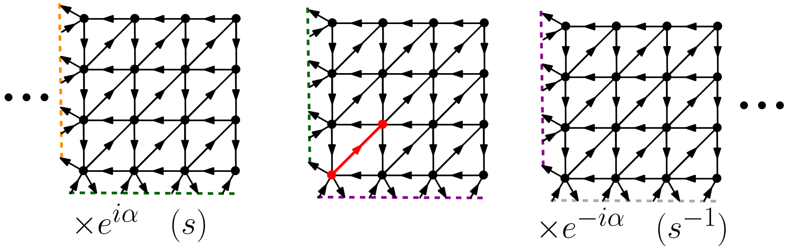

This example contributes txy.

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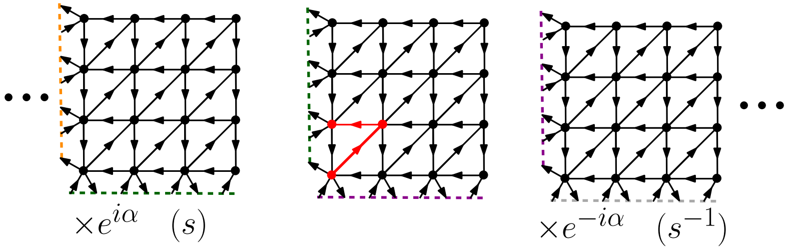

This example contributes $t^2 y$.

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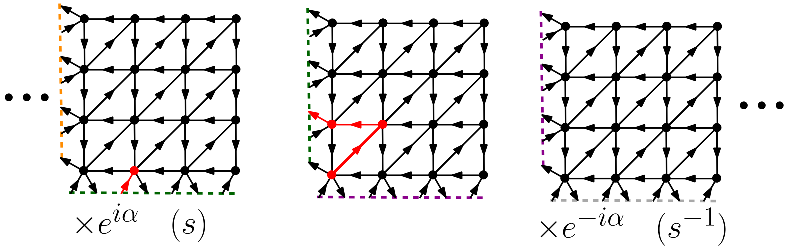

This example contributes $t^3xe^{i\alpha}$.

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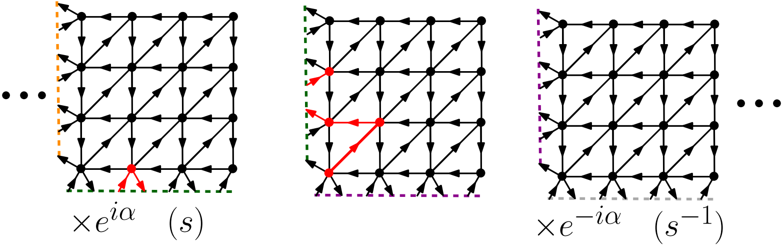

This example contributes $t^4 y^2$.

Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

Note: $Q(0, 0) = E(t, e^{i\alpha})$

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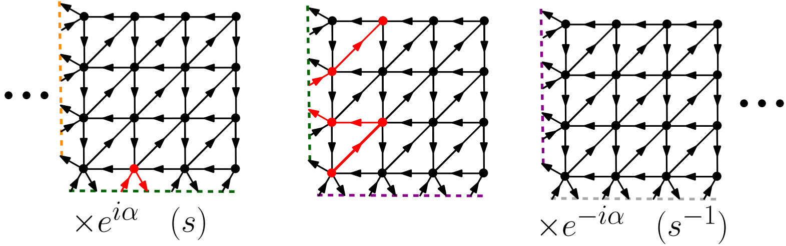

This example contributes t^5xy^3.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

Note: $Q(0,0) = E(t,e^{i\alpha})$

Kreweras walks by winding number

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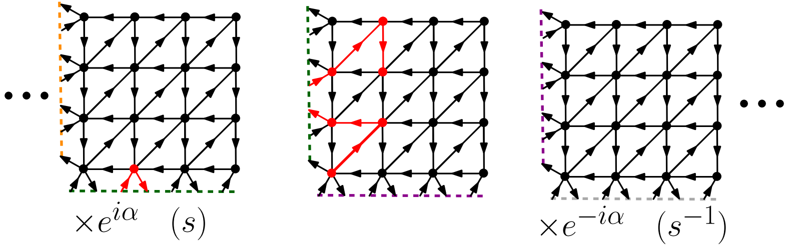

This example contributes $t^6 x y^2$.

Definition: $Q(t, \alpha, x, y) \equiv Q(x, y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

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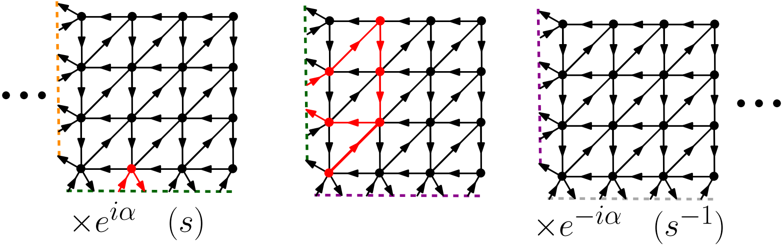

This example contributes t^7xy.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

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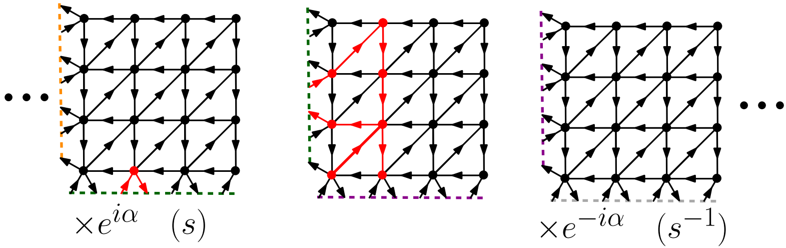

This example contributes t^8x.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

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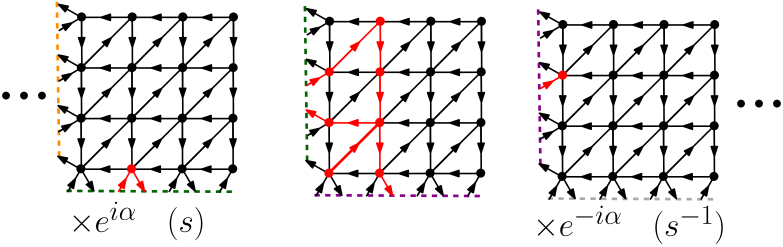

This example contributes $t^9 y^2 e^{-i\alpha}$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$

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Kreweras walks by winding number

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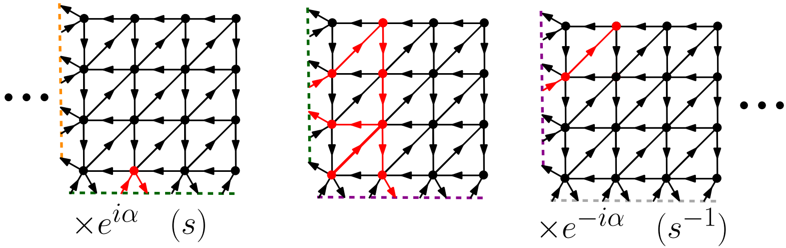


This example contributes $t^{10}xy^3e^{-i\alpha}$.

Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|}x^{x(p)}y^{y(p)}e^{i\alpha n(p)}$

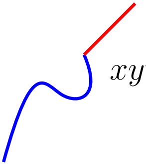
Note: $Q(0,0) = E(t, e^{i\alpha})$

Functional equation

Recursion → functional equation: separate by *type* of final step.

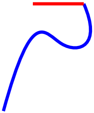
$$Q(x,y) = 1$$

$$+$$



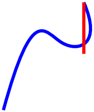
$$xytQ(x,y)$$

$$+$$



$$\frac{t}{x}(Q(x,y) - Q(0,y))$$

$$+$$



$$\frac{t}{y}(Q(x,y) - Q(x,0))$$

$$+ e^{i\alpha} tQ(0,x)$$

(Final step goes through left wall)

$$+ e^{-i\alpha} tyQ(y,0)$$

(Final step goes through bottom wall)

Kreweras walks by winding number

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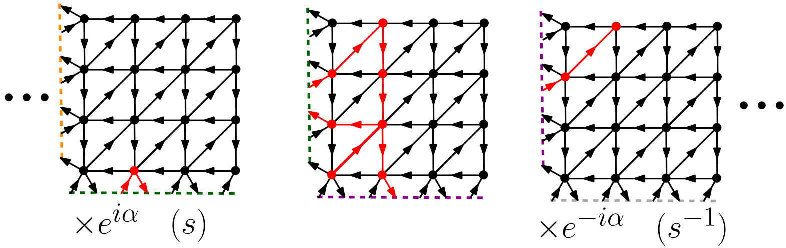


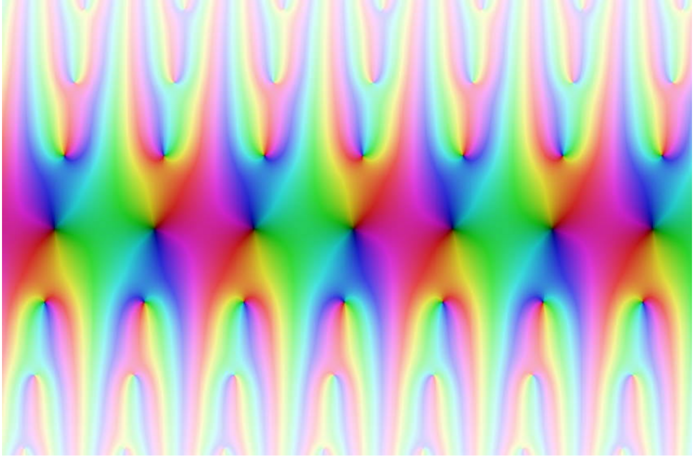
Definition: $Q(t,\alpha,x,y) \equiv Q(x,y) = \sum_{\text{paths } p} t^{|p|} x^{x(p)} y^{y(p)} e^{i\alpha n(p)}$.

Characterised by:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y)-Q(0,y)}{x} + t\frac{Q(x,y)-Q(x,0)}{y}$$

$$+e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Part 2: Solution (using theta functions)



Solution to Kreweras walks by winding number

Equation to solve:

$$Q(x,y) = 1 + txyQ(x,y) + t\frac{Q(x,y) - Q(0,y)}{x} + t\frac{Q(x,y) - Q(x,0)}{y}$$
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Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Solution to Kreweras walks by winding number

Equation to solve:

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Solution:

Step 1: Fix $t \in [0, 1/3), \alpha \in \mathbb{R}$. All series converge for $|x|, |y| < 1$.

Step 2: Write equation as $K(x,y)Q(x,y) = R(x,y)$, where

$$K(x,y) = 1 - txy - t/y - t/x$$
$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve:

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Step 3: Consider the curve $K(x,y) = 0$ (Then $R(x,y) = 0$).

Solution to Kreweras walks by winding number

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$$K(x,y) = 1 - txy - t/y - t/x$$

$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Step 3: Consider the curve $K(x,y) = 0$ (Then $R(x,y) = 0$).

Parameterisation involves the Jacobi theta function $\vartheta(z,\tau)$.

So far: Similar to [Kurkova, Raschel 12] and [Bernardi, Bousquet-Mélou, Raschel 17] for quadrant models (using $\wp$).

JACOBI THETA FUNCTION $\vartheta(z,\tau)$

Definition: For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$,

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

Useful facts (for fixed τ):

- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$

PARAMETERISATION OF $K(x,y)=0$ USING $\vartheta(z,\tau)$

Definition: For $\tau, z \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$,

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\left(\frac{2n+1}{2}\right)^2 i\pi\tau + (2n+1)iz}$$

Useful facts (for fixed τ):

- $\vartheta(z+\pi,\tau) = -\vartheta(z,\tau)$
- $\vartheta(z+\pi\tau,\tau) = -e^{-2iz-i\pi\tau}\vartheta(z,\tau)$

Parameterisation: The curve

$$K(x,y) := 1 - txy - t/y - t/x = 0$$

is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where τ is determined by $t = e^{-\frac{\pi\tau i}{3}}\dfrac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}$.

Solution to Kreweras walks by winding number

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

$$K(x,y) = 1 - txy - t/y - t/x,$$
$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$.

Solution to Kreweras walks by winding number

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

$$K(x,y) = 1 - txy - t/y - t/x,$$
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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$. Hence $R(X(z),X(z+\pi\tau)) = 0$ (assuming $|X(z)| \le 1$ and $|X(z+\pi\tau)| \le 1$).

Solution to Kreweras walks by winding number

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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}.$$

Then $K(X(z),X(z+\pi\tau)) = 0$. Hence $R(X(z),X(z+\pi\tau)) = 0$ (assuming $|X(z)| \leq 1$ and $|X(z+\pi\tau)| \leq 1$).
New equation to solve:

$$R(X(z),X(z+\pi\tau)) = 0,$$

Solution to Kreweras walks by winding number

Equation to solve:

$$K(x,y)Q(x,y) = R(x,y),$$

where

$$K(x,y) = 1 - txy - t/y - t/x,$$
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Define

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

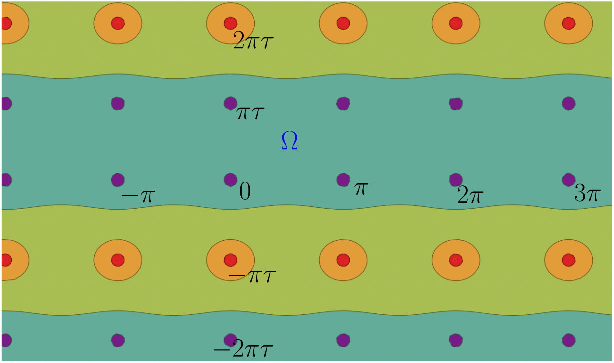
Then $K(X(z),X(z+\pi\tau)) = 0$. Hence $R(X(z),X(z+\pi\tau)) = 0$ (assuming $|X(z)| \leq 1$ and $|X(z+\pi\tau)| \leq 1$).

New equation to solve:

$$R(X(z),X(z+\pi\tau)) = 0,$$

Solution to Kreweras walks by winding number

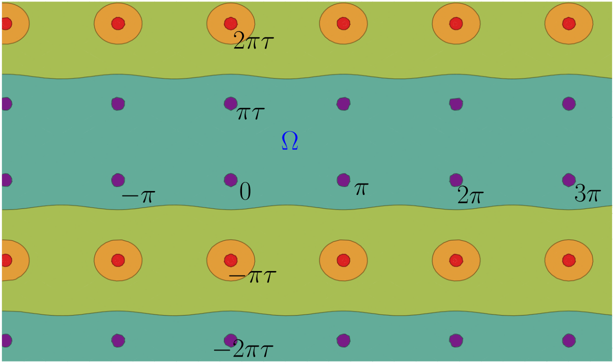
Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty]\right\}$.



For $z \in \Omega$, $\quad |X(z)| < 1 \Rightarrow Q(X(z),0)$ and $Q(0,X(z))$ are well defined.

Solution to Kreweras walks by winding number

Plot of $\left\{z : |X(z)| \in \left[0, \frac{1}{3}\right), \left(\frac{1}{3}, 1\right), (1,3), (3,9), (9,\infty]\right\}$.



For $z \in \Omega$, $|X(z)| < 1 \Rightarrow Q(X(z), 0)$ and $Q(0, X(z))$ are well defined.
Near $\mathrm{Re}(z) = 0$, we have $z \in \Omega$ and $z + \pi\tau \in \Omega$.

SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$R(X(z), X(z+\pi\tau)) = 0$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}.$$

$$R(x,y) = 1 - \frac{t}{x}Q(0,y) - \frac{t}{y}Q(x,0) + e^{i\alpha}tQ(0,x) + e^{-i\alpha}tyQ(y,0).$$

Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + \frac{t}{X(z + \pi\tau)} Q(X(z), 0) - e^{i\alpha} t Q(0, X(z)) - e^{-i\alpha} t X(z + \pi\tau) Q(X(z + \pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau)\, \vartheta(z - 2\pi\tau, 3\tau)}.$$

Equation to solve: (near $\mathrm{Re}(z)=0$)

$$1 = \frac{t}{X(z)}Q(0,X(z+\pi\tau)) + \frac{t}{X(z+\pi\tau)}Q(X(z),0) - e^{i\alpha}tQ(0,X(z)) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau),0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

For z near 0, define

$$L(z) = \frac{t}{X(z+\pi\tau)}Q(X(z),0) - e^{i\alpha}tQ(0,X(z)).$$

Both $L(z)$ and $L(z+\pi\tau)$ converge.

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + \frac{t}{X(z + \pi\tau)} Q(X(z), 0)$$
$$- e^{i\alpha} t Q(0, X(z)) - e^{-i\alpha} t X(z + \pi\tau) Q(X(z + \pi\tau), 0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}.$$

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Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z)=0$)

$$1 = \frac{t}{X(z)}Q(0,X(z+\pi\tau)) + L(z) - e^{-i\alpha}tX(z+\pi\tau)Q(X(z+\pi\tau),0),$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta(z-\pi\tau,3\tau)}{\vartheta(z+\pi\tau,3\tau)\vartheta(z-2\pi\tau,3\tau)}.$$

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SOLUTION TO KREWERAS WALKS BY WINDING NUMBER

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{t}{X(z)} Q(0, X(z + \pi\tau)) + L(z) - e^{-i\alpha} t X(z + \pi\tau) Q(X(z + \pi\tau), 0),$$

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$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta(z - \pi\tau, 3\tau)}{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - 2\pi\tau, 3\tau)}.$$

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Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

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We can solve this exactly:

$$L(z) = \frac{e^{3i\alpha}}{1 - e^{3i\alpha}} \left(1 - \frac{e^{i\alpha}}{X(z)} - e^{2i\alpha} X(z - \pi\tau) \right)$$
$$- \frac{e^{i\alpha + \frac{2i\pi\tau}{3}} \vartheta(\pi\tau, 3\tau) \vartheta'(0, \tau)}{(1 - e^{3i\alpha}) \vartheta(\frac{\alpha}{2} + \frac{\pi\tau}{3}, \tau) \vartheta'(0, 3\tau)} \frac{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - \frac{\alpha}{2} - \frac{\pi\tau}{3}, \tau)}{\vartheta(z, \tau) \vartheta(z, 3\tau)}$$

Solution to Kreweras walks by winding number

Equation to solve: (near $\mathrm{Re}(z) = 0$)

$$1 = \frac{e^{i\alpha}}{X(z)} L(z + \pi\tau) + L(z).$$

where

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}} \vartheta(z, 3\tau) \vartheta\left(z - \pi\tau, 3\tau\right)}{\vartheta\left(z + \pi\tau, 3\tau\right) \vartheta\left(z - 2\pi\tau, 3\tau\right)}.$$

We can solve this exactly:

$$L(z) = \frac{e^{3i\alpha}}{1 - e^{3i\alpha}} \left(1 - \frac{e^{i\alpha}}{X(z)} - e^{2i\alpha} X(z - \pi\tau)\right)$$
$$- \frac{e^{i\alpha + \frac{2i\pi\tau}{3}} \vartheta(\pi\tau, 3\tau) \vartheta'(0, \tau)}{(1 - e^{3i\alpha}) \vartheta(\frac{\alpha}{2} + \frac{\pi\tau}{3}, \tau) \vartheta'(0, 3\tau)} \frac{\vartheta(z + \pi\tau, 3\tau) \vartheta(z - \frac{\alpha}{2} - \frac{\pi\tau}{3}, \tau)}{\vartheta(z, \tau) \vartheta(z, 3\tau)}$$

We can extract $E(t, e^{i\alpha}) = Q(0, 0)$...

KREWERAS WALKS BY WINDING NUMBER: SOLUTION

Recall: τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}.$$

The gf $E(t,e^{i\alpha}) = Q(0,0) \equiv Q(t,\alpha,0,0)$ is given by:

$$E(t,e^{i\alpha}) = \frac{e^{i\alpha}}{t(1-e^{3i\alpha})}\left(e^{i\alpha} - e^{\frac{4\pi\tau i}{3}}\frac{\vartheta'(2\pi\tau,3\tau)}{\vartheta'(0,3\tau)} - e^{\frac{\pi\tau i}{3}}\frac{\vartheta(\pi\tau,3\tau)\vartheta'(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}{\vartheta'(0,3\tau)\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}\right).$$

KREWERAS WALKS BY WINDING NUMBER: SOLUTION

Recall: τ is determined by

$$t = e^{-\frac{\pi\tau i}{3}} \frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}.$$

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$$E(t,e^{i\alpha}) = \frac{e^{i\alpha}}{t(1-e^{3i\alpha})}\left(e^{i\alpha} - e^{\frac{4\pi\tau i}{3}}\frac{\vartheta'(2\pi\tau,3\tau)}{\vartheta'(0,3\tau)} - e^{\frac{\pi\tau i}{3}}\frac{\vartheta(\pi\tau,3\tau)\vartheta'(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}{\vartheta'(0,3\tau)\vartheta(\frac{\alpha}{2}-\frac{2\pi\tau}{3},\tau)}\right).$$

Equivalently:

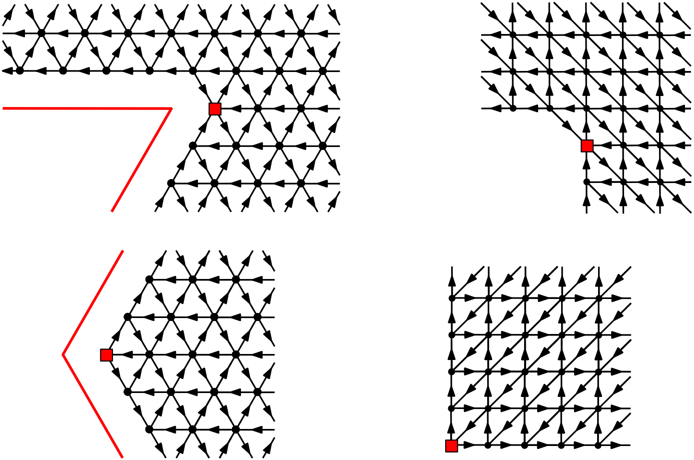
Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3)+6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

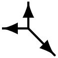
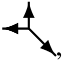
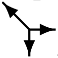
$$E(t,s) = \frac{s}{(1-s^3)t}\left(s - q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3}\frac{T_0(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T_0(sq^{-2/3},q)}\right).$$

Part 3: Walks in cones

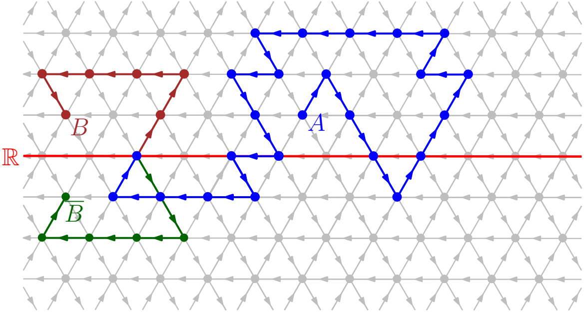


WALKS IN CONES WITH SMALL STEPS

- **Quarter plane walks:** Completely classified into rational, algebraic, D-finite, D-algebraic cases.
  [Mishna, Rechnitzer 09], [Bousquet-Mélou, Mishna 10], [Bostan, Kauers 10], [Fayolle, Raschel 10], [Kurkova, Raschel 12], [Melczer, Mishna 13], [Bostan, Raschel, Salvy 14], [Bernardi, Bousquet-Mélou, Raschel 17], [Dreyfus, Hardouin, Roques, Singer 18]
- **Walks avoiding a quadrant:**
  (Previously) solved in 5-10 of the 74 non-trivial cases
  [Bousquet-Mélou 16], [Raschel-Trotignon 19], [Budd 20], [Bousquet-Mélou, Wallner 20+]
- **Walks on the slit plane $\mathbb{C} \setminus \mathbb{R}_{<0}$:** solved exactly for simple walks [Bousquet-Mélou, Schaeffer, 02], but few other results.

New in this work: walks avoiding a quadrant with step set  and walks in the slit plane with step set ,  or .

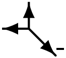
COUNTING KREWERAS WALKS IN A CONE

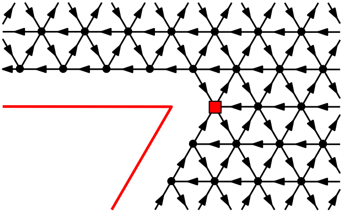


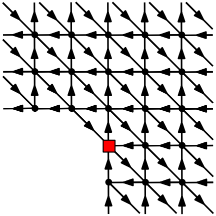
In the upper half plane: Use reflection principle

$$
\begin{aligned}
&\#(\text{Walks from } A \text{ to } B \text{ above } \mathbb{R}) \\
&= \#(\text{Walks from } A \text{ to } B) - \#(\text{Walks from } A \text{ to } B \text{ through } \mathbb{R}) \\
&= \#(\text{Walks from } A \text{ to } B) - \#(\text{Walks from } A \text{ to } \overline{B})
\end{aligned}
$$

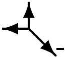
Counting Kreweras excursions in 5/6-plane

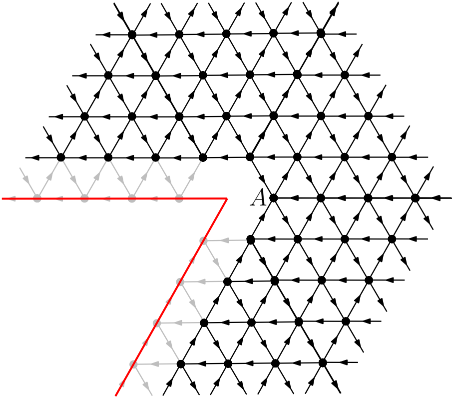
New model: -excursions avoiding a quadrant.



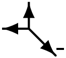


COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

New model: -excursions avoiding a quadrant.
First step: Transform to half plane



Counting Kreweras excursions in 5/6-plane

New model: -excursions avoiding a quadrant.

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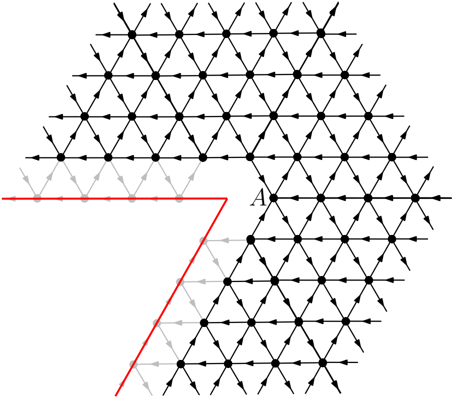

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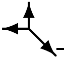
New model:  -excursions avoiding a quadrant.

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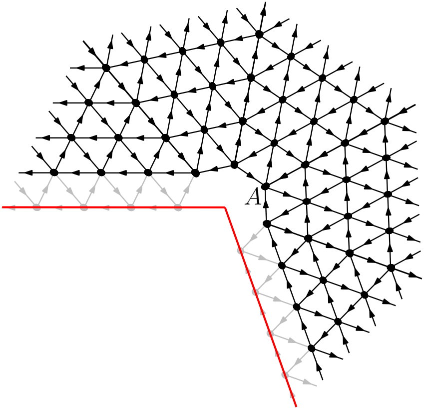
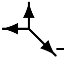
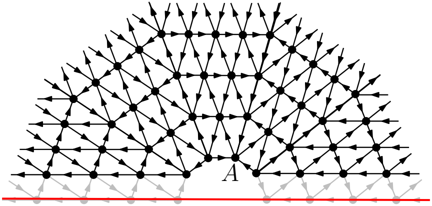
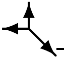


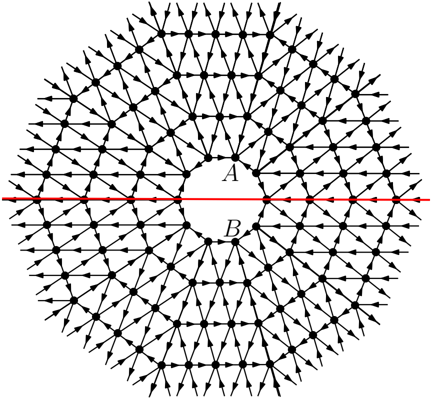
Counting Kreweras excursions in 5/6-plane

New model: -excursions avoiding a quadrant.
First step: Transform to half plane

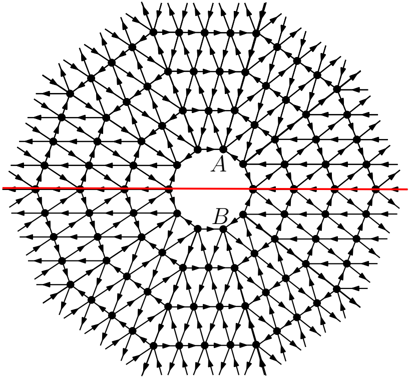


COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

New model: -excursions avoiding a quadrant.
First step: Transform to half plane → whole (punctured) plane



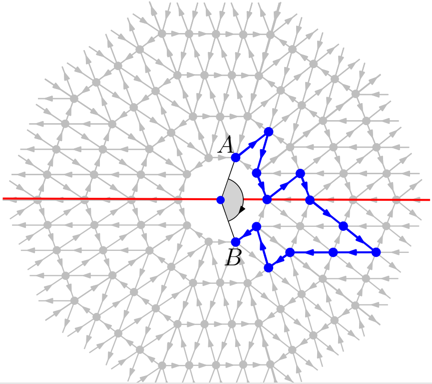
Counting Kreweras excursions in 5/6-plane

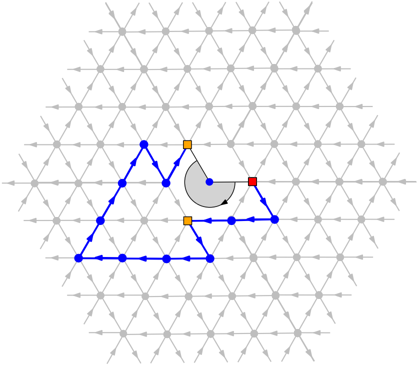


$$\#(\text{Kreweras excursions in } 5/6\text{-plane})$$
$$= \#(\text{Walks } A \to A \text{ in upper half plane})$$
$$= \#(\text{Walks } A \to A) - \#(\text{Walks } A \to B)$$

Counting Kreweras excursions in 5/6-plane

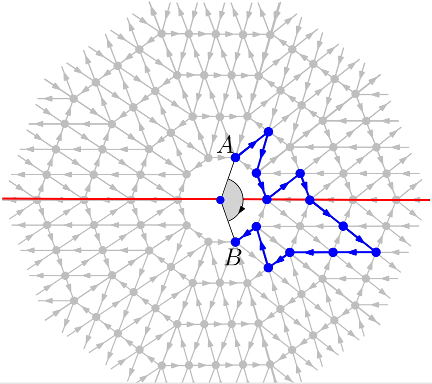
Walks $A \to B$ with winding angle β
$\equiv$ Kreweras almost-excursions with winding angle $\frac{5\beta}{3}$.

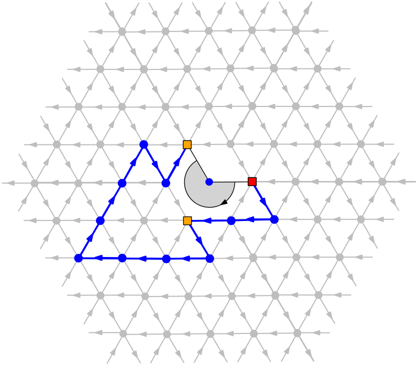




COUNTING KREWERAS EXCURSIONS IN 5/6-PLANE

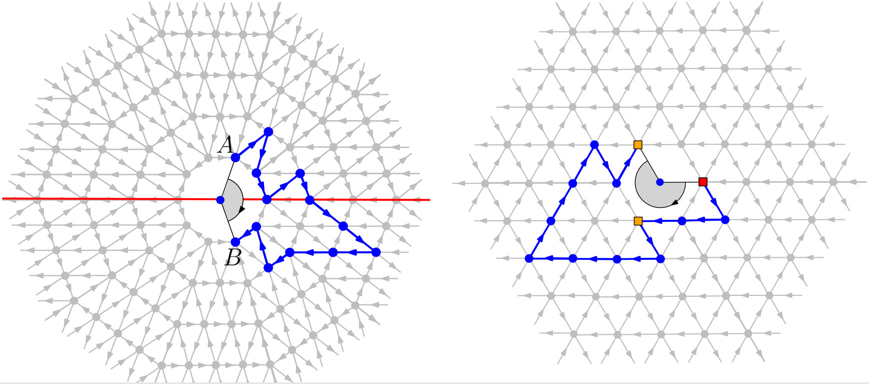
Walks $A \to B$ with winding angle $2\pi k - \frac{4\pi}{5}$
$\equiv$ Kreweras almost-excursions with winding angle $\frac{5}{3}\left(2\pi k - \frac{4\pi}{5}\right)$.





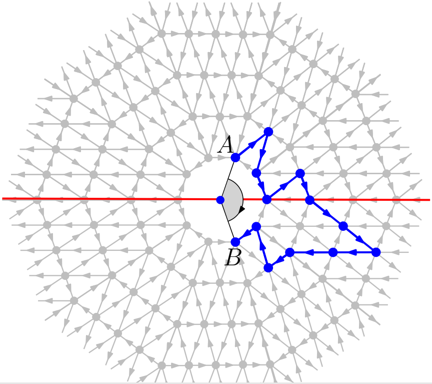
Counting Kreweras excursions in 5/6-plane

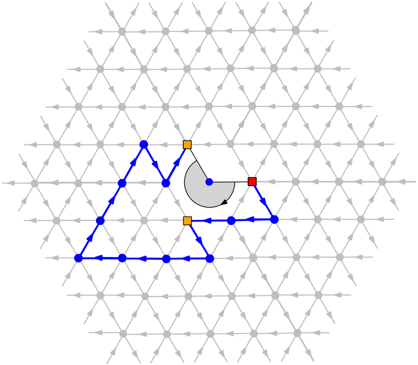
Walks $A \to B$ with winding angle $2\pi k - \frac{4\pi}{5}$
$\equiv$ Kreweras almost-excursions with winding angle $\frac{10\pi k}{3} - \frac{4\pi}{3}$.



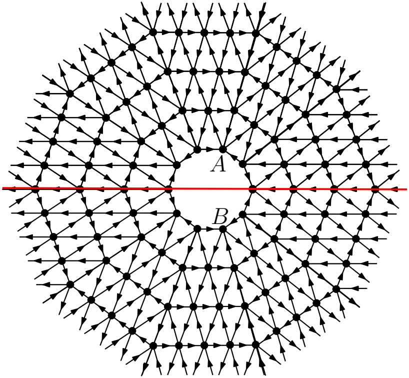
Counting Kreweras excursions in 5/6-plane

Walks $A \to B$ with winding angle $2\pi k - \frac{4\pi}{5}$
$\equiv$ Kreweras almost-excursions with winding angle $\frac{10\pi k}{3} - \frac{4\pi}{3}$.
Counted by: $s^{5k-2}\tilde{E}(t,s)$



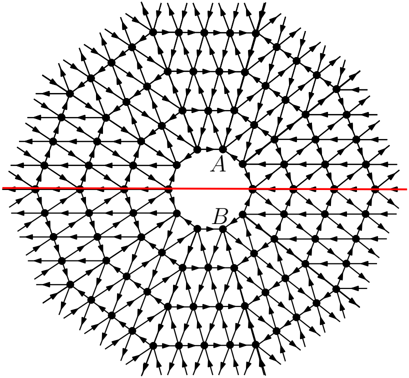


COUNTING KREWERAS EXCURSIONS IN $5/6$-PLANE



$$
\begin{aligned}
&\#(\text{Kreweras excursions in } 5/6\text{-plane}) \\
&= \#(\text{Walks } A \to A \text{ in upper half plane}) \\
&= \#(\text{Walks } A \to A) - \#(\text{Walks } A \to B) \\
&= \left(\sum_{k\in\mathbb{Z}} [s^{5k}]\tilde{E}(t,s)\right) - \left(\sum_{k\in\mathbb{Z}} [s^{5k-3}]\tilde{E}(t,s)\right) \\
&= \frac{1}{5}\sum_{j=1}^{4}\left(1 - e^{\frac{4\pi i j}{5}}\right)\tilde{E}\left(t, e^{\frac{2\pi i}{5}}\right)
\end{aligned}
$$

Counting Kreweras excursions in 5/6-plane



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\end{aligned}$$

More generally: The gf $C_{k,r}(t)$ for excursions in the $k/6$-plane is

$$C_{k,r}(t) = \frac{1}{k}\sum_{j=1}^{k-1}\left(1-e^{\frac{2\pi ijr}{k}}\right)\tilde{E}\left(t, e^{\frac{2\pi ij}{k}}\right).$$

Part 4: Analysis of solutions

Analysis of solution: Algebraicity

Recall: $\vartheta(z,\tau)$ is differentially algebraic $\rightarrow$ so are $\tilde{E}(t,s)$ and $Q(t,\alpha,x,y)$.

For $\alpha \in \frac{\pi}{3}(\mathbb{Q}\setminus\mathbb{Z})$ **we get algebraicity** (Ideas from [Zagier, 08] and [E.P., Zinn-Justin, 20+]):

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$$C_{k,r}(t) = \frac{1}{k}\sum_{j=1}^{k-1}\left(1 - e^{\frac{2\pi ijr}{k}}\right)\tilde{E}\left(t, e^{\frac{2\pi ij}{k}}\right).$$

ANALYSIS OF SOLUTION: ALGEBRAICITY

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Algebraic iff $3 \nmid k$. (always D-finite).

ASYMPTOTICS OF $\tilde{E}(t, e^{i\alpha})$ AND $C_{k,r}(t)$

Fix $\alpha \in (0, \pi) \setminus \{\frac{2\pi}{3}\}$.
Writing $\hat{\tau} = -\frac{1}{3\tau}$ and $\hat{q} = e^{2\pi i\hat{\tau}}$, the dominant singularity $t = 1/3$ corresponds to $\hat{q} = 0$.

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Series in $\hat{q}$:

$$t = \frac{1}{3} - 3\hat{q} + 18\hat{q}^2 + O(\hat{q}^3)$$

$$t\tilde{E}(t, e^{i\alpha}) = a_0 + a_1\hat{q} - \frac{27\alpha e^{i\alpha}}{2\pi(1 + e^{i\alpha} + e^{2i\alpha})}\hat{q}^{\frac{3\alpha}{2\pi}} + o\left(\hat{q}^{\frac{3\alpha}{2\pi}}\right),$$

$\rightarrow \tilde{E}(t, e^{i\alpha})$ as a series in $(1 - 3t)$,

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$$[t^n]\tilde{E}(t, e^{i\alpha}) \sim -\frac{3^{5-\frac{3\alpha}{\pi}} e^{\alpha i}\alpha}{2\pi(1 + e^{\alpha i} + e^{2\alpha i})\Gamma\left(-\frac{3\alpha}{2\pi}\right)} n^{-\frac{3\alpha}{2\pi}-1} 3^n,$$

$$[t^n]C_{k,r}(t) \sim -\frac{2 \cdot 3^{5-\frac{6}{k}} \sin^2\left(\frac{r\pi}{k}\right)}{\pi k^2 \left(1 + 2\cos\left(\frac{2\pi}{k}\right)\right)\Gamma\left(-\frac{3}{k}\right)} n^{-1-\frac{3}{k}} 3^n.$$

ASYMPTOTICS OF $\tilde{E}(t, e^{i\alpha})$ AND $C_{k,r}(t)$

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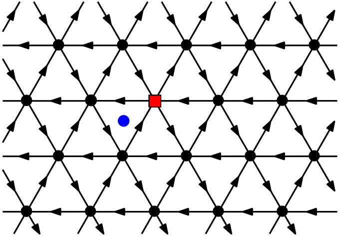
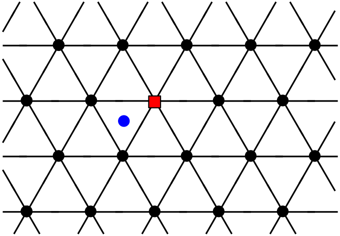
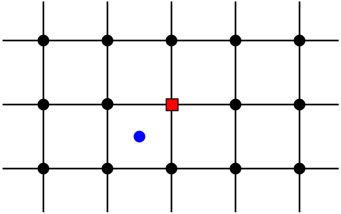
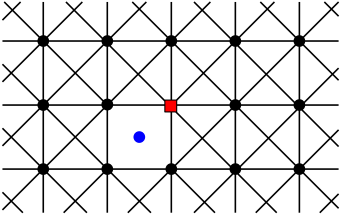
Alternatively: Terms 3^n and $n^{-1-\frac{3}{k}}$ known [Denisov, Wachtel, 2015].

Part 5: Other lattices



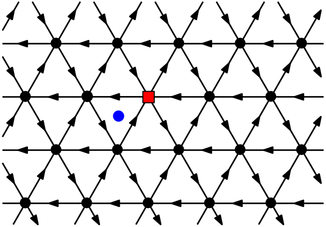




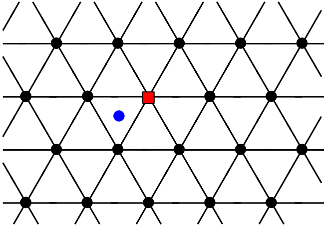


CELL-CENTRED LATTICES

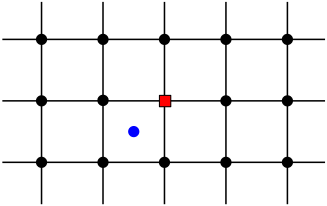
Important property: Decomposable into congruent sectors



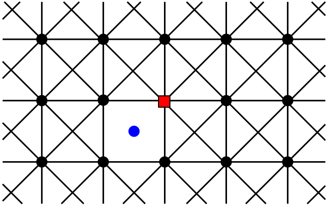
Kreweras lattice



Triangular Lattice



Square Lattice



King Lattice

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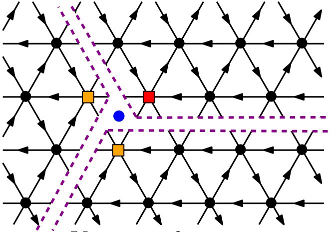
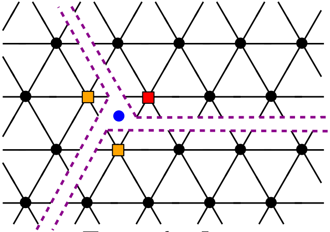
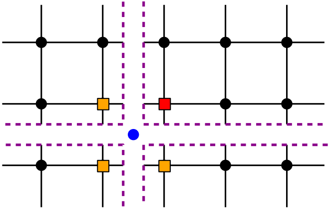


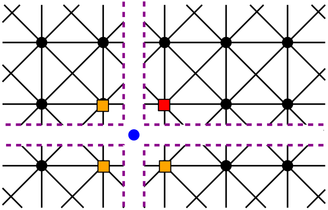
Kreweras lattice



Triangular Lattice



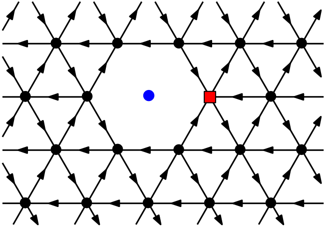
Square Lattice



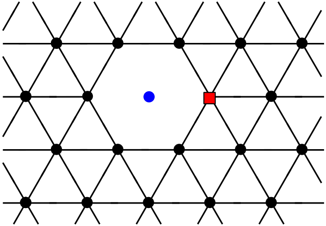
King Lattice

VERTEX-CENTRED LATTICES

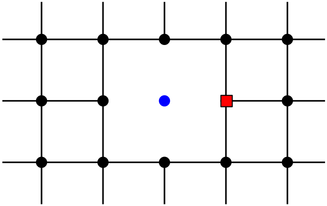
Decompose into rotationally congruent sectors



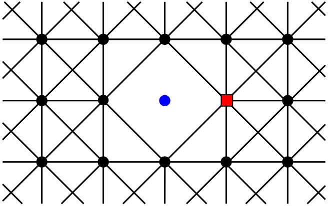
Kreweras lattice



Triangular Lattice



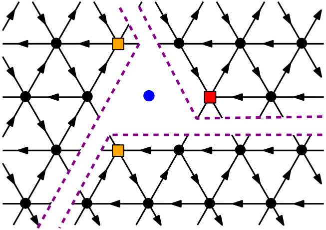
Square Lattice



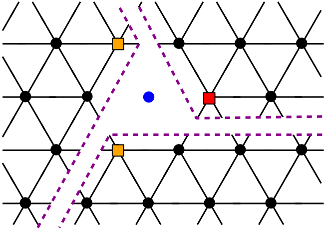
King Lattice

VERTEX-CENTRED LATTICES

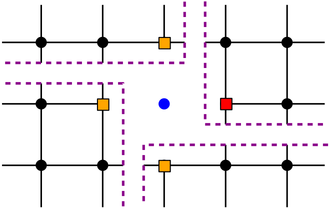
Decompose into rotationally congruent sectors



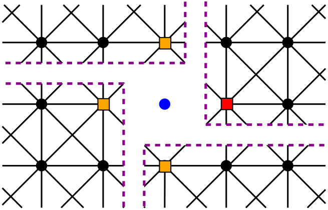
Kreweras lattice



Triangular Lattice



Square Lattice



King Lattice

Recall: Kreweras almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t^3 + 15t^6 + 279t^9 + \cdots$ satisfy

$$t = q^{1/3}\frac{T_1(1,q^3)}{4T_0(q,q^3) + 6T_1(q,q^3)}.$$

The gf for cell-centred Kreweras-lattice almost-excursions is:

$$E(t,s) = \frac{s}{(1-s^3)t}\left(s - q^{-1/3}\frac{T_1(q^2,q^3)}{T_1(1,q^3)} - q^{-1/3}\frac{T_0(q,q^3)T_1(sq^{-2/3},q)}{T_1(1,q^3)T_0(sq^{-2/3},q)}\right).$$

The gf for vertex-centred Kreweras-lattice almost-excursions is:

$$\tilde{E}(t,s) = \frac{s(1-s)q^{-\frac{2}{3}}}{t(1-s^3)}\frac{T_0(q,q^3)^2}{T_1(1,q^3)^2}\left(\frac{T_1(q,q^3)^2}{T_0(q,q^3)^2} - \frac{T_2(q,q^3)}{T_0(q,q^3)} - \frac{T_2(s,q)}{2T_0(s,q)} + \frac{T_3(1,q)}{6T_1(1,q)} + \frac{T_3(1,q^3)}{3T_1(1,q^3)}\right).$$

Square lattice almost-excursions

Define $T_k(u,q) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^k q^{n(n+1)/2}(u^{n+1} - (-1)^k u^{-n})$

$$= (u \pm 1) - 3^k q(u^2 \pm u^{-1}) + 5^k q^3(u^3 \pm u^{-2}) + O(q^6).$$

Let $q(t) \equiv q = t + 4t^3 + 34t^5 + 360t^7 + \cdots$ satisfy

$$t = \frac{qT_0(q^2,q^8)T_1(1,q^8)}{2T_0(q^4,q^8)(T_0(q^2,q^8) + 2T_1(q^2,q^8))}.$$

The gf for cell-centred Square-lattice almost-excursions is:

$$\frac{s^2}{(1-s^4)t}\left(s - s^{-1} + \frac{T_0(q^4,q^8)}{qT_1(1,q^8)} - \frac{T_0(q^4,q^8)T_1(s^{-1}q,q^2)}{qT_1(1,q^8)T_0(s^{-1}q,q^2)}\right).$$

The gf for vertex-centred Square-lattice almost-excursions is:

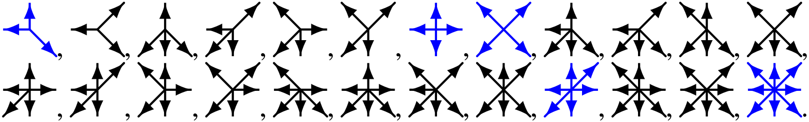
$$\frac{sT_0(q^4,q^8)}{qt(1+s^2)T_1(1,q^8)}\left(1 + \frac{2T_1(q^2,q^8)}{T_0(q^2,q^8)} + \frac{(1-s)T_1(s^{-1},q^2)}{(1+s)T_0(s^{-1},q^2)}\right).$$

Part 6: Final comments

OPEN QUESTION(S)

Choose any small step model and count walks by winding angle!

There are 24 distinct, non-trivial models:



In each case: Remove $(0, 0)$ and count excursions from some point $A \to A$ by length (t) winding angle (s).

- For which step sets (and which A) is this D-algebraic?
- For which step sets, which A and which fixed s is this Algebraic? D-finite?

Blue models solved (for some A)

FUNCTIONAL EQUATION THETA SOLUTION METHOD

Project: develop this method of solving functional equations.

Problems solved so far:

- Some walks by winding angle (this work)
- D-algebraic quadrant models [Bernardi, Bousquet-Mélou, Raschel 17]
- Six vertex model on planar maps [Kostov, 00], [E.P., Zinn-Justin, 20+], [Bousquet-Mélou, E.P., 20+].
- Properly q-coloured triangulations [E.P., 20+].

To do:

- Solve more problems.
- Streamline the method.
- Convert techniques to world of formal power series.
- find a good name for the method.

Thank you!

Bonus slide: Parameterization of $K(x,y)=0$

Write $K(x,y) = A(x)y^2 + B(x)y + C(x)$, then

$$Y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}$$

parameterizes $K(x, Y(x)) = 0$. Typically, $Y_+(x)$ is meromorphic on:

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

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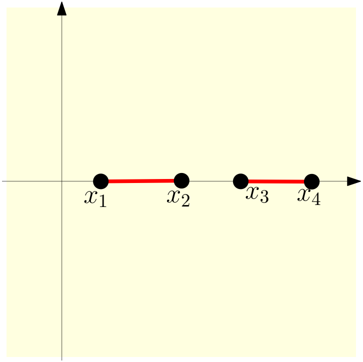

Bonus slide: Parameterization of $K(x,y)=0$

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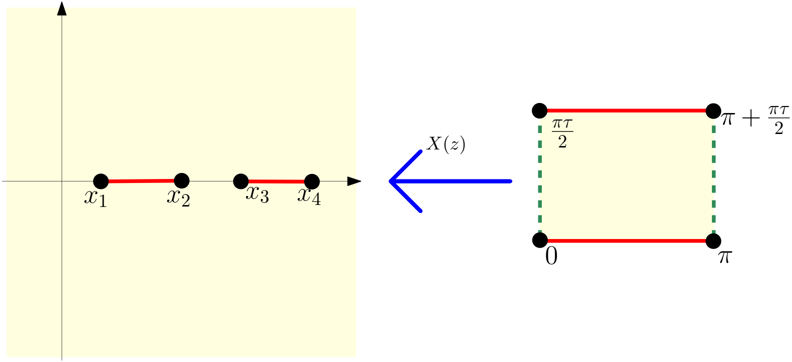
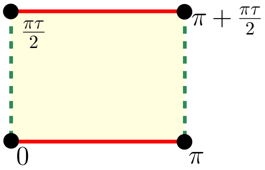


BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



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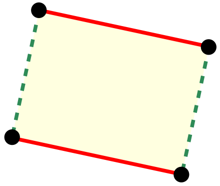
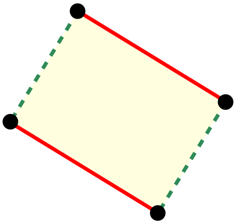


Bonus slide: Parameterization of $K(x, y) = 0$



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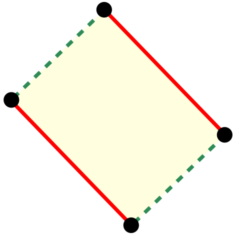

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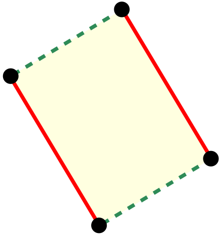
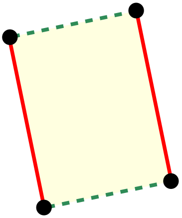


Bonus slide: Parameterization of $K(x, y) = 0$



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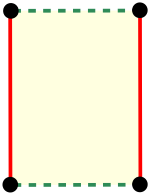
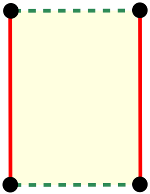


BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$



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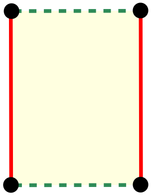

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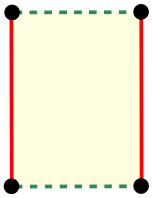

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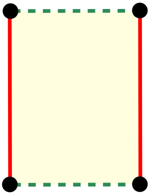

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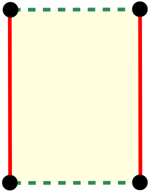

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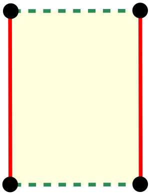

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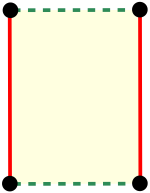

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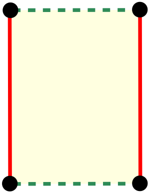
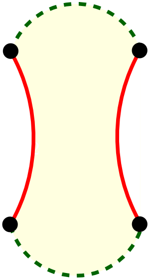


BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



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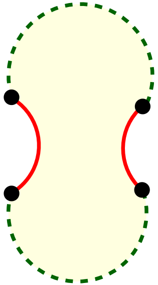

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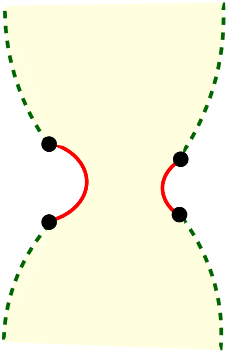

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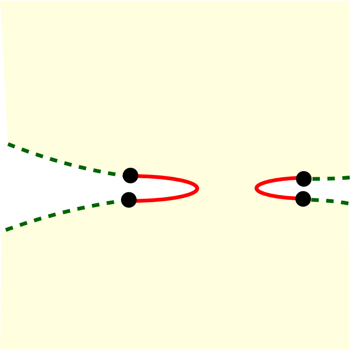
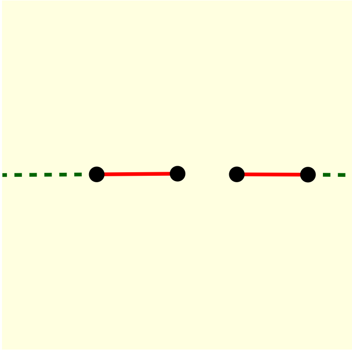


BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$



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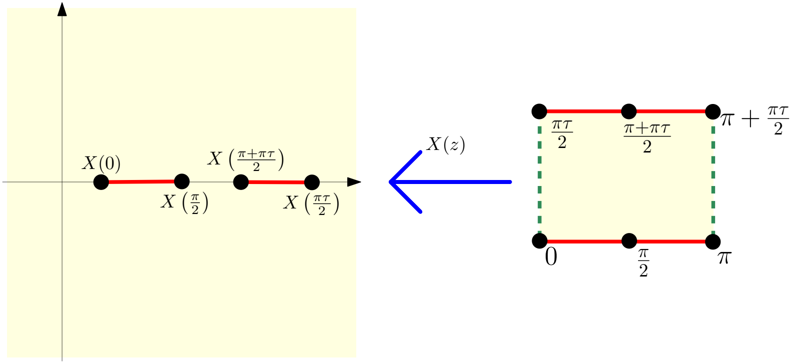

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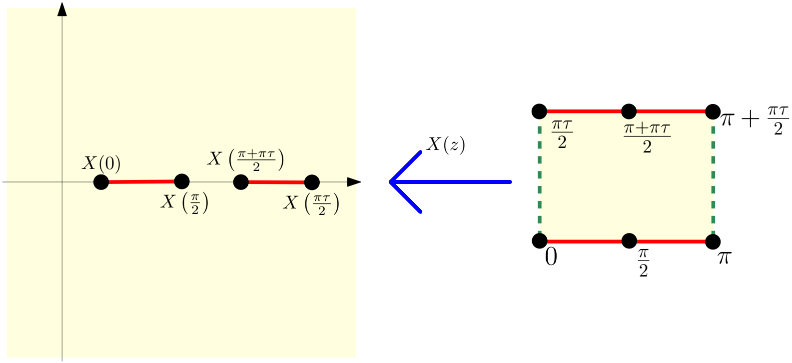
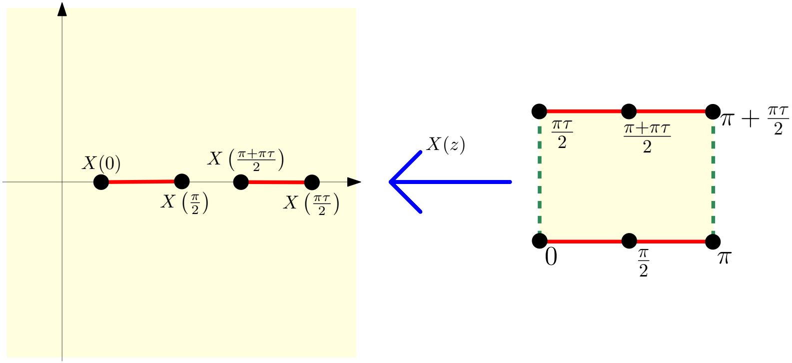


By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(\pi - r) = X(-r)$
- $X(\frac{\pi\tau}{2} + r) = X(\frac{\pi\tau}{2} - r)$

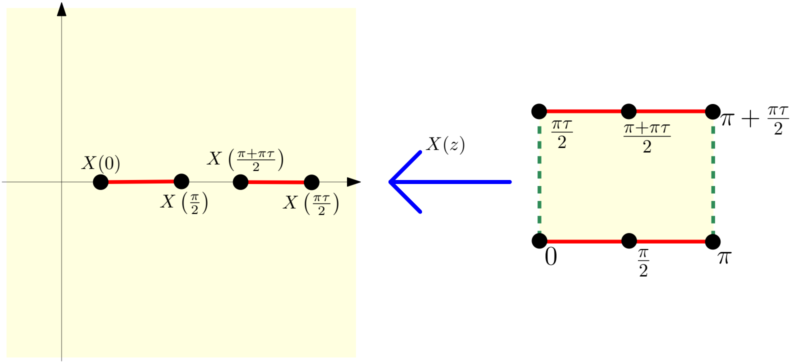
Bonus slide: Parameterization of $K(x,y)=0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z)$
- $X(z) = X(\pi\tau - z)$

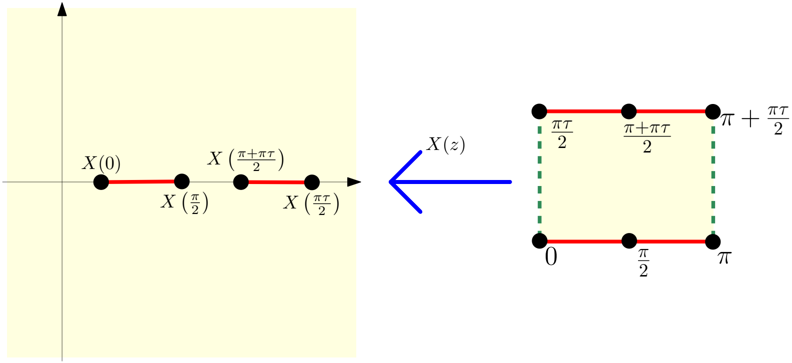
BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$
- $X(z) = X(\pi\tau - z)$

Bonus slide: Parameterization of $K(x, y) = 0$

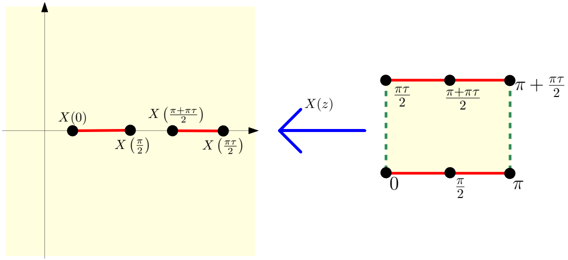


For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$

$$X(z) = c\frac{\vartheta(z-\alpha)\vartheta(z+\alpha)}{\vartheta(z-\beta)\vartheta(z+\beta)}$$

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



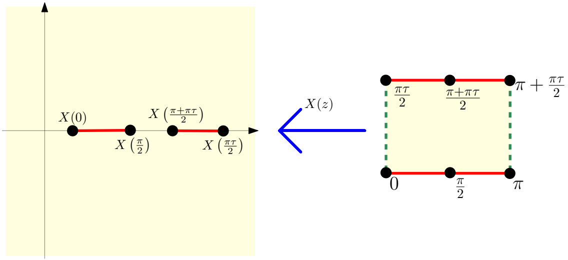
Recall:

$$y(x)=\frac{-B(x)\pm\sqrt{B(x)^2-4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z)=y(X(z))$. By symmetry, for $r\in\mathbb{R}$:

- $X(r)=X(-r)$, so $Y(r)+Y(-r)=-\frac{B(X(r))}{A(X(r))}$.
- Similarly, $Y\left(\frac{\pi\tau}{2}+r\right)+Y\left(\frac{\pi\tau}{2}-r\right)=-\frac{B\left(X\left(\frac{\pi\tau}{2}+r\right)\right)}{A\left(X\left(\frac{\pi\tau}{2}+r\right)\right)}$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



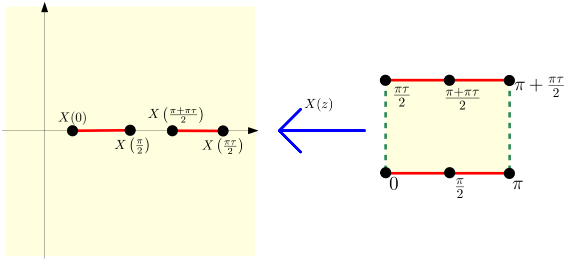
Recall:

$$y(x)=\frac{-B(x)\pm\sqrt{B(x)^2-4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z)=y(X(z))$. For $z\in\mathbb{C}$:

- $Y(z)+Y(-z)=-\frac{B(X(z))}{A(X(z))}$.
- $Y(z)+Y(\pi\tau-z)=-\frac{B(X(z))}{A(X(z))}$.

Bonus slide: Parameterization of $K(x,y)=0$



For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y(z) + Y(\pi\tau - z) = -\frac{B(X(z))}{A(X(z))}$.

So $Y(z) = Y(z+\pi\tau) = Y(z+\pi)$

$$\Rightarrow Y(z) = c\frac{\vartheta(z-\gamma)\vartheta(z-\delta)}{\vartheta(z-\epsilon)\vartheta(z-\gamma-\delta+\epsilon)}.$$

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

Equation characterising $Q(x, y) \equiv Q(t, x, y)$ for quadrant walks:

$$K(x, y)Q(x, y) + R(x, y) = 0.$$

$K(x, y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z - \alpha_1)\vartheta(z - \beta_1)}{\vartheta(z - \gamma_1)\vartheta(z - \delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z - \alpha_2)\vartheta(z - \beta_2)}{\vartheta(z - \gamma_2)\vartheta(z - \delta_2)},$$

where the constants satisfy $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.
So, $R(X(z), Y(z)) = 0$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

In general: $K(x, y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1+xytQ(x,y)+\frac{t}{x}\left(Q(x,y)-Q(0,y)\right)+\frac{t}{y}\left(Q(x,y)-Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

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with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.

Bonus slide: Parameterization of $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1+xytQ(x,y)+\frac{t}{x}\left(Q(x,y)-Q(0,y)\right)+\frac{t}{y}\left(Q(x,y)-Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.

- $K(0,0)=0$, so WLOG $\alpha_1=\alpha_2=0$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

For Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.
- as $x \to 0$, we have $y(x) \sim -x$ or $y(x) \sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z = \beta_1$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1+xytQ(x,y)+\frac{t}{x}\left(Q(x,y)-Q(0,y)\right)+\frac{t}{y}\left(Q(x,y)-Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2\frac{\vartheta(z)\vartheta(z-\beta_2)}{\vartheta(z-\beta_1)^2},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

- $K(0,0) = 0$, so WLOG $\alpha_1 = \alpha_2 = 0$.
- as $x \to 0$, we have $y(x) \sim -x$ or $y(x) \sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z = \beta_1$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

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Bonus slide: Parameterization of $K(x,y)=0$

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$$X(z) = c_1 \frac{\vartheta(z)\vartheta(z-\beta_1)}{\vartheta(z+\beta_1)\vartheta(z-2\beta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta(z-2\beta_1)}{\vartheta(z-\beta_1)^2},$$

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BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

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- So $3\beta_1 = \pi\tau$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

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Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z)\vartheta\left(z - \frac{\pi\tau}{3}\right)}{\vartheta\left(z + \frac{\pi\tau}{3}\right)\vartheta\left(z - \frac{2\pi\tau}{3}\right)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z)\vartheta\left(z - \frac{2\pi\tau}{3}\right)}{\vartheta\left(z - \frac{\pi\tau}{3}\right)^2},$$

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Bonus slide: Parameterization of $K(x,y)=0$

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Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1\frac{\vartheta(z)\vartheta\left(z-\frac{\pi\tau}{3}\right)}{\vartheta\left(z+\frac{\pi\tau}{3}\right)\vartheta\left(z-\frac{2\pi\tau}{3}\right)} \quad\text{and}\quad Y(z) = c_2\frac{\vartheta(z)\vartheta\left(z+\frac{\pi\tau}{3}\right)}{\vartheta\left(z-\frac{\pi\tau}{3}\right)\left(z+\frac{2\pi\tau}{3}\right)},$$

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BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

For Kreweras paths:

$$Q(x,y) = 1 + xytQ(x,y) + \frac{t}{x}\left(Q(x,y) - Q(0,y)\right) + \frac{t}{y}\left(Q(x,y) - Q(x,0)\right).$$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{9}}\vartheta(z)\vartheta\left(z-\frac{\pi\tau}{3}\right)}{\vartheta\left(z+\frac{\pi\tau}{3}\right)\vartheta\left(z-\frac{2\pi\tau}{3}\right)} \quad \text{and} \quad Y(z) = \frac{e^{-\frac{4\pi\tau i}{9}}\vartheta(z)\vartheta\left(z+\frac{\pi\tau}{3}\right)}{\vartheta\left(z-\frac{\pi\tau}{3}\right)\left(z+\frac{2\pi\tau}{3}\right)},$$

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- So $3\beta_1 = \pi\tau$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where

$$t = \frac{1}{X(z)Y(z) + X(z)^{-1} + Y(z)^{-1}}.$$

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Then $K(x,y)=xy-tx^2y^2-tx-ty=0$ is parameterised by

$$X(z)=\frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)}\quad\text{and}\quad Y(z)=X(z+\pi\tau),$$

where

$$t=e^{-\frac{\pi\tau i}{3}}\frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}.$$