Ising correlation $C(M, N)$ for $\nu = -k$

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Ising History

In 1976 Wu, McCoy, Tracy and Barouch used the concept of isomonodromic deformation to compute the scaling limit of the diagonal correlation function of the Ising model in terms of the solution of a Painlevé III equation.

In 1981 Jimbo and Miwa applied isomonodromic deformation theory to show that the diagonal correlation function satisfies a Painlevé VI equation.

It is therefore extremely natural to extend this program to the correlation function for an arbitrary position on the lattice. However, in the following 37 years this relation has not been discovered.

In this talk I will present recent progress.

Outline

1. Diagonal $C(N, N)$

   Jimbo-Miwa Painlevé VI.

2. Special case $\nu = -k$ for $C(M, N)$

   Toeplitz determinants for $C(0, N)$
   Difference equations for $C(M, N)$

3. Painlevé VI for $C(M, N)$ for $\nu = -k$

   Maple computations, Cosgrove,Okamoto

4. Toeplitz determinant comparisons

   Forrester-Witte
   Gamayun, Igorov and Lisovyy

5. Outstanding questions

1. Review of $C(N,N)$

The anisotropic Ising model on the square lattice is defined by

$$\mathcal{E} = -\sum_{j,k}\{E_v \sigma_{j,k}\sigma_{j+1.k} + E_h \sigma_{j.k}\sigma_{j.k+1}\}$$

and the diagonal correlation is

$$C(N,N) = \langle \sigma_{0,0}\sigma_{N,N}\rangle = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{in\theta}\left[\frac{1-\alpha e^{-i\theta}}{1-\alpha e^{i\theta}}\right]^{1/2}$$

$$\alpha = k = (\sinh 2E_v/kT \sinh 2E_h/kT)^{-1}$$

Singularities at T_c where

$$\sinh 2E_v/kT_c \sinh 2E_h/kT_c = 1$$

The Jimbo-Miwa Painlevé VI

$$\sigma = t(t-1)\frac{d}{dt}\ln C(N,N) - \frac{t}{4} \quad \text{with } t = k^2 \text{ for } T < T_c$$
$$\sigma = t(t-1)\frac{d}{dt}\ln C(N,N) - \frac{1}{4} \quad \text{with } t = k^{-2} \text{ for } T > T_c$$

and in both cases derived

$$\left(t(t-1)\frac{d^2\sigma}{dt^2}\right)^2 = N^2\left((t-1)\frac{d\sigma}{dt} - \sigma\right)^2 - 4\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt} - \sigma - \frac{1}{4}\right)\left(t\frac{d\sigma}{dt} - \sigma\right)$$

Boundary conditions at $t = 0$ are

$$C(N,N;t) = (1-t)^{1/4}\{1 + \lambda^2\frac{(1/2)_N(3/2)_N}{4[(N+1)!]^2}t^{N+1}(1+O(t))\} \text{ for } T < T_c$$

$$C(N,N;t) = (1-t)^{1/4}\{t^{N/2}\frac{(1/2)_N}{N!}\,{}_2F_1(\frac{1}{2}, N+\frac{1}{2}, N+1, t)$$
$$+\lambda^2\frac{(1/2)_N((3/2)_N)^2}{16(N+1)!(N+2)!}t^{3N/2+2}(1+O(t))\} \text{ for } T > T_c$$

with $\lambda = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$.

Comments

We note for both cases of $T < T_c$ and $T > T_c$ that there are solutions with boundary condition where $\lambda \neq 1$. Those solutions do not correspond to the determinants for $C(N, N)$ but rather for the lambda extended Fredholm determinants obtained from the form factor expansions. We also remark that for $T > T_c$ the term with $\lambda = 0$ is by itself an exact solution even though it is not a correlation function of the Ising model.

2. Special case $\nu = -k$ for $C(M,N)$

The general case $C(M,N)$ depends on the anisotropy

$$\nu = \frac{\sinh 2E_h/kT}{\sinh 2E_v/kT}$$

and moduli

$$k = (\sinh 2E_v/kT \sinh 2E_h/kT)^{-1} \;\; \text{for} \;\; T < T_c$$

$$k_> = \sinh 2E_v/kT \sinh 2E_h/kT \;\; \text{for} \;\; T > T_c$$

We will consider the special cases

$$\nu = -k \;\; \text{for} \;\; T < T_c$$

$$\nu = -k_> \;\; \text{for} \;\; T > T_c$$

To see why this case is special for $C(M,N)$ we first consider $C(0,1)$ and $C(0,N)$ separately.

$C(0,1)$

We define the complete elliptic integrals

$$\tilde{K}(k) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}} = {}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)$$

$$\tilde{E}(k) = \frac{2}{\pi}\int_0^{\pi/2} d\theta(1-k^2\sin^2\theta)^{1/2} = {}_2F_1(\frac{1}{2},-\frac{1}{2};1;k^2)$$

$$\tilde{\Pi}(-k\nu,k) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{(1+k\nu\sin^2\theta)(1-k^2\sin^2\theta)^{1/2}}$$

and recall the ancient results that for $T < T_c$

$$C(0,1) = \sqrt{1+\nu k}\{(1+k/\nu)\tilde{\Pi}(-\nu k,k) - (k/\nu)K(k)\}$$
$$= \sqrt{1+\nu k}\frac{2}{\pi}\int_0^{\pi/2} d\theta\frac{(1-k^2\sin^2\theta)^{1/2}}{1+k\nu\sin^2\theta}$$

and for $T > T_c$

$$C(0,1) = \frac{1}{\nu}\sqrt{1+\nu/k_>}\{(1+\nu k_>)\tilde{\Pi}(-\nu k_>,k_>) - \tilde{K}(k_>)\}$$
$$= k\sqrt{1+\nu/k_>}\frac{2}{\pi}\int_0^{\pi/2} d\theta\frac{1-\sin^2\theta}{(1+k_>\nu\sin^2\theta)(1-k_>^2\sin^2\theta)^{1/2}}$$

Specialize $\nu = -k$ **and** $\nu = -k_>$

For $T < T_c$ when $\nu = -k$

$$C(0,1) = \sqrt{1-k^2}\frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{1}{(1-k^2\sin^2\theta)^{1/2}}$$

$$= \sqrt{1-k^2}\tilde{K}(k)$$

and for $T > T_c$ when $\nu = -k_> = 1/k$

$$C(0,1) = 0$$

$C(O,N)$

The general row correlation is written as an $N \times N$ determinant

$$C(0,N) = \langle \sigma_{0,0}\sigma_{0,N}\rangle = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{in\theta}\left[\frac{(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})}{(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})}\right]^{1/2}$$

$$\alpha_1 = e^{-2E_v/kT}\tanh E_h/kT, \quad \alpha_2 = e^{-2E_v/kT}\coth E_h/kT$$

$$k = \frac{\alpha_2-\alpha_1}{1-\alpha_1\alpha_2} = (\sinh 2E_v kT \sinh 2E_h/kT)^{-1}$$

$$\nu = \frac{\sinh 2E_h/kT}{\sinh 2E_v/kT} = \frac{4\alpha_1\alpha_2}{(\alpha_2-\alpha_1)(1-\alpha_1\alpha_2)}$$

a_{2m} for $T < T_c$ and $\nu = -k$

Setting $\nu = -k$ we find $-\alpha_1 = \alpha_2 = \alpha$ and $k = \frac{2\alpha}{1+\alpha^2}$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{in\theta} \frac{1-\alpha^2+\alpha(e^{i\theta}-e^{-i\theta})}{\{(1-\alpha^2 e^{2i\theta})(1-\alpha^2 e^{-2i\theta})\}^{1/2}}$$

This has the symetry $a_{-n} = (-1)^n a_n$ and setting $2\theta = \phi$ we find with $m \geq 0$

$$a_{2m} = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{i|m|\phi} \frac{1-\alpha^2}{\{(1-\alpha^2 e^{i\phi})(1-\alpha^2 e^{-i\phi})\}^{1/2}}$$

$$a_{\pm(2|m|+1)} = \pm\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{i|m|\phi} \frac{\alpha(e^{i\phi}-1)}{\{(1-\alpha^2 e^{i\phi})(1-\alpha^2 e^{-i\phi})\}^{1/2}}$$

Reduce a_{2m} to a hypergeometric function

$$a_{2m} = (1-\alpha^2)\alpha^{2|m|}\frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!}\,{}_2F_1(|m|+\frac{1}{2},\frac{1}{2};|m|+1;\alpha^4)$$

To write in terms of k we use the quadratic transformation

$${}_2F_1(m+\frac{1}{2},m+\frac{1}{2},2m+1;k^2) = (1+\alpha^2)^{2m+1}\,{}_2F_1(m+\frac{1}{2},\frac{1}{2};m+1;\alpha^4)$$

$$a_{2m} = (k/2)^{2|m|}\sqrt{1-k^2}\frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!}\,{}_2F_1(|m|+1/2,|m|+1/2;2|m|+1;k^2)$$

a_{2m+1} for $T < T_c$ and $\nu = -k$

Similarly

$$a_{2m+1} = \left(\frac{k}{2}\right)^{2m+1} \frac{\Gamma(m+1/2)}{\pi^{1/2} m!} \{ \left(\frac{k}{2}\right)^{2} \frac{m+1/2}{m+1} {}_2F_1(m+\frac{3}{2}, m+\frac{3}{2}; 2m+3; k^2)$$
$$- {}_2F_1(m+\frac{1}{2}, m+\frac{1}{2}; 2m+1; k^2)\}$$

The two hypergeometric functions combine and thus

$$a_{\pm(2m+1)} = \mp \left(\frac{k}{2}\right)^{2|m|+1} \frac{\Gamma(|m|+1/2)}{\pi^{1/2}|m|!} {}_2F_1(|m|+\frac{1}{2}, |m|+\frac{1}{2}; 2|m|+2; k^2)$$

$C(0,N)$ for $T > T_c$ and $\nu = -k_>$

This case is more curious. Now we use $-\alpha_1 = \alpha_2^{-1} = \alpha$ and $k_> = \frac{2\alpha}{1+\alpha^2}$ and find

$$a_n = -\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{(n-1)i\theta}\left[\frac{1-\alpha^2 e^{2i\theta}}{1-\alpha^2 e^{-2i\theta}}\right]^{1/2}$$

By sending $\theta \to \theta + \pi$ we see that $a_n = (-1)^{n-1} a_n$ and thus $a_{2n} = 0$

$$a_{2n+1} = -\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{2ni\theta}\left[\frac{1-\alpha^2 e^{2i\theta}}{1-\alpha^2 e^{-2i\theta}}\right]^{1/2}$$
$$= -\frac{1}{2\pi}\int_0^{2\pi} d\phi e^{ni\phi}\left[\frac{1-\alpha^2 e^{i\phi}}{1-\alpha^2 e^{-i\phi}}\right]^{1/2}$$

which we recognize as the matrix elements a_{-n} of the diagonal correlation for $T < T_c$..

Factorization of $C(0,2N)$

It follows from $a_{2n}=0$ that $C(0,2N+1)=0$ and

$$C(0,2N)=\begin{vmatrix} a_{-1} & a_1 & \cdots & a_{2N-3} \\ a_{-3} & a_{-1} & \cdots & a_{2N-5} \\ \vdots & \vdots & & \vdots \\ a_{-(2N-1)} & a_{-(2N-3)} & \cdots & a_{-1} \end{vmatrix}$$

$$\times\begin{vmatrix} a_{1} & a_3 & \cdots & a_{2N-1} \\ a_{-1} & a_{1} & \cdots & a_{2N-3} \\ \vdots & \vdots & & \vdots \\ a_{-(2N-3)} & a_{-(2N-5)} & \cdots & a_{1} \end{vmatrix}$$

For example

$$\begin{aligned} C(0,2) &= k_>^{-2}\{\tilde{E}^2-(1-k_>^2)\tilde{K}^2\} \\ &= k_>^{-2}\{\tilde{E}-\sqrt{1-k_>^2}\,\tilde{K}\}\{\tilde{E}+\sqrt{1-k_>^2}\,\tilde{K}\} \end{aligned}$$

a_{2m+1} for $T > T_c$ and $\nu = -k_>$

Following the same reduction proceedure used for $T < T_c$ we find

$$a_{2n+1} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}\left(\frac{k_>}{2}\right)^{2(n-1)} 4(1-\sqrt{1-k_>^2})$$
$$\times\{{}_2F_1(n-\frac{1}{2},n+\frac{1}{2};2n+1;k_>^2)+\sqrt{1-k_>^2}\,{}_2F_1(n+\frac{1}{2},n+\frac{1}{2};2n+1;k_>^2)\}$$

and

$$a_{-(2n+1)} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!}\left(\frac{k_>}{2}\right)^{2(n-1)} 4(1+\sqrt{1-k_>^2})$$
$$\times\{{}_2F_1(n-\frac{1}{2},n+\frac{1}{2};2n+1;k_>^2)-\sqrt{1-k_>^2}\,{}_2F_1(n+\frac{1}{2},n+\frac{1}{2};2n+1;k_>^2)\}$$

The individual matrix elements contain the factor $1-\sqrt{1-k^2}$ but these factors cancel out in the expression for $C(0,2N)$ just as we saw for $C(0,2)$,

Quadratic difference equations for $C(M,N)$

$C(M,N)$ with $N > N$ can be written as an $N \times N$ determinant which is NOT Toeplitz. We will not use this but instead use quadratic difference equations relate the (high-temperature) correlation functions $C(M,N)$ for $T > T_c$ to the *dual correlation* $C_d(M,N)$ for $T > T_c$. defined as the low temperature correlation with the replacement: $s_v \longrightarrow \frac{1}{s_h}$ and$s_h \longrightarrow \frac{1}{s_v}$

$$s_h^2 \cdot [C_d(M,N)^2 - C_d(M,N-1) \cdot C_d(M.N+1)]$$
$$+[C(M,N)^2 - C(M-1,N) \cdot C(M+1,N)] = 0,$$
$$s_v^2 \cdot [C_d(M,N)^2 - C_d(M-1,N) \cdot C_d(M+1,N)]$$
$$+[C(M,N)^2 - C(M,N-1) \cdot C(M,N+1)] = 0$$
$$s_v s_h \cdot [C_d(M,N) \cdot C_d(M+1,N+1) - C_d(M,N+1) \cdot C_d(M+1,N)]$$
$$= C(M,N) \cdot C(M+1,N+1) - C(M,N+1) \cdot C(M+1,N),$$

which hold for all M and N, except $M = 0$, $N = 0$, where we have:

$$C(1,0) = (1+s_h^2)^{1/2} - s_h \cdot C_d(0,1),$$
$$C(0,1) = (1+s_v^2)^{1/2} - s_v \cdot C_d(1,0).$$

with $s_h = \sinh 2E_h/kT$ and $s_v = \sinh 2E_v/kT$

$C(1,2)$

For example for $T < T_c$ where $k = (s_v s_h)^{-1}$

$$C(1,2) = s_v^2 (s_v^{-2}+1)^{1/2} \Big(s_h^{-2}(s_v^{-2}s_h^{-2}-1)\tilde{K}^2 + (s_h^{-2}-1)\tilde{E}\tilde{K} + E^2$$
$$+(s_v^{-2}-1)(s_h^{-2}+1)\tilde{E}\tilde{\Pi} - (s_h^{-2}+1)(s_v^{-2}s_h^{-2}-1)\tilde{K}\tilde{\Pi}\Big).$$

and for $T > T_c$ where $k_> = s_v s_h$

$$C(1,2) = \frac{(s_v^2+1)^{1/2}}{s_h^2 s_v} \Big(\tilde{E}^2 - (s_h^2\, s_v^2 - 1)\tilde{K}^2 + (s_h^2\, s_v^2 + s_v^2 - 2)\tilde{E}\tilde{K}$$
$$(s_h^2+1)(s_v^2-1)\tilde{E}\tilde{\Pi} + (s_h^2\ +1)(s_h^2 s_v^2 - 1)\tilde{K}\tilde{\Pi}\Big).$$

For $T < T_c$ and $\nu = -k$ where $s_h = i,\ s_v = -i/k$

$$C(1,2) = -(1-k^2)^{1/2}k^{-2}\{(1-k^2)\tilde{K}^2 - 2\tilde{E}\tilde{K} + \tilde{E}^2\}$$

For $T > T_c$ and $\nu = -k_>$ where $s_h = -ik_>,\ s_v = i$

$$C(1,2) = 0$$

3. Painlevé VI for $\nu = -k$: Maple

I know of no reason why $C(M, N)$ at $\nu = -k$ must satisfy a nonlinear equation with the Painlevé property that the only singularites which depend on the boundary conditions are poles even though I do believe that there must be such an argument. Furthermore I have no idea how to analytically investigate this question.

However, recently a program called *guessfun* has been developed to search for nonlinear equations satsfied by long series expansions. My collaborators have used this program and find that indeed $C(M, N)$ at $\nu = -k$ does in fact satisfy a nonlinear equation.

Nonlinear equation for $C(M,N)$ with $T < T_c$

With $t = k^2$ and

$$\sigma = t(t-1)\frac{d\ln C(M,N)}{dt} - \frac{t}{4}$$

we have

$$[t(t-1)\sigma'']^2 + 4\{\sigma'(t\sigma'-\sigma)((t-1)\sigma'-\sigma)$$
$$-\frac{M^2}{4}(t\sigma'-\sigma)^2 - \frac{N^2}{4}\sigma'^2$$
$$+[\frac{M^2+N^2}{4} - \frac{1}{8}(1+(-1)^{M+N})]\sigma'(t\sigma'-\sigma)\} = 0$$

When $M = N$ this reduces to the Jimbo-Miwa equation for the diagonal correlation $C(N,N)$ for $T < T_c$.

Nonlinear equation for $C(M,N)$ with $T > T_c$

For $M+N$ odd $C(M,N)=0$

For $M+N$ even, $t=k^2$ and

$$\sigma = t(t-1)\frac{d \ln C(M,N)}{dt} - \frac{1}{4}$$

we have

$$\begin{aligned}
&[t(t-1)\sigma'']^2 + 4\{\sigma'(t\sigma'-\sigma)((t-1)\sigma'-\sigma)\} \\
&-M^2(t\sigma'-\sigma)^2 - (N^2+M^2-1)\sigma'(t\sigma'-\sigma) \\
&-N^2\sigma'^2 - \frac{1}{4}(N^2-M^2)\sigma'(t\sigma'-\sigma) \\
&-\frac{1}{4}(N^2-M^2)\sigma' - \frac{1}{16}(N^2-M^2) = 0
\end{aligned}$$

Second order second degree equations

The search for nonlinear equations with the Painlevé property is an ongoing field of research and is far from complete even for equations of second order. However for equations of the form $(y'')^2 = F(y, y', x)$ a solution was given by Cosgrove

$$\begin{aligned}
&(c_1x^3 + c_2x^2 + c_3x + c_4)^2(y'')^2 \\
&\quad = -4\{c_1(xy' - y)^3 + c_2y'(xy' - y)^2 \\
&\quad +c_3(y')^2(xy' - y) + c_4(y')^3 \\
&\quad +c_5(xy' - y)^2 + c_6y'(xy' - y) + c_7(y')^2 \\
&\quad +c_8(xy' - y) + c_9y' + c_{10}\}
\end{aligned}$$

This equation is invariant under the 6 parameter group of transformations

$$\bar{x} = \frac{a_1x + a_2}{a_3x + a_4}, \quad \bar{y} = \frac{a_5y + a_6x + a_7}{a_3x + a_4}$$

with $a_1a_4 - a_2a_3 = 1$ and $a_5 \neq 0$.

Okamoto's Painlevé VI equation

The canonical form of Painlevé VI of Okamoto which depends on 4 parameters $n_1,\ n_2,\ n_3,\ n_4$

$$h'\{t(t-1)h''\}^2 + \{h'(2h-(2t-1)h') + n_1n_2n_3n_4\}^2$$
$$-(h'-n_1^2)(h'-n_2^2)(h'-n_3^2)(h'-n_4^2) = 0$$

which when expanded and cancelling the common factor of h' is of the Cosgrove form with

$$c_1 = c_4 = c_5 = c_6 = 0, \ \ c_2 = -c_3 = 1$$
$$c_7 = -(n_1^2 + n_2^2 + n_3^2 + n_4^2)/4,$$
$$c_8 = -n_1n_2n_3n_4$$
$$c_9 = -(n_1^2n_2^2 + n_1^2n_3^2 + n_1^2n_4^2 + n_2^2n_3^2 + n_2^2n_4^2 + n_3^2n_4^2 - 2n_1n_2n_3n_4)/4$$
$$c_{10} = -(n_1^2n_2^2n_3^2 + n_1^2n_2^2n_4^2 + n_1^2n_3^2n_4^2 + n_2^2n_3^2n_4^2)/4$$

Okamoto and isomonodromic deformation

Okamoto showed that the sigma form of Painlevé VI can be obtained from isomonodromic deformation of the 2×2 linear system with 4 singularities

$$\frac{dY(x)}{dx} = \{\frac{A_0}{x} + \frac{A_t}{x-t} + \frac{A_1}{x-1}\}Y(x)$$

and $A_\infty = -(A_0 + A_t + A_1)$
where $\mathrm{Tr}A_k = 0$ and $\pm\theta_k$ are the eigenvalues of the residue matrices A_k of the linear system. The relation between the n_k and the θ_k is

$$n_1 = \theta_t + \theta_\infty,\ n_2 = \theta_t - \theta_\infty,\ n_3 = \theta_0 + \theta_1,\ n_4 = \theta_0 - \theta_1$$

We note, however, that because the sigma equation is invariant under permutations of n_k and the change of sign of any pair of n_k that there are several different sets of θ_k which lead to the same sigma equation.

Reduction to the Okomoto form $T < T_c$

Our equation is of the Cosgrove form with

$$c_1 = c_4 = 0, \;\; c_2 = -c_3 = 1$$
$$c_5 = -\frac{M^2}{4}, \; c_6 = \frac{M^2+N^2}{4} - \frac{1}{8}(1+(-1)^{M+N}), \; c_7 = -\frac{N^2}{4}$$
$$c_8 = c_9 = c_{10} = 0$$

To reduce our equation to the Okomoto form we need to find a linear shift $\sigma = h + At + B$ such that $c_5 = c_6 = 0$. This happens for
$A = M^2/4$ and $B = (N^2 - M^2)/8 - (1 + (-1)^{M+N})/16$
where

$$c_7 = -\frac{1}{8}\{N^2 + M^2 + \frac{1}{2}(1+(-1)^{M+N})\}$$
$$c_8 = \frac{1}{16}M^2\{N^2 - \frac{1}{2}(1+(-1)^{M+N})\}$$
$$c_9 = -\frac{1}{64}\{N^2 - M^2 - \frac{1}{2}(1+(-1)^{M+N})\}^2 - \frac{1}{8}N^2M^2$$
$$c_{10} = \frac{M^2}{16}B\{N^2 - \frac{1}{2}(1+(-1)^{M+N})\} - \frac{N^2M^4}{64}$$

The Okamoto and θ_k parameters

From these expressions for $c_7,\ c_8,\ c_9,\ c_{10}$ we obtain the following sets of Okamoto parameters for $C(M,N)$ with $T < T_c$ for $\nu = -k$ (unique up to permutations and the change of any two signs)

For $M+N$ odd
$n_1 = n_2 = \frac{N}{2},\ \ n_3 = -n_4 = \frac{M}{2}$

and for $M+N$ even
$n_1 = \frac{N+1}{2}\ \ \ n_2 = \frac{N-1}{2},\ \ n_3 = -n_4 = \frac{M}{2}$

and thus one set of θ_k for $M+N$ even is
$(\theta_0, \theta_t, \theta_1, \theta_\infty) = (0, N/2, M/2, 1/2)$

and for $M+N$ odd
$(\theta_0, \theta_t, \theta_1, \theta_\infty) = (0, N/2, M/2, 0)$

Note, however, if we permute $n_1 \leftrightarrow n_3$ that there is an equivalent set of θ_k for $M+N$ even
$(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{4}(N-M+1, N+M+1, N+M-1, M-N+1)$

and for $N+M$ odd
$(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{4}(N-M, N+M, N+M-1, M-N+1)$

4. The Forrester-Witte determinants

Forrester and Witte showed that the determinants

$$D_N^{(p,p',\eta,\xi)}(t) = \det\left[A_{j-k}^{(p,p',\eta,\xi)}(t)\right]_{j,k=0}^{N-1}$$

with

$$A_m^{(p,p',\eta,\xi)}(t) = \frac{\Gamma(1+p')t^{(\eta-m)/2}}{\Gamma(1+\eta-m)\Gamma(1-\eta+m+p')}\,{}_2F_1\left[\begin{matrix}-p, -p'+\eta-m \\ 1+\eta-m\end{matrix}, t\right]$$
$$+\frac{\xi\Gamma(1+p)t^{(m-\eta)/2}}{\Gamma(1-\eta+m)\Gamma(1+\eta-m+p)}\,{}_2F_1\left[\begin{matrix}-p', -p-\eta+m \\ 1-\eta+m\end{matrix}, t\right]$$

has the property that

$$\sigma = t(t-1)\frac{d}{dt}\ln(t^{\frac{\theta_0^2+\theta_t^2-\theta_1^2-\theta_\infty^2}{2}}(1-t)^{\frac{\theta_t^2+\theta_1^2-\theta_0^2-\theta_\infty^2+4\theta_1\theta_t}{2}}D_N)$$

with $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{2}(\eta, N, -N-p-p', p-p'+\eta)$

satisfies the Okamoto PVI equation with

$$n_1 = (N+p-p'+\eta)/2,\; n_2 = (N-p+p'-\eta)/2,$$
$$n_3 = (\eta-N-p-p')/2,\; n_4 = (\eta+N+p+p')/2$$

Comparison

Thus we see that for $M + N$ even the parameters n_k of $T < T_c$ of $C(M, N)$ for $\nu = -k$ agree with the parameters n_k of Forrester-Witte if

$$\eta = 0, \ \ p = \frac{M - N + 1}{2}, \ \ p' = \frac{M - N - 1}{2}$$

For $M + N$ odd the parameters n_k of $T < T_c$ of $C(M, N)$ agree with the parameters n_k of Forrester-Witte if

$$\eta = 0, \ \ p = p' = \frac{M - N}{2}$$

We note that for both $M + N$ even and odd that both p and p' are half an odd integer and for $N > M$ both p and p' are negative.
For $\eta = 0$ and $\xi = 0$ the determinants of Forrester and Witte reduce to Toeplitz determinants with the generating function
$C(\phi) = (1 - ke^{i\phi})^p(1 - ke^{-i\phi})^{p'}$
When $N = M$ then $p = 1/2$ and $p' = -1/2$ which is the generating function for $C(N, N)$.

5. Outstanding questions

1. How can we directly show that $C(M,N)$ for $T < T_c$ and $\nu = -k$ can be derived from isomonodromic deformation theory?

2.The comparison of the computer computation of the nonlinear equation with the nonlinear equation of the Forrester-Witte Toeplitz determinants reveals a striking equality which requires an explanation. Do ALL $C(M,N)$ with $\nu = -k$ for $T < T_c$ and $N > M$ have representations as $N \times N$ Toeplitz determinants even for $M \neq 0$ where no such representation has yet been found?

3. What are the implications of the factorization of $C(M,N)$ for $T > T_c$ at $\nu = -k_>$ and the quadratic transformation between the k and the α variables?

4. Why are $M + N$ even and odd different? The two cases have been recognized since the work of Ghosh and Shrock but is certainly not apparent in the Toeplitz determinant for $C(0,N)$.

5. What is the relation of the bordered determinants for $C(N, N+1)$ of Au-Yang and Perk to the PVI for $\nu = -k$?

6. How can $C(M,N)$ the general case be formulated as an isomonodromic deformation problem in two variables? Even for $C(0,N)$ this does not seem to be known.

7. Why do two different sets of θ_k give the same correlation function?