Diagonals of Rational Functions

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Diagonals

Given a rational function in n variables

$$R(x_1, \dots, x_n) = \frac{P(x_1, \dots, x_n)}{Q(x_1, \dots, x_n)},$$

where $P, Q \in \mathbb{Q}[x_1, \dots, x_n]$ such that $Q(0, \dots, 0) \neq 0$.

Definition: The diagonal of R is defined through its multi-Taylor expansion around $(0, \dots, 0)$:

$$R(x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} r_{m_1,\dots,m_n} \cdot x_1^{m_1} \cdots x_n^{m_n},$$

as the series in one variable x:

$$\mathrm{Diag}\big(R(x_1, \dots, x_n)\big) := \sum_{m=0}^{\infty} r_{m,m,\dots,m} \cdot x^m.$$

Example of a Diagonal

Consider the Taylor expansion of the bivariate rational function

$$
\begin{aligned}
f(x,y) &= \frac{1}{1-x-y-2xy} \\
&= 1 + x + y + x^2 + 4xy + y^2 + x^3 + 7x^2y + 7xy^2 + \dots
\end{aligned}
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Then the diagonal of f is

$$\mathrm{Diag}(f) = 1 + 4x + 22x^2 + 136x^3 + 886x^4 + 5944x^5 + \dots$$

Properties of Diagonals

The diagonal $f(x)$ of every rational function has the properties:

- **globally bounded:** there exist integers $c, d \in \mathbb{N}^*$, such that $df(cx) \in \mathbb{Z}[[x]]$, and $f(x)$ has nonzero radius of convergence.

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- This conjecture was first formulated in a paper in 1986 and is still widely open.
- It doesn't say anything about the number of variables in the rational function.
- One needs at least three variables, but no explicit example requiring more than three variables is known.

Christol's Conjecture

PROPOSITION : *Toute diagonale de fraction rationnelle f satisfait les propriétés suivantes :*

a) Elle est solution d'une équation différentielle linéaire L à coefficients dans $\mathbb{Q}[\lambda]$.

a') Cette équation différentielle est une équation de Picard Fuchs.

b) Pour toute place p (finie ou non) de $\mathbb{Q}$, le rayon de convergence $r_p(f)$ de la série f dans le corps $\mathbb{C}_p$ est non nul.

c) Pour presque toute place p de $\mathbb{Q}$, on a $r_p(f) = 1$.

c') Pour presque toute place p de $\mathbb{Q}$, la fonction f est bornée dans le disque $D_p(0,1) = \{x \in \mathbb{C}_p ; |x|<1\}$.

c") Pour presque toute place p de $\mathbb{Q}$,on a :

$\|f\|_p = \sup_{x \in D_p(0,1)} |f(x)| = 1.$

Seules les propriétés *a)* et *a')* ne sont pas immédiates. On en trouvera une démonstration dans [1] .

Dans cet article nous nous proposons de tester la conjecture suivante sur les fonctions hypergéométriques ${}_sF_{s-1}$:

CONJECTURE : *Une série entière f qui vérifie les propriétés a), b), c), c') et c") est la diagonale d'une fraction rationnelle.*

Hadamard Product

Definition: The Hadamard product of two series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \cdot x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} \beta_n \cdot x^n$$

is given by

$$f(x) \star g(x) = \sum_{n=0}^{\infty} \alpha_n \cdot \beta_n \cdot x^n.$$

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Note: Diagonals are closed under the Hadamard product, i.e., if two series are diagonals of rational functions, their Hadamard product is also a diagonal of a rational function.

Hypergeometric Series

Definition: Let $(a)_k := a(a+1)\cdots(a+k-1)$. Then

$$ {}_pF_q\big([a_1,\ldots,a_p],[b_1,\ldots,b_q],x\big) := \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\cdot\frac{x^k}{k!} $$

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Note: Any such hypergeometric function is D-finite, for example: the classical Gauß hypergeometric ${}_2F_1([a,b],[c],x)$ function satisfies Euler's differential equation:

$$ x(x-1)\,y''(x) + ((a+b+1)x-c)\,y'(x) + ab\,y(x) = 0. $$

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- If $q < p-1$ then the $_pF_q$ series has zero radius of convergence.
- If $q > p-1$ then the $_pF_q$ series cannot be globally bounded.

Confirmation

Certain classes of hypergeometric functions confirm Christol's conjecture.

Theorem (Christol): Let $f(x)$ be a hypergeometric series of the form

$$f(x) = {}_pF_{p-1}\big([a_1, \ldots, a_p], [b_1 \ldots, b_{p-1}], x\big)$$

of height

$$h = \big|\{1 \leqslant j \leqslant p \mid b_j \in \mathbb{Z}\}\big| - \big|\{1 \leqslant j \leqslant p \mid a_j \in \mathbb{Z}\}\big|$$

(where $b_p = 1$). If $f(x)$ can be written as the Hadamard product of h globally bounded series of height 1, then $f(x)$ is the diagonal of a rational function.

Example

The globally bounded hypergeometric series

$$f(x) = {}_3F_2\big(\big[\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\big], [1,1], x\big)$$

has height 3, and it can be written as the Hadamard product of three hypergeometric series of height 1:

$${}_1F_0\big(\big[\tfrac{1}{3}\big], [\,], x\big) \;\star\; {}_1F_0\big(\big[\tfrac{1}{3}\big], [\,], x\big) \;\star\; {}_1F_0\big(\big[\tfrac{1}{3}\big], [\,], x\big)$$

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By noting that ${}_1F_0([\frac{1}{3}], [\,], x) = (1-x)^{-1/3}$, we see that $f(x)$ is the diagonal of an algebraic function in three variables:

$$f(x) = \mathrm{Diag}\big((1-x)^{-1/3} \cdot (1-y)^{-1/3} \cdot (1-z)^{-1/3}\big).$$

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Theorem (Christol): A ${}_pF_{p-1}$ hypergeometric function of height 1 is globally bounded if and only if it is algebraic.

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- $c \in \mathbb{Q} \setminus \mathbb{Z}$: in this case the ${}_2F_1$ function is globally bounded if and only if it is algebraic.

Hence, this situation is not particularly interesting for our purposes.

Situation for 3F2 Functions

When is it easy to see that a globally bounded hypergeometric function ${}_3F_2([a,b,c],[d,e],x)$, $a,b,c \in \mathbb{Q} \setminus \mathbb{Z}$ is a diagonal of a rational function?

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Hence the interesting case occurs when only one of the two parameters d or e is rational, and the other is an integer.

But even in this case, a lot of the ${}_3F_2$ functions are easily seen to be diagonals of rational functions...

Situation for 3F2 Functions

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Then there are six ways to write this function as a Hadamard product of hypergeometric functions:

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- Then $f(x)$ is a diagonal if ${}_2F_1([c,1],[d],x)$ or ${}_2F_1([a,b],[d],x)$ is a diagonal (i.e. if one of them is algebraic).
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Thus if one of ${}_2F_1([a,b],[d],x)$, ${}_2F_1([b,c],[d],x)$, ${}_2F_1([a,c],[d],x)$ is algebraic, then $f(x)$ is the diagonal of a rational function.

Potential Counterexamples

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A longer list was generated by Christol and his co-authors in 2012.

- A. Bostan, S. Boukraa, G. Christol, S. Hassani, J-M. Maillard *Ising n-fold integrals as diagonals of rational functions and integrality of series expansions: integrality versus modularity.* Journal of Physics A: Mathematical and Theoretical **46**(18)

Potential Counterexamples

For example, these two hypergeometric functions are globally bounded, as they can be recast into series with integer coefficients:

$$ {}_3F_2\left(\left[\tfrac{2}{9}, \tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}, 1\right], 3^6 x\right) = 1 + 120x + 47124x^2 + 23483460x^3 + \dots $$

$$ {}_3F_2\left(\left[\tfrac{1}{9}, \tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}, 1\right], 3^6 x\right) = 1 + 84x + 32760x^2 + 16302000x^3 + \dots $$

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But they cannot be obtained as diagonals through Hadamard products, since the following series are not globally bounded:

$$ {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{5}{9}\right], \left[\tfrac{2}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{2}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{5}{9}, \tfrac{8}{9}\right], \left[\tfrac{2}{3}\right], x\right), $$

$$ {}_2F_1\left(\left[\tfrac{1}{9}, \tfrac{4}{9}\right], \left[\tfrac{1}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{4}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}\right], x\right), \quad {}_2F_1\left(\left[\tfrac{1}{9}, \tfrac{7}{9}\right], \left[\tfrac{1}{3}\right], x\right). $$

Not Globally Bounded

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$$ \ldots + \frac{2\cdot 11\cdot 20\cdots(9k-7)\cdot 5\cdot 14\cdot 23\cdots(9k-4)}{2\cdot 5\cdot 8\cdots(3k-1)\cdot 1\cdot 2\cdot 3\cdots k}\cdot\left(\frac{x}{27}\right)^k + \ldots $$

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There are infinitely many prime factors in the Taylor expansion, and therefore the function is not globally bounded.

Towards Christol

Theorem: The hypergeometric functions

$${}_3F_2\left(\left[\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}\right],\left[\tfrac{2}{3},1\right],27x\right) \quad \text{and} \quad {}_3F_2\left(\left[\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9}\right],\left[\tfrac{1}{3},1\right],27x\right)$$

are diagonals of rational functions.

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More precisely, we have:

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More generally, $\mathrm{Diag}\left(\frac{(1-x-y)^{a/b}}{1-x-y-z}\right)$ is shown to evaluate to

$${}_3F_2\left(\left[\frac{3a-b}{3a},\frac{2a-b}{3a},\frac{a-b}{3a}\right],\left[\frac{a-b}{a},1\right],27x\right).$$

Proof

The denominator of the algebraic function $\dfrac{(1-x-y)^{a/b}}{(1-x-y-z)}$ is expanded as a geometric series:

$$(1-x-y-z)^{-1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\binom{n}{m}\binom{m}{l}\cdot x^l y^{m-l} z^{n-m},$$

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while the numerator can be expanded as

$$\sum_{k=0}^{\infty}\frac{(-a/b)_k}{k!}\cdot(x+y)^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-a/b)_k}{k!}\cdot\binom{k}{j}x^j y^{k-j}.$$

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Multiplying these two sums and re-indexing, we obtain:

$$\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}x^s y^t z^u\sum_{j=0}^{s}\sum_{k=0}^{\infty}\frac{(-a/b)_k}{k!}\binom{k}{j}\binom{s+t+u-k}{s+t-k}\binom{s+t-k}{s-j}.$$

Proof

Hence the diagonal coefficient of $x^n y^n z^n$ is given by

$$\sum_{j=0}^{n} \sum_{k=0}^{\infty} \frac{(-a/b)_k}{k!} \cdot \binom{k}{j} \binom{3n-k}{2n-k} \binom{2n-k}{n-j},$$

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which by the Chu-Vandermonde identity

$$\binom{2n}{n}=\sum_{j=0}^{n}\binom{k}{j}\binom{2n-k}{n-j}$$

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Now use a computer algebra tool like Mathematica or Maple to simplify this sum further into a closed form...

Proof

More precisely, we employ Zeilberger's algorithm to find that

$$\binom{2n}{n} \cdot \sum_{k=0}^{2n} \frac{(-a/b)_k}{k!} \cdot \binom{3n-k}{2n-k} \quad =: S(n)$$

satisfies the first-order recurrence

$$\begin{aligned}&(a-3b-3bn)\cdot(a-2b-3bn)\cdot(a-b-3bn)\cdot S(n)\\&= b^2\cdot(n+1)^2\cdot(a-b-bn)\cdot S(n+1).\end{aligned}$$

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Together with the initial value $S(0)=1$, we get the closed form

$$S(n) = \frac{3^{3n}\cdot\left(\frac{b-a}{3b}\right)_n\cdot\left(\frac{2b-a}{3b}\right)_n\cdot\left(\frac{3b-a}{3b}\right)_n}{\left(\frac{b-a}{b}\right)_n\cdot\left(n!\right)^2},$$

yielding the hypergeom. function representation of the diagonal.

Diagonals as Integrals

Note that a diagonal $\mathrm{Diag}\big(R(x,y,z)\big)$ can also be expressed as

$$\langle y^0 z^0\rangle\, R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \operatorname{res}_{y,z}\frac{1}{yz}R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \oint \frac{1}{yz}R\Big(\frac{x}{y},\frac{y}{z},z\Big)\mathrm{d}y\,\mathrm{d}z.$$

where $\langle y^0 z^0\rangle$ denotes the constant coefficient w.r.t. y and z.

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Indeed, writing

$$R(x,y,z) = \sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{n\geqslant 0} r_{l,m,n}\, x^l y^m z^n$$

one obtains

$$R\Big(\frac{x}{y},\frac{y}{z},z\Big) = \sum_{l\geqslant 0}\sum_{m\geqslant 0}\sum_{n\geqslant 0} a_{l,m,n}\, x^l y^{m-l} z^{n-m}.$$

Proof by Creative Telescoping

Compute a linear differential operator that annihilates the diagonal of our algebraic function, by applying creative telescoping to

$$\oint \frac{1}{yz} R\Big(\frac{x}{y}, \frac{y}{z}, z\Big)\,\mathrm{d}y\,\mathrm{d}z = \oint \frac{(1 - x/y - y/z)^{a/b}}{yz - xz - y^2 - yz^2}\,\mathrm{d}y\,\mathrm{d}z$$

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We obtain the following telescoper of order three:

$$\begin{aligned} & b^3x^2(1-27x)\cdot D_x^3 + b^2x((27a-135b)\cdot x - a + 3b)\cdot D_x^2 \\ & - b\cdot((9a^2 - 63ab + 114b^2)\cdot x + ab - b^2)\cdot D_x \\ & + (a-3b)\cdot(a-2b)\cdot(a-b). \end{aligned}$$

Proof by Creative Telescoping

Compute a linear differential operator that annihilates the diagonal of our algebraic function, by applying creative telescoping to

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&+(a-3b)\cdot(a-2b)\cdot(a-b).
\end{aligned}$$

One of its solutions is the claimed ${}_3F_2$ hypergeometric function

$${}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big).$$

Software Demo

In[1]:=
```
<< RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

In[2]:=
```
alg = (1 - x - y) ^ (1 / 3) / (1 - x - y - z);
intg = ExpandAll[(alg /. {x → x / y, y → y / z}) / (y z)]
```

Out[3]= $\dfrac{\left(1-\frac{x}{y}-\frac{y}{z}\right)^{1/3}}{-y^2-x\,z+y\,z-y\,z^2}$

In[4]:=
```
CreativeTelescoping[intg, Der[y], {Der[x], Der[z]}][[1]]
```

Out[4]= $\{(144x^2z^2-72xz^3+9z^4+72xz^4-18z^5-36xz^5+9z^6)\,D_z^2+(-6x^2z-972x^3z-3xz^2+324x^2z^2-12xz^3-3xz^4)\,D_x+$
$(264x^2z-180xz^2-324x^2z^2+24z^3+366xz^3-66z^4-174xz^4+42z^5)\,D_z+(16x^2-46xz-540x^2z+6z^2+308xz^2-$
$(144x^2z-72xz^2+9z^3+72xz^3-18z^4-36xz^4+9z^5)\,D_x\,D_z+(24x^2-24xz+324x^2z+9z^2-6xz^2-27z^3-60xz^3+1$
$(48xz+6z^2+108xz^2-48z^3+6z^4)\,D_z+(8x+16z+180xz-74z^2+10z^3),\ (-144x^3+72x^2z-9xz^2-72x^2z^2+18$
$(-336x^2+138xz+108x^2z-9z^2-132xz^2+18z^3+48xz^3-9z^4)\,D_x+(-24xz+24z^2+36xz^2-30z^3+6z^4)\,D_z+(-64$

In[5]:=
```
CreativeTelescoping[%, Der[z]][[1]]
```

Out[5]= $\{(-27x^2+729x^3)\,D_x^3+(-72x+3402x^2)\,D_x^2+(-18+2538x)\,D_x+80\}$

In[6]:=
```
Annihilator[HypergeometricPFQ[{2 / 9, 5 / 9, 8 / 9}, {2 / 3, 1}, 27 x], Der[x]]
```

Out[6]= $\{(-27x^2+729x^3)\,D_x^3+(-72x+3402x^2)\,D_x^2+(-18+2538x)\,D_x+80\}$

From Algebraic to Rational

Denef and Lipshitz: For a given algebraic power series $f(x_1, \ldots, x_n)$ in n variables, construct a rational function $R(x_1, \ldots, x_{2n})$ in $2n$ variables such that

$$\operatorname{Diag}\big(R(x_1, \ldots, x_{2n})\big) = \operatorname{Diag}\big(f(x_1, \ldots, x_n)\big).$$

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Moreover, the "partial diagonal" of R, w.r.t. the pairs of variables

$$(x_1, x_{n+1}), \ldots, (x_{n-1}, x_{2n}),$$

yields the algebraic power series f.

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yields the algebraic power series f.

Example: We use the three-variable algebraic function

$$\begin{aligned} f(x,y,z) &= \frac{(1-x-y)^{1/3}}{1-x-y-z} \\ &= 1 + \tfrac{2}{3}x + \tfrac{2}{3}y + z + \tfrac{10}{9}xy + \tfrac{5}{3}xz + \tfrac{5}{3}yz + \tfrac{40}{9}xyz + \ldots \end{aligned}$$

Etale Extensions

The minimal polynomial of $f = \frac{(1-x-y)^{1/3}}{1-x-y-z}$ is given by

$$p(x, y, z, f) = \big((x + y + z - 1) \cdot f\big)^3 + 1 - x - y.$$

Denef and Lipshitz's theorem is formulated for étale extensions, which basically means that $\frac{\partial p}{\partial f}$ has a nonzero constant coefficient.

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Denef and Lipshitz's theorem is formulated for étale extensions, which basically means that $\frac{\partial p}{\partial f}$ has a nonzero constant coefficient.

By considering $\tilde{f} = f - 1$, i.e. by removing the constant term of f, we can achieve an étale extension. The minimal polynomial then reads

$$\tilde{p}(x, y, z, f) = \big((x + y + z - 1) \cdot (f + 1)\big)^3 + 1 - x - y.$$

and indeed, $\frac{\partial \tilde{p}}{\partial f}(0, 0, 0, 0) = -3 \neq 0$.

Special Diagonal

Now, the rational function

$$\tilde{r}(x,y,z,f) = f^2 \cdot \frac{\frac{\partial \tilde{p}}{\partial f}(xf,yf,zf,f)}{\tilde{p}(xf,yf,zf,f)}$$

has the property that $\mathcal{D}\big(\tilde{r}(x,y,z,f)\big) = \tilde{f}(x,y,z)$, where the operator $\mathcal{D}$ denotes a special kind of "diagonalization" with respect to the last variable:

$$\mathcal{D}\Big(\sum a_{i_1,\dots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n} y^j\Big) = \sum_{j=i_1+\cdots+i_n} a_{i_1,\dots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n}.$$

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Hence $\mathcal{D}\big(r(x,y,z,f)\big) = f(x,y,z)$ for $r(x,y,z,f) = \tilde{r}(x,y,z,f) + 1$.

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$$\tilde{r}(x,y,z,f) = f^2 \cdot \frac{\frac{\partial \tilde{p}}{\partial f}(xf,yf,zf,f)}{\tilde{p}(xf,yf,zf,f)}$$

has the property that $\mathcal{D}\big(\tilde{r}(x,y,z,f)\big) = \tilde{f}(x,y,z)$, where the operator $\mathcal{D}$ denotes a special kind of "diagonalization" with respect to the last variable:

$$\mathcal{D}\Big(\sum a_{i_1,\ldots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n} y^j\Big) = \sum_{j=i_1+\cdots+i_n} a_{i_1,\ldots,i_n,j} \cdot x_1^{i_1} \cdots x_n^{i_n}.$$

Hence $\mathcal{D}\big(r(x,y,z,f)\big) = f(x,y,z)$ for $r(x,y,z,f) = \tilde{r}(x,y,z,f) + 1$.

In our example we obtain:

$$r(x,y,z,f) = \frac{3f^2 \cdot (f+1)^2 \cdot (xf+yf+zf-1)^3}{(f+1)^3 \cdot (xf+yf+zf-1)^3 - xf - yf + 1} + 1.$$

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$$r_1(x, y, z, u_1, v_1) = \frac{u_1 \cdot r(x, y, z, u_1) - v_1 \cdot r(x, y, z, v_1)}{u_1 - v_1}$$

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Then r_2 is the desired rational function in six variables.

Final Result

The hypergeometric series

$$ {}_3F_2\Big(\Big[\frac{3a-b}{3a}, \frac{2a-b}{3a}, \frac{a-b}{3a}\Big], \Big[\frac{a-b}{a}, 1\Big], 27x\Big). $$

is the diagonal of the following rational function in the six variables x, y, z, u, v, w:

$$
\begin{aligned}
1 &+ \frac{au^3v\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-v)(v-w)} \\
&- \frac{av^4\,(1-vx-vy-vz)\,(1+v)^{a-1}(1-vx-vy-vz)^{a-1}}{(1+v)^a(1-vx-vy-vz)^a-(1-vx-vy)^b(u-v)(v-w)} \\
&- \frac{au^3w\,(1-ux-uy-uz)\,(1+u)^{a-1}(1-ux-uy-uz)^{a-1}}{(1+u)^a(1-ux-uy-uz)^a-(1-ux-uy)^b(u-w)(v-w)} \\
&- \frac{aw^4\,(1-wx-wy-wz)\,(1+w)^{a-1}(1-wx-wy-wz)^{a-1}}{(1+w)^a(1-wx-wy-wz)^a-(1-wx-wy)^b(u-w)(v-w)}
\end{aligned}
$$

Other Potential Counterexamples

Christol's original example:

$$ {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[\frac{1}{3}, 1\right], 27x\right) $$

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Note that our examples,

$$ {}_3F_2\left(\left[\tfrac{2}{9},\tfrac{5}{9},\tfrac{8}{9}\right],\left[\tfrac{2}{3},1\right],x\right) \quad \text{and} \quad {}_3F_2\left(\left[\tfrac{1}{9},\tfrac{4}{9},\tfrac{7}{9}\right],\left[\tfrac{1}{3},1\right],x\right), $$

have an arithmetic progression in the top parameters.

Integral Representation

Recalling the integral representation of the hypergeometric function

$$
{}_3F_2([a,b,c],[d,e],x) = \frac{\Gamma(d)\,\Gamma(e)}{\Gamma(a)\,\Gamma(b)\,\Gamma(d-a)\,\Gamma(e-b)} \times
$$

$$
\times \int_0^1 \int_0^1 y^{a-1} z^{b-1} (1-y)^{-a+d-1} (1-z)^{-b+e-1} (1-xyz)^{-c} \,\mathrm{d}y\,\mathrm{d}z
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For example, let

$$ A(x,y,z) = (1-y)^{d-b-1}\, y^b\, (1-xy^2)^{-a}\, (1-z)^{-c} $$

then the telescoper of

$$ \frac{1}{yz} A\Big(\frac{x}{y}, \frac{y}{z}, z\Big) $$

gives precisely the differential equation of ${}_3F_2([a,b,c],[d,1],x)$.

Integral Representation

Taking the parameter values $a = \frac{1}{9}, b = \frac{4}{9}, c = \frac{5}{9}, d = \frac{1}{3}$, one could hope that the diagonal of the algebraic function

$$\frac{y^{4/9}}{(1-y)^{10/9}\,(1-xy^2)^{1/9}\,(1-z)^{5/9}}$$

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Note: The diagonal of a rational function and a solution of the corresponding telescoper are close, yet distinct notions: the telescoper annihilates the n-fold integral over all integration cycles.

Open Problems

Future Work:

- Show that ${}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[\frac{1}{3}, 1\right], 27x\right)$ can be expressed as a diagonal of a rational function.
- Prove Christol's conjecture in general...

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Reference:

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