Algebraic Statistical Mechanics.

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La **loi Toubon** (loi n 94-665 du 4 août 1994 relative à l'emploi de la langue française) s'appuie sur une disposition introduite en 1992 dans la Constitution : "La langue de la République est le français" (article 2). Ainsi la loi reconnait le droit au citoyen français, pour les textes légaux, mais aussi au salarié pour tout ce qui touche au contrat de travail et au consommateur pour ce qui concerne la présentation des produits, les modes d'emploi et les garanties, de s'exprimer et de recevoir toute information utile en français. Corrélativement, elle crée l'obligation d'une **rédaction en langue française**. L'utilisation de l'anglais dans les entreprises a entrainé dans certains cas des problèmes de communication entre la direction et les salariés, ce qui a entrainé un certain nombre de réactions syndicales, particulièrement depuis 2004. Des entreprises **ont ainsi été condamnées pour usage illégal de l'anglais**.

Renormalization group in field theory, lattice stat. mech:

The iteration of the renormalization group generators can be seen as a discrete dynamical system acting in the parameter space of the model. The iteration of the renormalization group generators converges, **of course** to (fixed) **points** (at worst nice smooth manifolds).

Discrete dynamical systems:

The iteration of the mappings converges, **of course**, to all kind of stuff, **strange attractors**, stratified spaces, etc ...

What the difference between these two set of transformation ?

Answer: No difference. Diagnosis: soft schizophrenia.

It is OK in theoretical physics.

Vérité en deçà des Pyrénées, erreur au delà (in french in the text).

Ising n-fold integrals : the $\chi^{(n)}$'s

The magnetic susceptibility of the two-dimensional Ising model can be written as an **infinite sum** of n-folds integrals **holonomic functions**:

$$\chi(w) \;=\; \sum_{n=1}^{\infty} \chi^{(n)}(w).$$

The **magnetic susceptibility** χ is **not a holonomic function**, it is not D-finite: χ is not solution of a linear differential equation. It is much more involved.

We are even going to see that χ has a (unit circle) **natural boundary**, in the complex k-plane.

$|k| \;=\; 1$ is a natural boundary of $\chi(k)$.

Ising n-fold integrals : the $\chi^{(n)}$'s

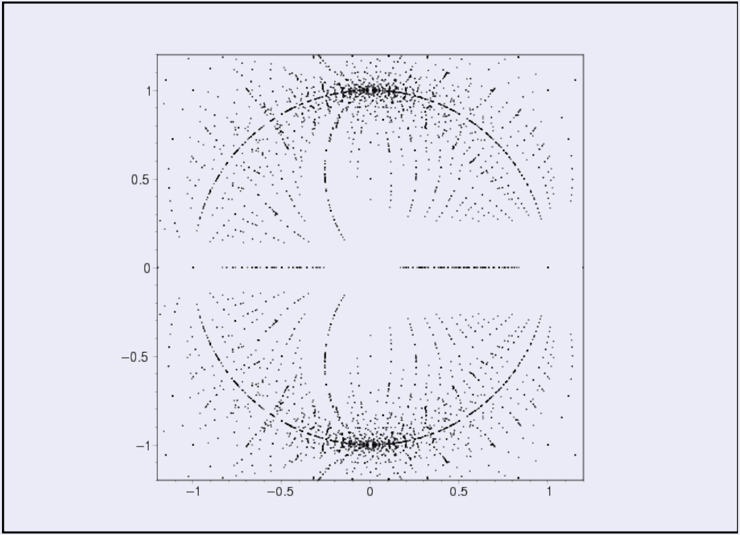
As far as series expansion are concerned, the holonomic $\tilde{\chi}^{(n)}$'s expand as a series with **integer coefficients**:

$$\tilde{\chi}^{(n)}(w) \quad = \quad 2^n \cdot w^{n^2} \cdot \kappa_n(w),$$

where:

$$\kappa_n(w) \; = \; 1 \; + 4\,n^2 \cdot w^2 \; + 2 \cdot (4\,n^4 \, + 13\,n^2 \, + 1) \cdot w^4 \\ + \frac{p_6(n)}{3} \cdot w^6 \; + \frac{p_8(n)}{3} \cdot w^8 \; + \frac{p_{10}(n)}{15} \cdot w^{10} \; + \cdots$$

$$p_6(n) \; = \; 8 \cdot \, (n^2 + 4)\,(4\,n^4 \, + 23\,n^2 + 3),$$
$$p_8(n) \; = \; \cdot (32\,n^8 \, + 624\,n^6 \, + 4006\,n^4 \, + 8643\,n^2 \, + 1404),$$
$$p_{10}(n) \; = \; 4 \cdot \, (n^2 + 8) \cdot \, (32\,n^8 \, + 784\,n^6 \, + 6238\,n^4 \\ + 16271\,n^2 \, + 3180).$$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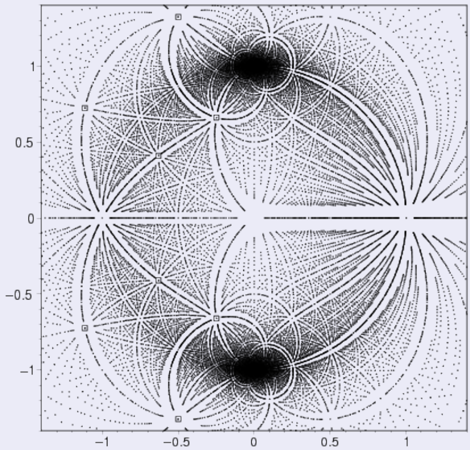
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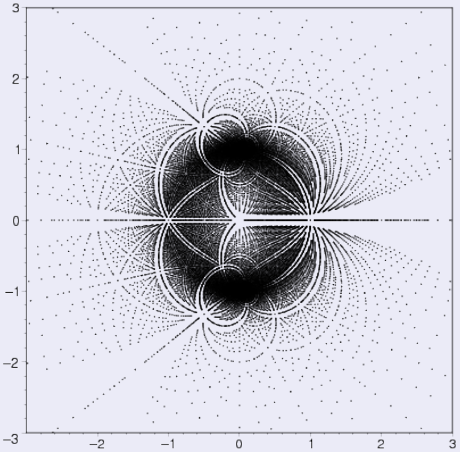
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Ising n-fold integrals : $\chi^{(5)}$

The five-particle contribution $\tilde{\chi}^{(5)}$ of the susceptibility of the Ising model is solution of an order-33 linear differential operator which has a direct-sum factorization (DFactorLCLM in Maple): the selected linear combination

$$\tilde{\chi}^{(5)} \; - \frac{1}{2}\,\tilde{\chi}^{(3)} \; + \frac{1}{120}\,\tilde{\chi}^{(1)},$$

is solution of an order-29 (globally nilpotent) linear differential operator

$$L_{29} \; = \quad L_5 \cdot L_{12} \cdot \tilde{L}_1 \cdot L_{11},$$

where:

$$L_{11} \; = \quad (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s).$$

Ising n-fold integrals : $\chi^{(6)}$

Similarly $\tilde{\chi}^{(6)}$ is solution of an order-52 linear differential operator which has a direct-sum factorization: the selected linear combination

$$\tilde{\chi}^{(6)} \; - \frac{2}{3}\tilde{\chi}^{(4)} \; + \frac{2}{45}\tilde{\chi}^{(2)},$$

is solution of an order-46 (globally nilpotent) linear differential operator

$$L_{46} \;\; = \;\; L_6 \cdot L_{23} \cdot L_{17},$$

where: $$L_{17} \; = \; \tilde{L}_5 \oplus L_3 \oplus \, (L_4 \cdot \tilde{L}_3 \cdot L_2),$$

$$\tilde{L}_5 \; = \; \left(\frac{d}{dx} - \frac{1}{x}\right) \oplus \Big(L_{1,3} \cdot \, (L_{1,2} \oplus L_{1,1} \oplus D_x)\Big).$$

The "Quarks" in $\chi^{(5)}$ and $\chi^{(6)}$

Quasi-trivial order-one (globally nilpotent) linear differential operators: $\tilde{L}_1, \; N_1, \; L_1^s, \; L_{1,n} \quad \longrightarrow \quad D_x \; - \frac{1}{N} \cdot \frac{d \ln(R(x))}{dx}$

$V_2, \; L_2, \; L_3, \; L_5$ and L_6 are respectively equivalent (homomorphic) to L_E, to the symmetric square of L_E and to the *symmetric fourth and fifth power* of L_E.

Remain to understand the "very nature" of:

$F_2, \, F_3, \, \tilde{L}_3, \quad L_4$ and: $L_{12}, \, L_{23}$

The order-12 operator L_{12} has been shown to be irreducible and not equivalent to symmetric product of differential operators of smaller orders.

L_{23}: beyond current computational resources ?

Modular Forms

Let us consider the second order linear differential operator

$$\frac{d^2}{dz^2} \;+\; \frac{(z^2 + 56\,z + 1024)}{z \cdot (z + 16)\,(z + 64)} \cdot \frac{d}{dz} \;-\; \frac{240}{z \cdot (z + 16)^2\,(z + 64)}.$$

which has the (modular form) solution:

$$\begin{aligned} &{}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; 1728\, \frac{z}{(z + 16)^3}\Big) \\ &= \; 2 \cdot \Big(\frac{z + 256}{z + 16}\Big)^{-1/4} \cdot \; {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1]; 1728\, \frac{z^2}{(z + 256)^3}\Big). \end{aligned}$$

Fundamental modular curve

The **two pull-backs** in the previous modular form

$$u \;=\; u(z) \;=\; \frac{1728\; z}{(z+16)^3}, \qquad v \;=\; \frac{1728\; z^2}{(z+256)^3} \;=\; u\Big(\frac{2^{12}}{z}\Big).$$

are related by a *Atkin-Lehner involution* $z \quad \leftrightarrow \quad 2^{12}/z$, and correspond to a rational parametrization of the **fundamental modular curve** $X_0(2)$:

$$\begin{aligned} & 5^9\, v^3\, u^3 \;-\; 12 \cdot 5^6\, u^2\, v^2 \cdot (u+v) \\ & +375\; u\, v \cdot (16\, u^2 \;+\, 16\, v^2 \;-\; 4027\, v\, u) \\ & -64\, (v+u) \cdot (v^2 + 1487\, v\, u \;+\, u^2) \;\; + 2^{12} \cdot 3^3 \cdot u\, v \;=\; 0. \end{aligned}$$

relating two **Hauptmoduls** u and v.

Dedekind η function

Getting rid of this $(2\pi)^{12}$ factor, (q is the nome of the elliptic curve) $\Delta(q) = q \cdot \prod_{n=1}^{\infty} (1 - q^n)^{24}$, one can now introduce a "second layer" of parametrization identifying the previous z with the (well-known) j-function and writing it as a ratio of **Dedekind eta function**

$$z = j(q) = \Delta(q)/\Delta(q^2), \tag{1}$$

The *Atkin-Lehner involutive* transformation $j \rightarrow 2^{12}/j$ and transformation $q \rightarrow q^2$ are *actually compatible* thanks to the remarkable "Ramanujan-like" functional identity on Dedekind η functions

$$\begin{aligned} 4096 \cdot \Delta(q) \cdot \Delta(q^4)^2 \; - \Delta(q^2)^3 \; + (\Delta(q) \\ +48 \cdot \Delta(q^2)) \cdot \Delta(q) \cdot \Delta(q^4) \;\; &= \;\; 0. \end{aligned}$$

Isogenies, Landen transformation, Renormalization group

The *exact* generators of the *renormalization group* must necessarily identify with various isogenies which amounts to multiplying, or dividing, τ the ratio of the two periods of the elliptic curves, by an integer. The simplest example is the *Landen transformation*:

$$k \quad \longleftrightarrow \quad k_L \,=\, \frac{2\sqrt{k}}{1+k}, \qquad\qquad \tau \,\longleftrightarrow\, 2\,\tau.$$

which corresponds to the previous *genus zero fundamental modular curve* two **Hauptmoduls** $u \,=\, 12^3/j$ and $v \,=\, 12^3/j'$, and relating the two j-functions

$$j(k) \,=\, 256 \cdot \frac{(1-k^2+k^4)^3}{(1-k^2)^2 \cdot k^4}, \qquad j(k_L) \,=\, 16 \cdot \frac{(1+14\,k^2+k^4)^3}{(1-k^2)^4 \cdot k^2}.$$

Deconstruction of scaling laws (pas selon Derrida)

The magnetic susceptibility χ is believed to diverge like $|t|^{\gamma}$, where t is the distance to criticality. The temperature variable t is a **real variable of course**, is it ? Scaling law: $\gamma = \beta \cdot (\delta - 1) = (2 - \eta) \cdot \nu$, $\alpha + 2\beta + \gamma = 2$, hyperscaling law: $d\nu = 2 - \alpha$. 2-D Ising: $\gamma = 7/4$. All these critical exponents are always **rational numbers of course**, are they ? You know conformal field theory, that sort of stuff, where everything is deduced from conformal invariance on a complex variable, you know z, because the variables have to be **complex variables of course**, is it ? Look at Fonseca Zamolodchikov paper ...

The well-suited temperature variable for Ising is $1 - k$ or $k - 1$.

So, k is real **and** complex. Diagnosis: soft schizophrenia.

It is OK in theoretical physics.

Deconstruction of scaling laws

Question: when k is a **complex variable** should we write $\chi \simeq (1-k)^\gamma$ (resp. $\chi \simeq (k-1)^\gamma$), because when k is real it is $\chi \simeq |1-k|^\gamma$.

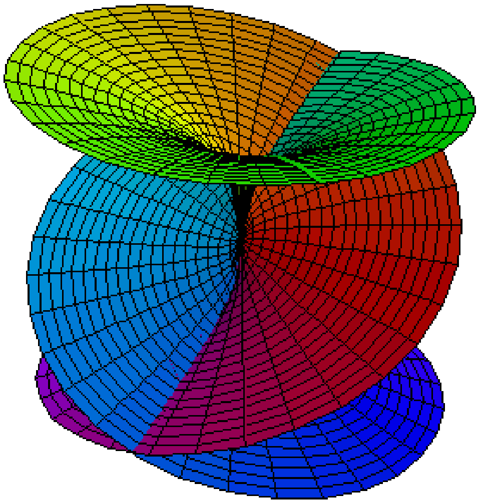
Recalling the unit k-circle **natural boundary**, $|k| = 1$, for the susceptibility, one sees that when k is a complex variable, one has

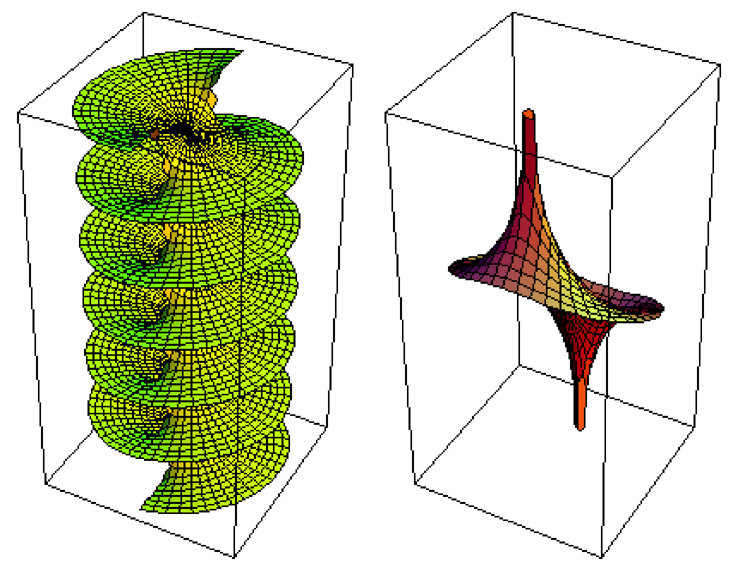
$$\chi \simeq (1-k)^\gamma, \qquad \textbf{only when} \qquad |k| < 1$$

and

$$\chi \simeq (k-1)^\gamma, \qquad \textbf{only when} \qquad |k| > 1.$$

This is not exactly a nice **Riemann surface** ...





Deconstruction of scaling laws

An **exact generator of the renormalization group must preserve** the three "points" (actually algebraic varieties): $k = 0, 1, \infty$, namely the zero and infinite temperature fixed points and the critical temperature fixed point.

Such an exact generator must also be compatible with all the exact symmetries of the model: gauge-like (linear) symmetries $SL(2, C) \times SL(2, C)$, the **infinite set of birational** (non-linear) symmetries, the **lattice of periods** of the elliptic parametrization.

The **Landen transformation** and the other **isogenies** actually satisfy **all** these constraints. They are **the only transformations satisfying these drastic constraints**.

Fixed points of the renormalization group. Heegner numbers. complex multiplication.

Seeing the Landen transformation as a transformation on **complex variables** (instead of real variables) provides, beyond $k = 0, 1,$ two other *complex* fixed points which actually correspond to *complex multiplication* for the elliptic curve, and are, actually, fundamental new singularities: $1 + 3\,w + 4\,w^2 = 0,$

$j = (-15)^3, \quad \tau = 1 - \frac{2}{\tau} \simeq \frac{\tau}{2}, \quad \tau^2 - \tau + 2 = 0$

where $w = (1 + s^2)/s/2$ with $s = \sinh(2\,K)$.

The modulus of the elliptic functions is $k = s^2$.

The fixed points of the isogenies $\tau \longleftrightarrow N \cdot \tau$ are **Heegner numbers** with **complex multiplication**.

The puzzling L_4: preliminary results on L_4

Preliminary calculations show that L_4 **cannot be reduced to elliptic functions**, modular forms, and it is not ${}_4F_3$-solvable *if one restricts to rational pull-backs*.
Is this operator going to be a **counter-example** to our favourite "mantra" that the Ising model is nothing but the theory of elliptic curves and other modular forms ?

Computing the exterior square of the linear differential operator L_4, one finds an order-six linear differential operator with the *direct sum decomposition*

$$\mathrm{ext}^{(2)}(L_4) \quad = \quad \tilde{N}_1 \oplus N_5,$$

where $\tilde{N}_1$ has a **rational function solution**. Along this line L_4 has a symplectic differential Galois group $SP(4, \mathbb{C})$.

The puzzling L_4: preliminary results on L_4

Along a globally nilpotent line, the L_4 operator is "more" than a *G-operator*, with its associated G-series. The series solution (analytical at $x = 0$) $sol(L_4)$ is a **series with integer coefficients** in the variable $y = x/2$:

$$
\begin{aligned}
sol(L_4) \;=\;\; & 175 + 34398\,y + 4017125\,y^2 + 362935156\,y^3 \\
& + 28020752579\,y^4 + 1943802285620\,y^5 + 124761498220195\,y^6 \\
& + 7549851868859190\,y^7 + 436341703365296321\,y^8 \\
& + 24309515324321362986\,y^9 + 1314618756208478845353\,y^{10} \\
& + 69377289961823319909960\,y^{11} \\
& + 3588051829563766082490527\,y^{12} \\
& + 182471551181260556637299032\,y^{13} \\
& + 9150139649421210256395488775\,y^{14} + \cdots
\end{aligned}
$$

L_4 is a Hadamard product of two elliptic curves:

it is a Calabi-Yau operator !

Seeking for $_4F_3$ hypergeometric functions up to homomorphisms, and assuming an **algebraic pull-back** with the *square root extension*, $(1 - 16 \cdot w^2)^{1/2}$, we actually found that the solution of L_4 *can be expressed in terms of a selected* $_4F_3$

$$ {}_4F_3\Big([1/2,\, 1/2,\, 1/2,\, 1/2], [1,1,1];\; z\Big) $$

$$ = \quad {}_2F_1\Big([1/2,\, 1/2], [1];\; z\Big) \star {}_2F_1\Big([1/2,\, 1/2], [1];\; z\Big), $$

where: $$ z \;=\; \Big(\frac{1 \,+\, (1 \,-\, 16 \cdot w^2)^{1/2}}{1 \,-\, (1 \,-\, 16 \cdot w^2)^{1/2}} \Big)^4 $$

where the pull-back z *is nothing but* the fourth power of the modulus k of the elliptic functions !

The $\chi^{(n)}$'s are diagonal of rational functions.

Let us consider the series of $\tilde{\chi}^{(3)}/8/w^9$

$$1 \; + \; 36\,w^2 \; + \; 4\,w^3 \; + \; 884\,w^{13} \; + \; 196\,w^5 \; + \; 18532\,w^6 \quad + \; \cdots$$

Let us now consider this very series *modulo the prime* $p \; = \; 2$. It reads the quite lacunary series

$$1 \; + w^8 \; + w^{24} \; + w^{56} \; + w^{120} \; + w^{248} \; + w^{504} \; + w^{1016} \; + \cdots \, ,$$

In fact,*modulo the prime* $p \; = \; 2$, $H(w) \; = \; \tilde{\chi}^{(3)}/8$ is, actually, an **algebraic function**, solution of the quadratic equation:

$$H(w)^2 \; + w \cdot \, H(w) \; + w^{10} \quad = \quad 0 \qquad \mod 2.$$

The $\chi^{(n)}$'s are diagonal of rational functions.

In fact, the series for $\tilde{\chi}^{(3)}$, or for **any** $\tilde{\chi}^{(n)}$, modulo **any prime**, reduces to an algebraic function (the complexity of the algebraic functions growing with p).
This is, in fact, the consequence of the fact that the $\chi^{(n)}$'s are **diagonal of rational functions**.

Definition of the diagonal of series of several complex variables:

$$\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big) \;=\; \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1, m_2, \ldots, m_n} \cdot z_1^{m_1}\, z_2^{m_2}\, \cdots\, z_n^{m_n},$$

$$Diag\Big(\mathcal{F}\Big(z_1,\, z_2,\, \ldots,\, z_n\Big)\Big) \;=\; \sum_{m=0}^{\infty} F_{m, m, \ldots, m} \cdot z^m.$$

Pedagogical examples of diagonal of rational functions.

Let us consider the rational function of three complex variables $\mathcal{F} = 1/(1 - z_2 - z_3 - z_1 z_2 - z_1 z_3)$. Its diagonal reads:

$$1 \quad + 4z \; + 36z^2 \; + 400z^3 \; + 4900z^4 \; + 63504z^5 \; + \; \cdots$$

which is nothing but the complete elliptic integral of the first kind

$$\sum_{m \geq 0} \binom{2m}{m}^2 \cdot z^m \quad = \quad {}_2F_1\Big([\frac{1}{2}, \frac{1}{2}], [1], 16\, z\Big)$$

Such diagonals of rational functions are **highly selected functions**: they are solutions of G-operators. They are also functions that are always algebraic mod. **any prime** p. They fill the gap between algebraic functions and G-series: they can be seen as **generalisations of algebraic functions**.

Towards a conclusion

• We are extemely close to achieve our journey "from Onsager to Wiles" (and now Calabi-Yau ...), where we will, finally, be able to say that the Ising model is nothing but the theory of elliptic curves, modular forms and other mirror maps and Calabi-Yau

• Do note that all the ideas, displayed here, *are not specific of the Ising model* and can be generalized to most of the problems occurring in exact lattice statistical mechanics, enumerative combinatorics, particle physics, ..., the elliptic curves being replaced by more general algebraic varieties, the Hauptmoduls being replaced by the corresponding (mirror symmetries) generalizations.

• We do hope that these ideas will, eventually yield the emergence of a New Algebraic Statistical Mechanics (NASM) classifying all the problems of theoretical physics on a completely (effective) algebraic geometry basis.

Almost the conclusion

Interplay between different domains of physics (field theory, enumerative combinatorics, lattice statistical mechanics, condensed matter, particle physics, ...) and different domains of mathematics: **Algebraic Geometry, Differential Algebra, Differential Geometry**, (differential Galois groups), **Arithmetics, Number Theory**.

Not surpringly for Yang-Baxter integrability experts, the deepest ideas do not come from continuous symmetries but do emerge with infinite discrete symmetries.

Doing physics is not doing less mathematics. Paradoxically, doing (good) physics is (without knowing it ...) doing quite fundamental mathematics, working, in a quite deep way, precisely at the crossroad of different domains of mathematics,

as Monsieur Jourdain talked prose, without knowing it.

Conclusion

Since these fascinating interplay correspond to **discrete symmetries**, let us pay, in conclusion, tribute to **Evariste Galois**.

Let us recall that the **group of permutations** of N elements can be generated by elementary transpositions:

AERES $\rightarrow$ **EARES** $\rightarrow$ **ERAES** $\rightarrow$ **ERASE**

LRU, AERES $\rightarrow$ YES WE CAN.